MATH 153 — FINAL EXAM

December 6, 2005

NAME: Solutions

1. Do not open this exam until you are told to begin.
2. This exam has 11 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so I will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the I can see not only the answer but also how you obtained it. **Include units in your answers where appropriate.**
6. You may use your calculator. However, please indicate if it is used in any significant way.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	14	
2	16	
3	12	
4	10	
5	12	
6	15	
7	6	
8	15	
TOTAL	100	

1. (2 points each) Circle "True" or "False" for each of the following problems. Circle "True" only is the statement is *always* true. No explanation is necessary.

(a) Two vectors in three-space are parallel if and only if their cross product is 0.

True False

(b) The smaller the radius of a circle, the larger the curvature of the circle.

True False

(c) The dot product of two vectors is another vector.

True False

(d) The series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}$ converges.

True False

(e) The equation $(\mathbf{u} \times \mathbf{v}) + (\mathbf{v} \bullet \mathbf{w})$ makes sense.

True False

(f) The vectors $\mathbf{i} + \mathbf{j}$ and $2\mathbf{i} - 2\mathbf{j}$ are orthogonal.

True False

(g) One has $(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{v} = 0$ for all vectors $\mathbf{u}$ and $\mathbf{v}$.

True False

2. (4 points each) Let $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $v = -\mathbf{i} + \mathbf{j} - 4\mathbf{k}$.

(a) Find $2\mathbf{u} + \mathbf{v}$.

$$\begin{aligned} 2\mathbf{u} + \mathbf{v} &= 2(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) + -\mathbf{i} + \mathbf{j} - 4\mathbf{k} \\ &= 4\mathbf{i} + 6\mathbf{j} - 8\mathbf{k} - \mathbf{i} + \mathbf{j} - 4\mathbf{k} \\ &= 3\mathbf{i} + 7\mathbf{j} - 12\mathbf{k}. \end{aligned}$$

(b) Find $\mathbf{u} \bullet \mathbf{v}$.

$$\begin{aligned} \mathbf{u} \bullet \mathbf{v} &= 2(-1) + 3(1) + (-4)(-4) \\ &= 17. \end{aligned}$$

(c) Find $|\mathbf{u}|$ and $|\mathbf{v}|$.

$$\begin{aligned} |\mathbf{u}| &= \sqrt{2^2 + 3^2 + (-4)^2} \\ &= \sqrt{29}. \end{aligned}$$

$$\begin{aligned} |\mathbf{v}| &= \sqrt{(-1)^2 + 1^2 + (-4)^2} \\ &= \sqrt{18}. \end{aligned}$$

(d) What is the angle between $\mathbf{u}$ and $\mathbf{v}$?
Recall the formula $\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta)$. Solving this for θ and inserting what we have found above we obtain

$$\theta = \cos^{-1}\left(\frac{17}{\sqrt{29}\sqrt{18}}\right) = 0.732.$$

3. (4+6 points) As the holidays approach you decide to rearrange the furniture in your living room to accomodate more people. You decide to push your couch into a new location 3 meters away from its current position. You push downward on the couch at an angle of 35 degrees with a force of 200 N for 20 seconds and are unable to budge the couch. (The 35 degrees is the angle between your force and the horizontal.)

(a) How much work have you done on the couch?

Since you are unable to move the couch, the displacement vector $\mathbf{D}$ is $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j}$, therefore, regardless of the force, one has $\mathbf{F} \bullet \mathbf{D} = \mathbf{F} \bullet (0\mathbf{i} + 0\mathbf{j}) = 0$. So there is no work done.

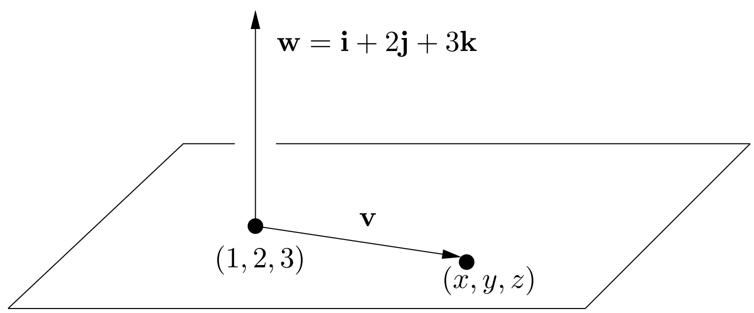
(b) A friend stops over and helps by pulling the couch while you push at the same angle with the same force as before. Together you are able to move the couch into the new position. How much work have YOU done on the couch this time?

In this situation the force you apply is actually important since there is a displacement given by $\mathbf{D} = 3\mathbf{i}$. The force you exert on the couch is given by $\mathbf{F} = 200\cos(35)\mathbf{i} - 200\sin(35)\mathbf{j}$. Therefore, taking the dot product to find the work you exert on the couch we find

$$\begin{aligned} \mathbf{F} \bullet \mathbf{D} &= (200\cos(35)\mathbf{i} - 200\sin(35)\mathbf{j}) \bullet (3\mathbf{i}) \\ &= 491.49 \text{ Nm.} \end{aligned}$$

4. (6 points each) **(a)** Find the equation of the plane through $(1, 2, 3)$ and perpendicular to the vector $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

The following picture may be helpful in visualizing what is going on.



The vector $\mathbf{v}$ is given by the formula

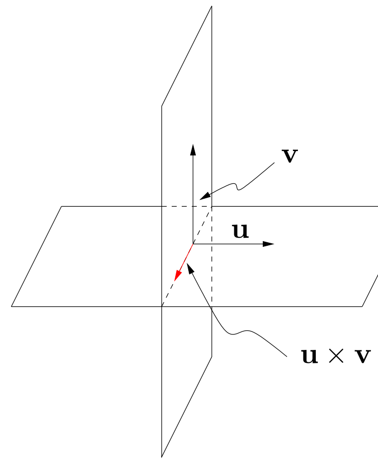
$$\mathbf{v} = (x - 1)\mathbf{i} + (y - 2)\mathbf{j} + (z - 3)\mathbf{k}.$$

In order for the plane to be perpendicular to the vector $\mathbf{w}$, it is necessary that $\mathbf{v} \bullet \mathbf{w} = 0$ for all values of x, y, and z. Therefore, we have that the plane is given by the equation

$$0 = (x - 1) + 2(y - 2) + 3(z - 3).$$

(b) Find the equation of the plane through $(-1, -2, 3)$ and perpendicular to both the planes $x - 3y + 2z = 7$ and $2x - 2y - z = -3$.

We begin by reducing this to the same type of problem as that solved in part (a) by finding a vector that is perpendicular to the plane we are trying to construct. The following picture may be helpful in visualizing what is going on.



where we note that the red vector is the vector that takes the place of $\mathbf{w}$ in part (a). The vector $\mathbf{u}$ is the normal vector to the plane $x - 3y + 2z = 7$ and so is given by $\mathbf{u} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$. Similarly, the vector $\mathbf{v}$ is the normal vector to the plane $2x - 2y - z = -3$ and so is given by $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

Therefore, the vector $\mathbf{w}$ is their cross product and is given by

$$\begin{aligned}
\mathbf{w} &= \mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ 2 & -2 & -1 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} -3 & 2 \\ -2 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -3 \\ 2 & -2 \end{vmatrix} \\
&= 7\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}
\end{aligned}$$

where we used the fact that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Now we proceed as we did in part (a). The vector in the plane here is given by

$$(x+1)\mathbf{i} + (y+2)\mathbf{j} + (z-3)\mathbf{k}.$$

Dotting this with the normal vector we obtain the plane

$$7(x+1) + 5(y+2) + 4(z-3) = 0.$$

5. (3 points each) Suppose that the function $f(x)$ has the degree 12 Taylor polynomial centered at $x = 3$ given by

$$P_{12}(x) = 5 + 6(x-3) + 34(x-3)^4 - 2(x-3)^5 + (x-3)^7 + 120(x-3)^{10} - \frac{1}{2}(x-3)^{11}.$$

Suppose that we know that $|f^{(13)}(x)| < 100$ for all $0 < x < 10$.

(a) What is the slope of the tangent line to $f(x)$ at $x = 3$?

The coefficient of $(x-3)$ is the slope of the tangent line, i.e., $f'(3) = 6$.

(b) What is the value of $f^{(7)}(3)$?

In this case we have $\frac{f^{(7)}(3)}{7!} = 1$, i.e., $f^{(7)}(3) = 7!$.

(c) Using the Taylor polynomial given, give an approximation of $f(3.5)$.

$f(3.5) \approx P_{12}(3.5) = 5 + 6(.5) + 34(.5)^4 - 2(.5)^5 + (.5)^7 + 120(.5)^{10} - (.5)^{12}$.

(d) Bound the error obtained in part (c).

The error in this case is bounded by

$$\begin{aligned}
|\text{error}| &\leq \frac{|f^{(13)}(c)|}{13!}(3.5-3)^{13} \qquad \text{for some } 3 \leq c \leq 3.5 \\
&< \frac{100}{13!}(.5)^{13} \approx 1.96 \cdot 10^{12}
\end{aligned}$$

where we used that $|f^{(13)}(c)| < 100$ since c is necessarily between 0 and 10.

6. (3 points each) Twelve inches of water sit in your sink. A foolish little bug lands on the surface of the water just as you release the drain. The bug's path along the water rushing from the sink can be modelled by the vector $\mathbf{r}(t) = \dfrac{(12-t)}{12}\cos(t)\mathbf{i} + \dfrac{(12-t)}{12}\sin(t)\mathbf{j} + (12-t)\mathbf{k}$ where t is measured in seconds and distances are measured in inches.

(a) What is the bug's initial position?

The bug's initial position occurs when $t = 0$. Plugging $t = 0$ into the equation $\mathbf{r}(t)$ giving the bug's position, we have that the bug is at the point $(1, 0, 12)$.

(b) At what time does the bug reach the drain and how did you arrive at this answer?

The bug will reach the drain when it has a height of 0 inches. This occurs when the $\mathbf{k}$ component of the position vector is 0, i.e., when $t = 12$ seconds.

(c) At time $t = \pi$ seconds, what is the bug's velocity?

To find the bug's velocity at any time t we differentiate the position vector $\mathbf{r}(t)$ with respect to time:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left(-\frac{1}{12}\cos(t) - \frac{12-t}{12}\sin(t)\right)\mathbf{i} + \left(-\frac{1}{12}\sin(t) + \frac{12-t}{12}\cos(t)\right)\mathbf{j} - \mathbf{k}.$$

To find the bug's velocity at time $t = \pi$ seconds, we plug π into the equation for $\mathbf{v}(t)$ and obtain

$$\mathbf{v}(\pi) = \left(\frac{1}{12}\right)\mathbf{i} - \left(\frac{12-\pi}{12}\right)\mathbf{j} - \mathbf{k}$$

where the velocity is measured in inches/second.

(d) At time $t = \pi$ seconds, what is the bug's speed?

We know that speed is the magnitude of the velocity, so we need only find the length of the velocity vector $\mathbf{v}(\pi)$ found in part (c).

$$\begin{aligned} |\mathbf{v}(\pi)| &= \sqrt{\left(\frac{1}{12}\right)^2 + \left(\frac{12-\pi}{12}\right)^2 + (-1)^2} \\ &\approx 1.25 \text{ inches/second.} \end{aligned}$$

(Continued on the next page)

(e) What is the curvature of the bug's path at $t = \pi$ seconds? The following may be helpful:

$$\mathbf{r}''(\pi) = \left(\frac{12-\pi}{12}\right)\mathbf{i} + \frac{1}{6}\mathbf{j}.$$

Here we use the formula provided at the back of the exam that the curvature is given by

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}.$$

We are given $\mathbf{r}''(\pi)$, found $\mathbf{r}'(\pi)$ in part (c), and found $|\mathbf{r}'(\pi)|$ in part(d), so we need only take the cross product of $\mathbf{r}'$ and $\mathbf{r}''$ and find the resulting vector's magnitude. The cross product is given by

$$\begin{aligned}
\mathbf{r}'(\pi) \times \mathbf{r}''(\pi) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{12} & \frac{\pi-12}{12} & -1 \\ \frac{12-\pi}{12} & \frac{1}{6} & 0 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} \frac{\pi-12}{12} & -1 \\ \frac{1}{6} & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{1}{12} & -1 \\ \frac{12-\pi}{12} & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{1}{12} & \frac{\pi-12}{12} \\ \frac{12-\pi}{12} & \frac{1}{6} \end{vmatrix} \\
&= \left(\frac{1}{6}\right)\mathbf{i} + \left(\frac{12-\pi}{12}\right)\mathbf{j} + 0.56\mathbf{k}.
\end{aligned}$$

Finding the length of this vector as above we obtain

$$|\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)| = 0.94.$$

Therefore the curvature at $t = \pi$ is given by

$$\kappa = \frac{0.94}{(1.25)^3} = 0.482.$$

7. (6 points) Three people decide to slice up a pie into 4 equal pieces, each of them removing a quarter of the pie. They then slice up the remaining quarter of the pie into 4 equal pieces, each of them removing a quarter. They continue slicing up and removing the remaining part in this way. Show that each person ends up with one third of the pie in the end.

After the first division, each person has $\frac{1}{4}$ of the pie. There is $\frac{1}{4}$ of the pie remaining to divide. They divide this $\frac{1}{4}$ of the pie into quarters, so pies of size $\frac{1}{16}$ the pie. After their remove their pieces after the second division each person has $\frac{1}{4} + \frac{1}{16}$ of the pie. This can be written more suggestively as $\frac{1}{4} + \left(\frac{1}{4}\right)^2$. After the third division, each person removes $\frac{1}{4}$ of the $\frac{1}{16}$ that was remaining. Thus, after three divisions each person has $\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3$ of the pie. Continuing in this pattern indefinitely each person will have $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$ of the pie. This is a geometric series with $a = \frac{1}{4}$ and $r = \frac{1}{4}$, so we see the sum is given by

$$\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}.$$

Thus each person ends up with $\frac{1}{3}$ of the pie in the long run.

8. (3+4+4+4 points) Consider the circle of radius a centered at $(a, 0)$.

(a) Write the Cartesian equation for this circle.

$(x-a)^2 + y^2 = a^2$.

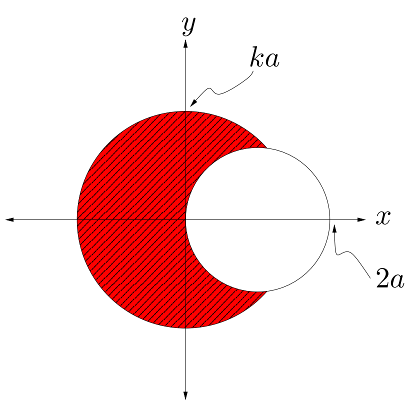
(b) Show that one can write the equation from part (a) as $r = 2a\cos(\theta)$.

Substitute the equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into the equation from part (a).

$$\begin{aligned} a^2 &= (r\cos(\theta) - a)^2 + r^2\sin^2(\theta) \\ a^2 &= r^2\cos^2(\theta) - 2ar\cos(\theta) + a^2 + r^2\sin^2(\theta) \\ 0 &= r^2(\cos^2(\theta) + \sin^2(\theta)) - 2ar\cos(\theta) \\ r^2 &= 2ar\cos(\theta). \end{aligned}$$

(c) A goat is tethered to the edge of a circular pond of radius a by a rope of length ka $(0 < k \leq 2)$. Graph on the axes below the grazing area of the goat. Be sure to label your graph appropriately. (Hint: This was exactly a homework problem. The pond is represented by the circle from part (a). The rope should sweep out a circle of radius ka.)

The red shaded area is the grazing area of the goat.



(d) Set up but do NOT evaluate an integral for the grazing area of the goat.

Recall that the area bounded between angles α and β by a function $r = f(\theta)$ is given by

$$\frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta.$$

We apply this equation to this problem, finding the red area shaded above. First we find the intersection points of the circles so as to let us know the bounds of our integration. To accomplish this we set $ka = 2a\cos(\theta)$ and solve for θ, obtaining $\theta = \cos^{-1}\left(\frac{k}{2}\right)$. We use the fact that $0 < k \leq 2$ to assure that this is the angle we see in the first quadrant. There is a bit of a trick here; we did an example somewhat like this in class. From the picture it looks as if the upper limit of our integration should be $\theta = \pi$. However, if you investigate what this means in terms of the circle representing the pond, one sees that this would correspond to going all the way around the circle. The circle representing the pond is back at $(0, 0)$ when $\theta = \frac{\pi}{2}$. So the area in the first quadrant is given by

$$\int_{\cos^{-1}\left(\frac{k}{2}\right)}^{\frac{\pi}{2}} \left(\frac{1}{2}\right) \left((ka)^2 - (2a\cos(\theta))^2\right) d\theta.$$

The area in the second quadrant is one-quarter the area of a circle of radius ka, so is $\frac{\pi(ka)^2}{4}$. Therefore, the total grazing area is given by

$$\frac{\pi(ka)^2}{2} + \int_{\cos^{-1}\left(\frac{k}{2}\right)}^{\frac{\pi}{2}} \left((ka)^2 - (2a\cos(\theta))^2\right) d\theta.$$

Of course, one could leave it in integral form as

$$\int_{\cos^{-1}\left(\frac{k}{2}\right)}^{\pi} (ka)^2 d\theta - \int_{\cos^{-1}\left(\frac{k}{2}\right)}^{\frac{\pi}{2}} (2a\cos(\theta))^2 d\theta.$$