MATH 153 — FINAL EXAM

December 6, 2005

NAME: ______________________________

1. Do not open this exam until you are told to begin.
2. This exam has 11 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so I will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the I can see not only the answer but also how you obtained it. **Include units in your answers where appropriate.**
6. You may use your calculator. However, please indicate if it is used in any significant way.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	14	
2	16	
3	12	
4	10	
5	12	
6	15	
7	6	
8	15	
TOTAL	100	

1. (2 points each) Circle "True" or "False" for each of the following problems. Circle "True" only is the statement is *always* true. No explanation is necessary.

(a) Two vectors in three-space are parallel if and only if their cross product is 0.

True False

(b) The smaller the radius of a circle, the larger the curvature of the circle.

True False

(c) The dot product of two vectors is another vector.

True False

(d) The series $\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}$ converges.

True False

(e) The equation $(\mathbf{u} \times \mathbf{v}) + (\mathbf{v} \bullet \mathbf{w})$ makes sense.

True False

(f) The vectors $\mathbf{i} + \mathbf{j}$ and $2\mathbf{i} - 2\mathbf{j}$ are orthogonal.

True False

(g) One has $(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{v} = 0$ for all vectors $\mathbf{u}$ and $\mathbf{v}$.

True False

2. (4 points each) Let $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $v = -\mathbf{i} + \mathbf{j} - 4\mathbf{k}$.

(a) Find $2\mathbf{u} + \mathbf{v}$.

(b) Find $\mathbf{u} \bullet \mathbf{v}$.

(c) Find $|\mathbf{u}|$ and $|\mathbf{v}|$.

(d) What is the angle between $\mathbf{u}$ and $\mathbf{v}$?

3. (4+6 points) As the holidays approach you decide to rearrange the furniture in your living room to accomodate more people. You decide to push your couch into a new location 3 meters away from its current position. You push downward on the couch at an angle of 35 degrees with a force of 200 N for 20 seconds and are unable to budge the couch. (The 35 degrees is the angle between your force and the horizontal.)

(a) How much work have you done on the couch?

(b) A friend stops over and helps by pulling the couch while you push at the same angle with the same force as before. Together you are able to move the couch into the new position. How much work have YOU done on the couch this time?

4. (6 points each) **(a)** Find the equation of the plane through $(1, 2, 3)$ and perpendicular to the vector $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(b) Find the equation of the plane through $(-1, -2, 3)$ and perpendicular to both the planes $x - 3y + 2z = 7$ and $2x - 2y - z = -3$.

5. (3 points each) Suppose that the function $f(x)$ has the degree 12 Taylor polynomial centered at $x = 3$ given by

$$P_{12}(x) = 5 + 6(x-3) + 34(x-3)^4 - 2(x-3)^5 + (x-3)^7 + 120(x-3)^{10} - \frac{1}{2}(x-3)^{11}.$$

Suppose that we know that $|f^{(13)}(x)| < 100$ for all $0 < x < 10$.

(a) What is the slope of the tangent line to $f(x)$ at $x = 3$?

(b) What is the value of $f^{(7)}(3)$?

(c) Using the Taylor polynomial given, give an approximation of $f(3.5)$.

(d) Bound the error obtained in part (c).

6. (3 points each) Twelve inches of water sit in your sink. A foolish little bug lands on the surface of the water just as you release the drain. The bug's path along the water rushing from the sink can be modelled by the vector $\mathbf{r}(t) = \dfrac{(12-t)}{12}\cos(t)\mathbf{i} + \dfrac{(12-t)}{12}\sin(t)\mathbf{j} + (12-t)\mathbf{k}$ where t is measured in seconds and distances are measured in inches.

(a) What is the bug's initial position?

(b) At what time does the bug reach the drain and how did you arrive at this answer?

(c) At time $t = \pi$ seconds, what is the bug's velocity?

(d) At time $t = \pi$ seconds, what is the bug's speed?

(Continued on the next page)

(e) What is the curvature of the bug's path at $t = \pi$ seconds? The following may be helpful:

$$\mathbf{r}''(\pi) = \left(\frac{12 - \pi}{12}\right)\mathbf{i} + \frac{1}{6}\mathbf{j}.$$

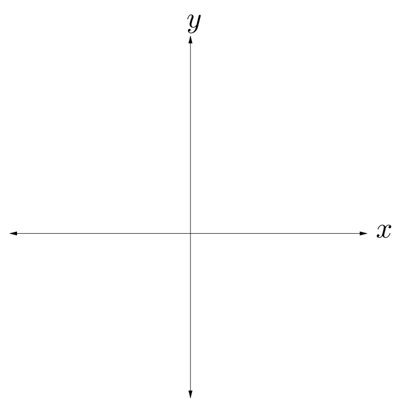
7. (6 points) Three people decide to slice up a pie into 4 equal pieces, each of them removing a quarter of the pie. They then slice up the remaining quarter of the pie into 4 equal pieces, each of them removing a quarter. They continue slicing up and removing the remaining part in this way. Show that each person ends up with one third of the pie in the end.

8. (3+4+4+4 points) Consider the circle of radius a centered at $(a, 0)$.

(a) Write the Cartesian equation for this circle.

(b) Show that one can write the equation from part (a) as $r = 2a\cos(\theta)$.

(c) A goat is tethered to the edge of a circular pond of radius a by a rope of length ka $(0 < k \leq 2)$. Graph on the axes below the grazing area of the goat. Be sure to label your graph appropriately. (Hint: This was exactly a homework problem. The pond is represented by the circle from part (a). The rope should sweep out a circle of radius ka.)



(d) Find the grazing area of the goat.

Theorem 1. *(Integral Test) Let f be a continuous, positive, nonincreasing function on the interval $[1, \infty)$ and suppose that $a_k = f(k)$ for all positive integers k. Then the infinite series*

$$\sum_{k=1}^{\infty} a_k$$

converges if and only if the improper integral

$$\int_1^{\infty} f(x)dx$$

converges.

Theorem 2. *(p-series) The series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

Theorem 3. *(Ordinary Comparison Test) Suppose that $0 \leq a_n \leq b_n$ for all $n \geq N$.*
(i) If $\sum b_n$ converges, so does $\sum a_n$.
(ii) If $\sum a_n$ diverges, so does $\sum b_n$.

Theorem 4. *(Limit Comparison Test) Suppose that $a_n \geq 0$, $b_n > 0$, and*

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L.$$

If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ converge of diverge together. If $L = 0$ and $\sum b_n$ is convergent, then $\sum a_n$ converges.

Theorem 5. *(Alternating Series Test) Let*

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

be an alternating series with $a_n > a_{n+1} > 0$. If $\lim_{n \to \infty} a_n = 0$, then the series converges. Moreover, the error made by using the sum S_n of the first n terms to approximate the sum S of the series is not more than a_{n+1}.

Theorem 6. *(Absolute Ratio Test) Let $\sum a_n$ be a series of nonzero terms and suppose that*

$$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \rho.$$

(i) If $\rho < 1$, the series converges absolutely (hence converges).
(ii) If $\rho > 1$, the series diverges.
(iii) If $\rho = 1$, the test is inconclusive.

Some Potentially Useful Formulas

$$
\begin{aligned}
1 &= \cos^2\theta + \sin^2\theta \\
\sec^2\theta &= \tan^2\theta + 1 \\
\csc^2\theta &= 1 + \cot^2\theta \\
\sin(\theta \pm \phi) &= \sin\theta\cos\phi \pm \cos\theta\sin\phi \\
\cos(\theta \pm \phi) &= \cos\theta\cos\phi \mp \sin\theta\sin\phi \\
\sin^2\theta &= \frac{1}{2}(1 - \cos 2\theta) \\
\cos^2\theta &= \frac{1}{2}(1 + \cos 2\theta) \\
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta
\end{aligned}
$$

$$\text{Work} = \mathbf{F} \bullet \mathbf{D}$$

$$\kappa = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}}$$

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$$