An Advanced Course in Linear Algebra

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Chapter 1

Introduction

These notes are the product of teaching Math 8530 several times at Clemson University. This course serves as the first year breadth course for the Algebra and Discrete Mathematics subfaculty and is the only algebra course many of our graduate students will see. As such, this course was designed to be a serious introduction to linear algebra at the graduate level giving students a flavor of what modern abstract algebra consists of. While undergraduate abstract algebra is not a formal prerequisite, it certainly helps. To aid with this an appendix reviewing essential undergraduate concepts is included.

It is assumed students have had an undergraduate course in basic linear algebra and are comfortable with concepts such as multiplying matrices, Gaussian elimination, finding the null and column space of matrices, etc. In this course we work with abstract vector spaces over an arbitrary field and linear transformations. We do not limit ourselves to $\mathbb{R}^n$ and matrices, but do rephrase the more general results in terms of matrices for the convenience of the reader. Once a student has completed and mastered the material in these notes s/he should have no trouble translating these results into the results typically presented in a first or second semester undergraduate linear algebra or matrix analysis course.

While it certainly would be helpful to use modules at several places in these notes, the main content of these notes is presented without modules. There are some sections dealing with modules (and more forthcoming), but these sections are for the interested reader and are not presented in the actual course.

Please report any errors you find in these notes to me so that they may be corrected.

The motivation for typing these notes was provided by Kara Stasikelis. She provided her typed class notes from this course Summer 2013. These were greatly appreciated and gave me the starting point from which I typed this vastly expanded version.

Chapter 2

Vector spaces

In this chapter we give the basic definitions and facts having to do with vector spaces that will be used throughout the rest of the course. Many of the results in this chapter are covered in an undergraduate linear algebra class in terms of matrices. We often convert the language back to that of matrices, but the focus is on abstract vector spaces and linear transformations.

Throughout this chapter F is a field.

2.1 Getting started

We begin with the definition of a vector space.

Definition 2.1.1. Let V be a non-empty set with an operation

$$\begin{aligned} V \times V &\to V \\ (v, w) &\mapsto v + w \end{aligned}$$

referred to as *addition* and an operation

$$\begin{aligned} F \times V &\to V \\ (c, v) &\mapsto cv \end{aligned}$$

referred to as *scalar multiplication* so that V satisfies the following properties:

1. $(V, +)$ is an abelian group;

2. $c(v + w) = cv + cw$ for all $c \in F$, $v, w \in V$;

3. $(c + d)v = cv + dv$ for all $c, d \in F$, $v \in V$;

4. $(cd)v = c(dv)$ for all $c, d \in F$, $v \in V$;

5. $1_F \cdot v = v$ for all $v \in V$.

We say V is a *vector space* (or an *F-vector space*). If we need to emphasize the addition and scalar multiplication we write $(V, +, \cdot)$ instead of just V. We call elements of V *vectors* and elements of F *scalars.*

We now recall some familiar examples from undergraduate linear algebra. The verification that these are actually vector spaces is left as an exercise.

Example 2.1.2. Set $F^n = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_i \in F \right\}$. Then F^n is a vector space with addition given by

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

and scalar multiplication given by

$$c \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{pmatrix}.$$

Example 2.1.3. Let F and K be fields and suppose $F \subseteq K$. Then K is an F-vector space. The typical example of this one sees in undergraduate linear algebra is $\mathbb{C}$ as an $\mathbb{R}$-vector space.

Example 2.1.4. Let $m, n \in \mathbb{Z}_{>0}$. Set $\mathrm{Mat}_{m,n}(F)$ to be the m by n matrices with entries in F. This forms an F-vector space with addition being matrix addition and scalar multiplication given by multiplying each entry of the matrix by the scalar. In the case $m = n$ we write $\mathrm{Mat}_n(F)$ for $\mathrm{Mat}_{n,n}(F)$.

Example 2.1.5. Let $n \in \mathbb{Z}_{\geq 0}$. Define

$$P_n(F) = \{f \in F[x] | \deg(f) \leq n\}.$$

This is an F-vector space. We also have that

$$F[x] = \bigcup_{n \geq 0} P_n(F)$$

is an F-vector space.

Example 2.1.6. Let U and V be subsets of $\mathbb{R}$. Define $C^0(U, V)$ to be the set of continuous functions from U to V. This forms an $\mathbb{R}$-vector space. Let $f, g \in C^0(U, V)$ and $c \in \mathbb{R}$. Addition is defined point-wise, namely, $(f + g)(x) = f(x) + g(x)$ for all $x \in U$. Scalar multiplication is given point-wise as well: $(cf)(x) = cf(x)$ for all $x \in U$.

More generally, for $k \in \mathbb{Z}_{\geq 0}$ we let $C^k(U,V)$ be the functions from U to V so that $f^{(j)}$ is continuous for all $0 \leq j \leq k$ where $f^{(j)}$ denotes the jth derivative of f. This is an $\mathbb{R}$-vector space. If $U = V$ we write $C^k(U)$ for $C^k(U,U)$.

We set $C^\infty(U,V)$ to be the set of smooth functions, i.e., $f \in C^\infty(U,V)$ if $f \in C^k(U,V)$ for all $k \geq 0$. This is an $\mathbb{R}$-vector space. If $U = V$ we write $C^\infty(U)$ for $C^\infty(U,U)$.

Example 2.1.7. Set

$$F^\infty = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix} : a_i \in F, a_i = 0 \text{ for all but finitely many } i \right\}$$

is an F-vector space.

Set

$$F^{\mathbb{N}} = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix} : a_i \in F \right\}.$$

This is an F-vector space.

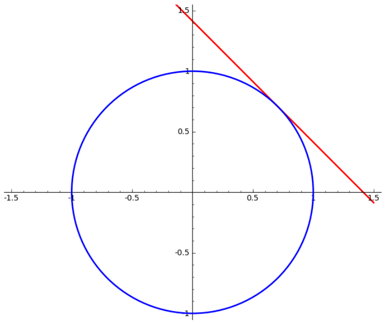
Example 2.1.8. Consider the sphere

$$S^{n-1} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}.$$

Let $p = (a_1, \ldots, a_n)$ be a point on the sphere. We can realize the sphere as the $w = 1$ level surface of the function $w = f(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$. The gradient of f is $\nabla f = 2x_1e_1 + 2x_2e_2 \cdots + 2x_ne_n$ where $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \cdots, 0, 1)$. This gives that the tangent plane at the point p is given by

$$2a_1(x_1 - a_1) + 2a_2(x_2 - a_2) + \cdots + 2a_n(x_n - a_n) = 0.$$

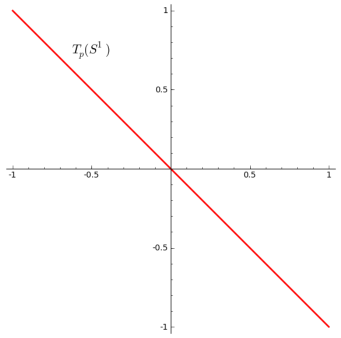
This is pictured in the $n = 2$ case here:



Note the tangent plane is not a vector space because it does not contain a zero vector, but if we shift it to the origin we have a vector space called the tangent space of S^{n-1} at p:

$$T_p(S^{n-1}) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : 2a_1x_1 + \cdots + 2a_nx_n = 0\}.$$

The shifted tangent space from the graph pictured above is given here:



Lemma 2.1.9. *Let V be an F-vector space.*

1. *The zero element $0_V \in V$ is unique.*
2. *We have $0_F \cdot v = 0_V$ for all $v \in V$.*
3. *We have $(-1_F) \cdot v = -v$ for all $v \in V$.*

Proof. Suppose that $0, 0'$ both satisfy the conditions necessary to be 0_V, i.e.,

$$\begin{gathered} 0 + v = v = v + 0 \text{ for all } v \in V \\ 0' + v = v = v + 0' \text{ for all } v \in V. \end{gathered}$$

We apply this with $v = 0$ to obtain $0 = 0 + 0'$, and now use $v = 0'$ to obtain $0 + 0' = 0'$. Thus, we have that $0 = 0'$.

Observe that $0_F \cdot v = (0_F + 0_F) \cdot v = 0_F \cdot v + 0_F \cdot v$. So if we subtract $0_F \cdot v$ from both sides we get that $0_V = 0_F \cdot v$.

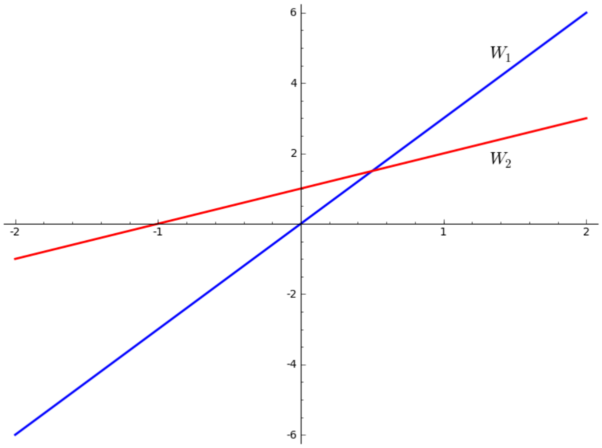
Finally, we have $(-1_F) \cdot v + v = (-1_F) \cdot v + (1_F) \cdot v = (-1_F + 1_F) \cdot v = 0_F \cdot v = 0_V$, i.e., $(-1_F) \cdot v + v = 0_V$. So $(-1_F) \cdot v$ is the unique additive identity of v, i.e., $(-1_F) \cdot v = -v$. □

Note we will often drop the subscripts on 0 and 1 when they are clear from context.

Definition 2.1.10. Let V be an F-vector space and let W be a nonempty subset of V. If W is an F-vector space with the same operations as V we call W a *subspace* (or an *F-subspace*) of V.

Example 2.1.11. We saw above that $V = \mathbb{C}$ is an $\mathbb{R}$-vector space. Set $W = \{x + 0 \cdot i : x \in \mathbb{R}\}$. Then W is an $\mathbb{R}$-subspace of V. We have that V is also a $\mathbb{C}$-vector space. However, W is not a $\mathbb{C}$-subspace of V as it is not closed under scalar multiplication by i.

Example 2.1.12. Let $V = \mathbb{R}^2$. In the graph below W_1 is a subspace but W_2 is not. It is easy to check that any line passing through the origin is a subspace, but a line not passing through the origin cannot be a subspace because it does not contain the zero element.



Example 2.1.13. Let $n \in \mathbb{Z}_{\geq 0}$. We have $P_j(F)$ is a subspace of $P_n(F)$ for all $0 \leq j \leq n$.

Example 2.1.14. The F-vector space F^∞ is a subspace of $F^{\mathbb{N}}$.

Example 2.1.15. Let $V = \mathrm{Mat}_n(F)$. The set of diagonal matrices is a subspace of V.

The following lemma gives easy to check criterion for when one has a subspace. The proof is left as an exercise.

Lemma 2.1.16. *Let V an F-vector space and $W \subseteq V$. Then W is a subspace of V if*

1. *W is nonempty;*
2. *W is closed under addition;*
3. *W is closed under scalar multiplication.*

As is customary in algebra, in order to study objects we first introduce the subobjects (subspaces in our case) and then the appropriate maps between the objects of interest. In our case these are linear transformations.

Definition 2.1.17. Let V and W be F-vector spaces and let $T : V \to W$ be a map. We say T is a *linear transformation* (or *F-linear*) if

1. $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in V$;
2. $T(cv) = cT(v)$ for all $c \in F$, $v \in V$.

The collection of all F-linear maps from V to W is denoted $\mathrm{Hom}_F(V, W)$.

Example 2.1.18. Define $\mathrm{id}_V : V \to V$ by $\mathrm{id}_V(v) = v$ for all $v \in V$. Then $\mathrm{id}_V \in \mathrm{Hom}_F(V, V)$. We refer to this as the *identity transformation*.

Example 2.1.19. Let $T : \mathbb{C} \to \mathbb{C}$ be defined by $T(z) = \overline{z}$. Since $\mathbb{C}$ is a $\mathbb{C}$ and an $\mathbb{R}$-vector space, it is natural to ask if T is $\mathbb{C}$-linear or $\mathbb{R}$-linear. Observe we have

$$\begin{aligned} T(z+w) &= \overline{z+w} \\ &= \overline{z} + \overline{w} \\ &= T(z) + T(w) \end{aligned}$$

for all $z, w \in \mathbb{C}$. However, if we let $c \in \mathbb{C}$ we have

$$\begin{aligned} T(cv) &= \overline{cv} \\ &= \overline{c}T(v). \end{aligned}$$

Note that if $T(v) \neq 0$, then $T(cv) = cT(v)$ if and only if $c = \overline{c}$. Thus, T is not $\mathbb{C}$-linear but is $\mathbb{R}$-linear.

Example 2.1.20. Let $m, n \in \mathbb{Z}_{\geq 1}$. Let $A \in \mathrm{Mat}_{m,n}(F)$. Define $T_A : F^n \to F^m$ by $T_A(x) = Ax$. This is an F-linear map.

Example 2.1.21. Set $V = C^\infty(\mathbb{R})$. In this example we give several linear maps that arise in calculus.

Given any $a \in \mathbb{R}$, define

$$\begin{aligned} E_a : V &\to \mathbb{R} \\ f &\mapsto f(a). \end{aligned}$$

We have $E_a \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ for every $a \in \mathbb{R}$.

Define

$$\begin{aligned} D : V &\to V \\ f &\mapsto f'. \end{aligned}$$

We have $D \in \mathrm{Hom}_{\mathbb{R}}(V, V)$.

Let $a \in \mathbb{R}$. Define

$$\begin{aligned} I_a : V &\to V \\ f &\mapsto \int_a^x f(t)dt. \end{aligned}$$

We have $I_a \in \operatorname{Hom}_{\mathbb{R}}(V, V)$ for every $a \in \mathbb{R}$.

Let $a \in \mathbb{R}$. Define

$$\begin{aligned} \widetilde{E}_a : V &\to V \\ f &\mapsto f(a) \end{aligned}$$

where here we view $f(a)$ as the constant function. We have $\widetilde{E}_a \in \operatorname{Hom}_{\mathbb{R}}(V, V)$.

We can use these linear maps to rephrase the fundamental theorems of calculus as follows:

1. $D \circ I_a = \mathrm{id}_V$;
2. $I_a \circ D = \mathrm{id}_V - \widetilde{E}_a$.

Exercise 2.1.22. Let V and W be F-vector spaces. Show that $\operatorname{Hom}_F(V, W)$ is an F-vector space.

Lemma 2.1.23. *Let $T \in \operatorname{Hom}_F(V, W)$. Then $T(0_V) = 0_W$.*

Proof. This is proved using the properties of the additive identity element:

$$\begin{aligned} T(0_V) &= T(0_V + 0_V) \\ &= T(0_V) + T(0_V), \end{aligned}$$

i.e., $T(0_V) = T(0_V) + T(0_V)$. Now, subtract $T(0_V)$ from both sides to obtain $0_W = T(0_V)$ as desired. □

Exercise 2.1.24. Let U, V, W be F-vector spaces. Let $S \in \operatorname{Hom}_F(U, V)$ and $T \in \operatorname{Hom}_F(V, W)$. Then $T \circ S \in \operatorname{Hom}_F(U, W)$.

In the next chapter we will focus on matrices and their relation with linear maps much more closely, but we have the following elementary result that we can prove immediately.

Lemma 2.1.25. *Let $m, n \in \mathbb{Z}_{\geq 1}$.*

1. *Let $A, B \in \operatorname{Mat}_{m,n}(F)$. Then $A = B$ if and only if $T_A = T_B$.*
2. *Every $T \in \operatorname{Hom}_F(F^n, F^m)$ is given by T_A for some $A \in \operatorname{Mat}_{m,n}(F)$.*

Proof. If $A = B$ then clearly we must have $T_A = T_B$ from the definition. Conversely, suppose $T_A = T_B$. Consider the standard vectors $e_1 = {}^t(1, 0, \ldots, 0), e_2 = {}^t(0, 1, 0, \ldots, 0), \ldots, e_n = {}^t(0, \ldots, 0, 1)$. We have $T_A(e_j) = T_B(e_j)$ for $j = 1, \ldots, n$. However, it is easy to see that $T_A(e_j)$ is the jth column of A. Thus, $A = B$.

Let $T \in \mathrm{Hom}_F(F^n, F^m)$. Set

$$A = \left(T(e_1) \vdots \cdots \vdots T(e_n) \right).$$

It is now easy to check that $T = T_A$. □

In general, we do not multiply vectors. However, if $V = \mathrm{Mat}_n(F)$, we can multiply vectors in here! So V is a vector space, but also a ring. In this case, V is an example of an F-algebra. Though we will not be concerned with algebras in these notes, we give the definition here for the sake of completeness.

Definition 2.1.26. An *F-algebra* is a ring A with a multiplicative identity together with a ring homomorphism $f : F \to A$ mapping 1_F to 1_A so that $f(F)$ is contained in the center of A, i.e., if $a \in A$ and $f(c) \in f(F)$, then $af(c) = f(c)a$.

Exercise 2.1.27. Show that $F[x]$ is an F-algebra.

The fact that we can multiply in $\mathrm{Mat}_n(F)$ is due to the fact that we can compose linear maps. In fact, it is natural to define matrix multiplication to be the matrix associated to the composition of the associated linear maps. This definition explains the “bizarre” multiplication rule defined on matrices.

Definition 2.1.28. Let m, n and p be positive integers. Let $A \in \mathrm{Mat}_{m,n}(F)$, $B \in \mathrm{Mat}_{n,p}(F)$. Then $AB \in \mathrm{Mat}_{m,p}(F)$ is the matrix corresponding to $T_A \circ T_B$.

Exercise 2.1.29. Show this definition of matrix multiplication agrees with that given in undergraduate linear algebra class.

Definition 2.1.30. Let $T \in \mathrm{Hom}_F(V, W)$ be invertible, i.e., there exists $T^{-1} \in \mathrm{Hom}_F(W, V)$ so that $T \circ T^{-1} = \mathrm{id}_W$ and $T^{-1} \circ T = \mathrm{id}_V$. We say T is an *isomorphism* and we say V and W are *isomorphic* and right $V \cong W$.

Exercise 2.1.31. Let $T \in \mathrm{Hom}_F(V, W)$. Show that T is an isomorphism if and only if T is bijective.

Example 2.1.32. Let $V = \mathbb{R}^2, W = \mathbb{C}$. These are both $\mathbb{R}$-vector spaces. Define

$$\begin{aligned} T : \mathbb{R}^2 &\to \mathbb{C} \\ (x, y) &\mapsto x + iy. \end{aligned}$$

We have $T \in \mathrm{Hom}_{\mathbb{R}}(V, W)$. It is easy to see this is an isomorphism with inverse given by $T^{-1}(x + iy) = (x, y)$. Thus, $\mathbb{C} \cong \mathbb{R}^2$ as $\mathbb{R}$-vector spaces.

Example 2.1.33. Let $V = P_n(F)$ and $W = F^{n+1}$. Define a map $T : V \to W$ by sending $a_0 + a_1 x + \cdots + a_n x^n$ to ${}^t(a_0, \ldots, a_n)$. It is elementary to check this is an isomorphism.

Example 2.1.34. Let $V = \mathrm{Mat}_n(F)$ and $W = F^{n^2}$. Define $T : V \to W$ by sending $A = (a_{i,j})$ to ${}^t(a_{1,1}, a_{1,2}, \ldots, a_{n,n})$. This gives an isomorphism between V and W.

Definition 2.1.35. Let $T \in \operatorname{Hom}_F(V, W)$. The *kernel of* T is given by

$$\ker T = \{v \in V : T(v) = 0_W\}.$$

The image of T is given by

$$\operatorname{Im} T = \{w \in W : \text{there exists } v \in V \text{ with } T(v) = w\} = T(V).$$

One should note that in undergraduate linear algebra one often refers to $\ker T$ as the null space and $\operatorname{Im} T$ the column space when $T = T_A$ for some $A \in \operatorname{Mat}_{m,n}(F)$.

Lemma 2.1.36. *Let $T \in \operatorname{Hom}_F(V, W)$. Then $\ker T$ is a subspace of V and $\operatorname{Im} T$ is a subspace of W.*

Proof. First, observe that $0_V \in \ker T$ so $\ker T$ is nonempty. Now let $v_1, v_2 \in \ker T$ and $c \in F$. Then we have

$$\begin{aligned} T(v_1 + v_2) &= T(v_1) + T(v_2) \\ &= 0_W + 0_W \\ &= 0_W \end{aligned}$$

and

$$\begin{aligned} T(cv_1) &= cT(v_1) \\ &= c \cdot 0_W \\ &= 0_W. \end{aligned}$$

Thus, $\ker T$ is a subspace of V.

Next we show that $\operatorname{Im} T$ is a subspace. We have $\operatorname{Im} T$ is nonempty because $T(0_V) = 0_W \in \operatorname{Im} T$. Let $w_1, w_2 \in \operatorname{Im} T$ and $c \in F$. There exists $v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$. Thus,

$$\begin{aligned} w_1 + w_2 &= T(v_1) + T(v_2) \\ &= T(v_1 + v_2). \end{aligned}$$

and

$$\begin{aligned} cw_1 &= cT(v_1) \\ &= T(cv_1). \end{aligned}$$

Thus, $w_1 + w_2$ and cw_1 are both in $\operatorname{Im} T$, i.e., $\operatorname{Im} T$ is a subspace of W. □

Example 2.1.37. Let $m, n \in \mathbb{Z}_{>0}$ with $m > n$. Define $T : F^m \to F^n$ by

$$T\left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}\right) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

The image of this map is F^n and the kernel is given by

$$\ker T = \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{n+1} \\ \vdots \\ a_m \end{pmatrix} : a_i \in F \right\}.$$

It is easy to see that $\ker T \cong F^{m-n}$.

Define $S : F^m \to F^n$ by

$$S\left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}\right) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where there are $m-n$ zeroes. The image of this map is isomorphic to F^{m-n} and the kernel is trivial.

Example 2.1.38. Define $T : \mathbb{Q}[x] \to \mathbb{R}$ by $T(f(x)) = f(\sqrt{3})$. This is a $\mathbb{Q}$-linear map. The kernel of this map consists of those polynomials $f \in \mathbb{Q}[x]$ satisfying $f(\sqrt{3}) = 0$, i.e., those polynomials f for which $(x^2-3) \mid f$. Thus, $\ker T = (x^2-3)\mathbb{Q}[x]$. The image of this map clearly contains $\mathbb{Q}(\sqrt{3}) = \{a+b\sqrt{3} : a, b \in \mathbb{Q}\}$. Let $\alpha \in \operatorname{Im}(T)$. Then there exists $f \in \mathbb{Q}[x]$ so that $f(\sqrt{3}) = \alpha$. Write $f = q(x^2-3) + r$ with $\deg r < 2$. Then

$$\begin{aligned} \alpha &= f(\sqrt{3}) \\ &= q(\sqrt{3})((\sqrt{3})^2 - 3) + r(\sqrt{3}) \\ &= r(\sqrt{3}). \end{aligned}$$

Since $\deg r < 2$, we have $\alpha = r(\sqrt{3}) = a + b\sqrt{3}$ for some $a, b \in \mathbb{Q}$. Thus, $\operatorname{Im}(T) = \mathbb{Q}(\sqrt{3})$.

2.2 Bases and dimension

One of the most important features of vector spaces is the fact that they have bases. This essentially means that one can always find a nice subset (in many interesting cases a finite set) that will completely describe the space. Many of the concepts of this section are likely familiar from undergraduate linear algebra.

Throughout this section V denotes an F-vector space unless indicated otherwise.

Definition 2.2.1. Let $\mathcal{B} = \{v_i\}_{i \in I}$ be a subset of V. We say $v \in V$ is an F-*linear combination* of $\mathcal{B}$ and write $v \in \operatorname{span}_F \mathcal{B}$ if there exists a finite collection $\{c_1, \dots, c_n\} \subset F$ such that

$$v = \sum_{i=1}^{n} c_i v_i.$$

Definition 2.2.2. Let $\mathcal{B} = \{v_i\}_{i \in I}$ be a subset of V. We say $\mathcal{B}$ is F-*linearly independent* if whenever we have a finite linear combination $\sum c_i v_i = 0$ for some $c_i \in F$ we must have $c_i = 0$ for all i.

Definition 2.2.3. Let $\mathcal{B} = \{v_i\}$ be a subset of V. We say $\mathcal{B}$ is an F-*basis* of V if

1. $\operatorname{span}_F \mathcal{B} = V$;
2. $\mathcal{B}$ is linearly independent.

As is usual, if F is clear from context we drop it from the notation and simply say "linear combination", "linearly independent", and "basis".

The first goal of this section is to show every vector space has a basis. We will need Zorn's lemma to prove this. We recall Zorn's lemma for the convenience of the reader.

Theorem 2.2.4 (Zorn's Lemma)**.** *Let X be any partially ordered set with the property that every chain in X has an upper bound. Then X contains at least one maximal element.*

Theorem 2.2.5. *Let V be a vector space. Let $\mathcal{A} \subseteq \mathcal{C}$ be subsets of V. Furthermore, assume $\mathcal{A}$ linearly independent and $\mathcal{C}$ spans V. Then there exists a basis $\mathcal{B}$ of V satisfying $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$. In particular, if we set $\mathcal{A} = \emptyset$, $\mathcal{C} = V$, then this says that V has a basis.*

Proof. Let

$$X = \{\mathcal{B}' \subseteq V : \mathcal{A} \subseteq \mathcal{B}' \subseteq \mathcal{C}, \mathcal{B}' \text{ is linearly independent}\}.$$

We have X is a partially ordered set under inclusion and is nonempty because $\mathcal{A} \in X$. We also have that $\mathcal{C}$ provides an upper bound on X. This allows us to apply Zorn's Lemma to the chain X to conclude it has a maximal element, say $\mathcal{B}$. If $\operatorname{span}_F \mathcal{B} = V$, we are done. Suppose $\operatorname{span}_F \mathcal{B} \neq V$. Then there exists $v \in \mathcal{C}$ such that $v \notin \operatorname{span}_F \mathcal{B}$. However, this gives $\mathcal{B}' = \mathcal{B} \cup \{v\}$ is an element of X that properly contains $\mathcal{B}$, a contradiction. Thus, $\mathcal{B}$ is the desired basis. □

One should note that the above proof gives that every vector space has a basis, but it does not provide an algorithm for constructing a basis. We will come back to this issue momentarily. We first deal with the easier case that there is a finite collection of vectors $\mathcal{B}$ so that $\operatorname{span}_F \mathcal{B} = V$. In this case we say V is a *finite dimensional F-vector space.* We will make use of the following fact from undergraduate linear algebra.

Lemma 2.2.6. *([3, Theorem 1.1]) A homogeneous system of m linear equations in n unknowns with $m < n$ always has nontrivial solutions.*

Corollary 2.2.7. *Let $\mathcal{B} \subseteq V$ be such that* $\operatorname{span}_F \mathcal{B} = V$ *and $\#\mathcal{B} = m$. Then any set with more than m elements cannot be linearly independent.*

Proof. Let $\mathcal{C} = \{w_1, \dots, w_n\}$ with $n > m$. Let $\mathcal{B} = \{v_1, \dots, v_m\}$ be a spanning set for V. For each i write

$$w_i = \sum_{j=1}^{m} a_{ji} v_j.$$

Consider the equation

$$\sum_{i=1}^{n} a_{ji} x_i = 0.$$

The previous lemma gives a solution $(c_1, \dots, c_n)$ to this equation with $(c_1, \dots, c_n) \neq (0, \dots, 0)$. We have

$$\begin{aligned} 0 &= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ji} c_i \right) v_j \\ &= \sum_{i=1}^{n} c_i \left(\sum_{j=1}^{m} a_{ji} v_j \right) \\ &= \sum_{i=1}^{n} c_i w_i. \end{aligned}$$

This shows $\mathcal{C}$ is not linearly independent. □

Theorem 2.2.8. *Let V be a finite dimensional F-vector space. Any two bases of V have the same number of elements.*

Proof. Let $\mathcal{B}$ and $\mathcal{C}$ be bases of V. Suppose that $\#\mathcal{B} = m$ and $\#\mathcal{C} = n$. Since $\mathcal{B}$ is a basis, it is a spanning set and since $\mathcal{C}$ is a basis it is linearly independent. The previous result now gives so $n \leq m$. We now reverse the roles of $\mathcal{B}$ and $\mathcal{C}$ to get the other direction. □

Now that we have shown any two bases of a finite dimensional vector space have the same number of elements we can make the following definition.

Definition 2.2.9. Let V be a finite dimensional vector space. The number of elements in a F-basis of V is called the *F-dimension of V* and written $\dim_F V$.

Note that if V is not finite dimensional we will write $\dim_F V = \infty$. The notion of basis is not very useful in this context, as we will see below, so we do not spend much time on it here.

The following theorem does not contain anything new, but it is useful to summarize some of the important facts to this point.

Theorem 2.2.10. *Let V be a finite dimensional F-vector space with $\dim_F V = n$. Let $\mathcal{C} \subseteq V$ with $\#\mathcal{C} = m$.*

1. *If $m > n$, then $\mathcal{C}$ is not linearly independent.*
2. *If $m < n$, then $\operatorname{span}_F \mathcal{C} \neq V$.*
3. *If $m = n$, then the following are equivalent:*
   - *$\mathcal{C}$ is a basis;*
   - *$\mathcal{C}$ is linearly independent;*
   - *$\mathcal{C}$ spans V.*

This theorem immediately gives the following corollary.

Corollary 2.2.11. *Let $W \subseteq V$ be a subspace. Then $\dim_F W \leq \dim_F V$. If $\dim_F V < \infty$, then $V = W$ if and only if $\dim_F V = \dim_F W$.*

We now give several examples of bases of some familiar vector spaces. It is left as an exercise to check these are actually bases.

Example 2.2.12. Let $V = F^n$. Set $e_1 = {}^t(1, 0, \ldots, 0), e_2 = {}^t(0, 1, 0, \ldots, 0), \ldots, e_n = {}^t(0, \ldots, 0, 1)$. Then $\mathcal{E} = \{e_1, \ldots, e_n\}$ is a basis for V and is often referred to as the *standard basis.* We have $\dim_F F^n = n$.

Example 2.2.13. Let $V = \mathbb{C}$. From the previous example we have $\mathcal{B} = \{1\}$ is a basis of V as a $\mathbb{C}$-vector space and $\dim_{\mathbb{C}} \mathbb{C} = 1$. We saw before that $\mathbb{C}$ is also a $\mathbb{R}$-vector space. In this case a basis is given by $\mathcal{C} = \{1, i\}$ and $\dim_{\mathbb{R}} \mathbb{C} = 2$.

Example 2.2.14. Let $V = \operatorname{Mat}_{m,n}(F)$. Let $e_{i,j}$ to be the matrix with 1 in (i, j)th position and zeros elsewhere. Then $\mathcal{B} = \{e_{i,j}\}$ is a basis for V over F and $\dim_F \operatorname{Mat}_{m,n}(F) = mn$.

Example 2.2.15. Set $V = \mathfrak{sl}_2(\mathbb{C})$ where

$$\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{Mat}_2(\mathbb{C}) : a + d = 0 \right\}.$$

One can check this is a $\mathbb{C}$-vector space. Moreover, it is a proper subspace because $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is not in $\mathfrak{sl}_2(\mathbb{C})$. Set

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

It is easy to see that $\mathcal{B}$ is linearly independent. We know that $\dim_{\mathbb{C}} \mathfrak{sl}_2(\mathbb{C}) < 4$ because it is a proper subspace of $\operatorname{Mat}_2(\mathbb{C})$. This gives $\dim_{\mathbb{C}} \mathfrak{sl}_2(\mathbb{C}) = 3$ and $\mathcal{B}$ is a basis. The vector space $\mathfrak{sl}_2(\mathbb{C})$ is actually the Lie algebra associated to the Lie group $\mathrm{SL}_2(\mathbb{C})$. There is a very rich theory of Lie algebras and Lie groups, but this is too far afield to delve into here. (Note the algebra multiplication on $\mathfrak{sl}_2(\mathbb{C})$ is not matrix multiplication, it is given by $X \cdot Y = [X, Y] = XY - YX$.)

Example 2.2.16. Let $f(x) \in F[x]$ be an polynomial of degree n. We can use this polynomial to split $F[x]$ into equivalence classes analogously to how one creates the field $\mathbb{F}_p$. For details of this construction, please see Example A.0.25 found in Appendix A. This is an F-vector space under addition given by $[g(x)] + [h(x)] := [g(x) + h(x)]$ and scalar multiplication given by $c[g(x)] := [cg(x)]$.

Note that given any $g(x) \in F[x]$ we can use the Euclidean algorithm on $F[x]$ (polynomial long division) to find unique polynomials $q(x)$ and $r(x)$ satisfying $g(x) = f(x)q(x) + r(x)$ where $r(x) = 0$ or $\deg r < \deg f$. This shows that each nonzero equivalence class has a representative $r(x)$ with $\deg r < \deg f$. Thus, we can write

$$F[x]/(f(x)) = \{[r(x)] : r(x) \in F[x] \text{ with } r = 0 \text{ or } \deg r < \deg f\}.$$

From this one can see that a spanning set for $F[x]/(f(x))$ is given by $\{[1], [x], \ldots, [x^{n-1}]\}$. This is also seen to be linearly independent by observing if there exists $a_0, \ldots, a_{n-1} \in F$ so that $[a_0 + a_1x + \cdots + a_{n-1}x^{n-1}] = [0]$, this means $f(x)$ divides $a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$. However, this is impossible unless $a_0 + a_1x + \cdots + a_{n-1}x^{n-1} = 0$ because $\deg f = n$. Thus $a_0 = a_1 = \cdots = a_{n-1} = 0$. This gives that $F[x]/(f(x))$ is an F-vector space of degree n.

Exercise 2.2.17. Show that $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$ as $\mathbb{R}$-vector spaces.

The following lemma gives an alternate way to check if a set is a basis.

Lemma 2.2.18. *Let V be an F-vector space and let $\mathcal{C} = \{v_j\}$ be a subset of V. We have $\mathcal{C}$ is a basis of V if and only if every vector in V can be written uniquely as a linear combination of elements in $\mathcal{C}$.*

Proof. First suppose $\mathcal{C}$ is a basis. Let $v \in V$. Since $\operatorname{span}_F \mathcal{C} = V$, we can write any vector as a linear combination of elements in $\mathcal{C}$. Suppose we can write $v = \sum a_i v_i = \sum b_i v_i$ for some $a_i, b_i \in F$. Subtracting these equations gives $0 = \sum (a_i - b_i)v_i$. We now use that the v_i are linearly independent to conclude $a_i - b_i = 0$, i.e., $a_i = b_i$ for every i.

Now suppose every $v \in V$ can be written uniquely as a linear combination of the elements in $\mathcal{C}$. This immediately gives $\operatorname{span}_F \mathcal{C} = V$. It remains to show $\mathcal{C}$ is linearly independent. Suppose there exists $a_i \in F$ with $\sum a_i v_i = 0$. We also have $\sum 0v_i = 0$, so the uniqueness gives $a_i = 0$ for every i, i.e., $\mathcal{C}$ is linearly independent. □

Before we go further, we briefly delve into these concepts for infinite dimensional spaces. We begin with two familiar examples. The first works out exactly as one would hope; the second shows things are not as nice in general.

Example 2.2.19. Consider the vector space $F[x]$. This cannot be a finite dimensional vector space. For instance, if $\{f_1, \ldots, f_n\}$ were a basis, the element x^{M+1} for $M = \max_{1 \leq j \leq n} \deg f_j$ would not be in the span of these vectors. We can find a basis for this space though. Consider the collection $\mathcal{B} = \{1, x, x^2, \ldots\}$. It is clear this set is linearly independent and spans $F[x]$, thus it forms a basis.

Example 2.2.20. Recall the vector space $V = \mathbb{R}^{\mathbb{N}}$ defined earlier. This can be identified with sequences $\{a_n\}$ of real numbers. One might be interested in a basis for this vector space. At first glance the most obvious choice would be $\mathcal{E} = \{e_1, e_2, \dots, \}$. However, it is immediate that this set does not span V as $v = (1, 1, \dots)$ can not be represented as a finite linear combination of these elements. Now we know since v is not in $\operatorname{span}_{\mathbb{R}} \mathcal{E}$, that $\mathcal{E} \cup \{v\}$ is a linearly independent set. However, it is clear this does not span either as $(1, 2, 3, 4, \dots)$ is not in the span of this set. We know that V has a basis, but it can be shown that no countable collection of vectors forms a basis for this space. Thus, one cannot construct a basis of this space by adding one vector at a time. The next thing one might try to do is to allow oneself to add infinitely many vectors. However, without some notion of convergence this does not make sense. For instance, how would one define $(1, 1, \dots) + (2, 2, \dots) + (3, 3, \dots) + \cdots$?

The previous example shows that while we know every vector space has a basis, it may not be practical to construct such a basis. In fact, this definition of basis is not very useful for infinite dimensional spaces and is given the name *Hamel basis* since other more useful concepts are often referred to as bases in this setting. We will not say more about other notions here as these are more appropriate for a functional analysis course. We will deal mostly with finite dimensional vector spaces in this course.

The following proposition will be used repeatedly throughout the course. It says that a linear transformation between vector spaces is completely determined by what it does to a basis. One way we will often use this is to define a linear transformation by only specifying what it does to a basis. This provides another important application of the fact that vector spaces are completely determined by their bases.

Proposition 2.2.21. *Let V, W be vector spaces.*

1. *Let $T \in \operatorname{Hom}_F(V, W)$. Then T is determined by its values on a basis of V.*
2. *Let $\mathcal{B} = \{v_i\}$ be a basis of V and $\mathcal{C} = \{w_i\}$ be any collection of vectors in W so that $\#\mathcal{B} = \#\mathcal{C}$. There is a unique linear transformation $T \in \operatorname{Hom}_F(V, W)$ satisfying $T(v_i) = w_i$.*

Proof. Let $\mathcal{B} = \{v_i\}$ be a basis of V. Given any $v \in V$ there are elements $a_i \in F$ so that $v = \sum a_i v_i$. We have

$$\begin{aligned} T(v) &= T\left(\sum a_i v_i\right) \\ &= \sum T(a_i v_i) \\ &= \sum a_i T(v_i). \end{aligned}$$

Thus, if one knows the elements $T(v_i)$, one knows $T(v)$ for any $v \in V$. This gives the first claim.

The second part follows immediately from the first. Set $T(v_i) = w_i$ for each i. For $v = \sum a_i v_i \in V$, define $T(v)$ by

$$T(v) = \sum a_i w_i.$$

It is now easy to check this is a linear map from V to W and is unique. $\square$

Example 2.2.22. Let $V = W = \mathbb{R}^2$. It is easy to check that $\mathcal{B} = \left\{ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is a basis of V. The previous results says to define a map from V to W it is enough to say where to send v_1 and v_2. For instance, let $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ -10 \end{pmatrix} \right\}$ be a subset of W. Then we have a unique linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(v_1) = w_1$, $T(v_2) = w_2$. Note it is exactly this property that allows us to represent a linear map $T : F^n \to F^m$ as a matrix.

Corollary 2.2.23. *Let $T \in \mathrm{Hom}_F(V, W)$, $\mathcal{B} = \{v_i\}$ be a basis of V, and $\mathcal{C} = \{w_i = T(v_i)\} \subseteq W$. Then $\mathcal{C}$ is a basis for W if and only if T is an isomorphism.*

Proof. Suppose $\mathcal{C}$ is a basis for W. The previous result allows us to define $S : W \to V$ such that $S(w_i) = v_i$. This is an inverse for T, so T is an isomorphism.

Now suppose T is an isomorphism. We need to show that $\mathcal{C}$ spans W and it is linearly independent. Let $w \in W$. Since T is an isomorphism there exists a $v \in V$ with $T(v) = w$. Using that $\mathcal{B}$ is a basis of V we can write $v = \sum a_i v_i$ for some $a_i \in F$. Applying T to v we have

$$\begin{aligned} w &= T(v) \\ &= T\left(\sum a_i v_i\right) \\ &= \sum a_i T(v_i) \\ &= \sum a_i w_i. \end{aligned}$$

Thus, $w \in \mathrm{span}_F \mathcal{C}$ and since w was arbitrary we have $\mathrm{span}_F \mathcal{C} = W$. Suppose there exists $a_i \in F$ with $\sum a_i w_i = 0$. We have

$$\begin{aligned} T(0) &= 0 \\ &= \sum a_i w_i \\ &= \sum a_i T(v_i) \\ &= \sum T(a_i v_i) \\ &= T\left(\sum a_i v_i\right). \end{aligned}$$

Applying that T is injective we have $0 = \sum a_i v_i$. However, $\mathcal{B}$ is a basis so it must be the case that $a_i = 0$ for all i. This gives $\mathcal{C}$ is linearly independent and so a basis. $\square$

The following result ties the dimension of the kernel of a linear transformation, the dimensions of the image, and the dimension of the domain vector space. This is extremely useful as one often knows (or can bound) two of the three.

Theorem 2.2.24. *Let V be a finite dimensional vector space and let $T \in \mathrm{Hom}_F(V, W)$. Then*

$$\dim_F \ker T + \dim_F \operatorname{Im} T = \dim_F V.$$

Proof. Let $\dim_F \ker T = k$ and $\dim_F V = n$. Let $\mathcal{A} = \{v_1, \dots, v_k\}$ be a basis of $\ker T$ and extend this to a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of V. It is enough to show that $\mathcal{C} = \{T(v_{k+1}), \dots, T(v_n)\}$ is a basis for $\operatorname{Im} T$.

Let $w \in \operatorname{Im} T$. There exists $v \in V$ so that $T(v) = w$. Since $\mathcal{B}$ is a basis for V there exists a_i such that $v = \sum_{i=1}^{n} a_i v_i$. This gives

$$\begin{aligned} w &= T(v) \\ &= T\left(\sum_{i=1}^{n} a_i v_i\right) \\ &= \sum_{i=1}^{n} a_i T(v_i) \\ &= \sum_{i=k+1}^{n} a_i T(v_i) \end{aligned}$$

where we have used $T(v_i) = 0$ for $i = 1, \dots, k$ because $\mathcal{A}$ is a basis for $\ker T$. Thus, $\mathrm{span}_F\, \mathcal{C} = \operatorname{Im} T$. It remains to show $\mathcal{C}$ is linearly independent. Suppose there exists $a_i \in F$ such that $\sum_{i=k+1}^{n} a_i T(v_i) = 0$. Note

$$\begin{aligned} 0 &= \sum_{i=k+1}^{n} T(a_i v_i) \\ &= T\left(\sum_{i=k+1}^{n} a_i v_i\right). \end{aligned}$$

Thus, $\sum_{i=k+1}^{n} a_i v_i \in \ker T$. However, $\mathcal{A}$ spans $\ker T$ so there exists $a_1, \dots, a_k$ in F such that

$$\sum_{i=k+1}^{n} a_i v_i = \sum_{i=1}^{k} a_i v_i,$$

i.e.,

$$\sum_{i=k+1}^{n} a_i v_i + \sum_{i=1}^{k} (-a_i) v_i = 0.$$

Since $\mathcal{B} = \{v_1, \dots, v_n\}$ is a basis of V we must have $a_1 = \cdots = a_k = -a_{k+1} = \cdots = -a_n = 0$. In particular, $a_{k+1} = \cdots = a_n = 0$ as desired. □

This theorem allows to prove the following important results.

Corollary 2.2.25. *Let V, W be F-vector spaces with $\dim_F V = n$. Let $V_1 \subseteq V$ be a subspace of dimension k and $W_1 \subseteq W$ be a subspace of dimension $n - k$. Then there exists $T \in \operatorname{Hom}_F(V, W)$ such that $V_1 = \ker T$ and $W_1 = \operatorname{Im} T$.*

Proof. Let $\mathcal{B} = \{v_1, \dots, v_k\}$ be a basis of V_1. Extend this to a basis $\{v_1, \dots, v_k, \dots, v_n\}$ of V. Let $\mathcal{C} = \{w_{k+1}, \dots, w_n\}$ be a basis of W_1. Define T by $T(v_1) = \cdots = T(v_k) = 0$ and $T(v_{k+1}) = w_{k+1}, \dots, T(v_n) = w_n$. This is the required linear map. □

The previous corollary says the only limitation on a subspace being the kernel or image of a linear transformation is that the dimensions add up properly. One should contrast this to the case of homomorphisms in group theory for example. There in order to be a kernel one requires the subgroup to satisfy the further property of being a normal subgroup. This is another way in which vector spaces are very nice to work with.

The following corollary follows immediately from Theorem 2.2.24. This corollary makes checking a map between two vector spaces of the same dimension is an isomorphism much easier as one only needs to check the map is injective or surjective, not both.

Corollary 2.2.26. *Let $T \in \operatorname{Hom}_F(V, W)$ and $\dim_F V = \dim_F W$. Then the following are equivalent:*

1. *T is an isomorphism;*

2. *T is surjective;*

3. *T is injective.*

We can also rephrase this result in terms of matrices.

Corollary 2.2.27. *Let $A \in \operatorname{Mat}_n(F)$. Then the following are equivalent:*

1. *A is invertible;*

2. *There exists $B \in \operatorname{Mat}_n(F)$ with $BA = 1_n$;*

3. *There exists $B \in \operatorname{Mat}_n(F)$ with $AB = 1_n$.*

One should prove the previous two corollaries as well as the following corollary as exercises.

Corollary 2.2.28. *Let V, W be F-vector spaces and let $\dim_F V = m$, $\dim_F W = n$.*

1. *If $m < n$ and $T \in \mathrm{Hom}_F(V, W)$, then T is not surjective.*
2. *If $m > n$ and $T \in \mathrm{Hom}_F(V, W)$, then T is not injective.*
3. *We have $m = n$ if and only if $V \cong W$. In particular, $V \cong F^m$.*

Note that while it is true that if $\dim_F V = \dim_F W = n < \infty$ then $V \cong W$, it is not the case that every linear map from V to W is an isomorphism. This result is only saying there is a map $T : V \to W$ that is an isomorphism. Clearly we can define a linear map $T : V \to W$ by $T(v) = 0_W$ for all $v \in V$ and this is not an isomorphism.

The previous corollary gives a very important fact: if V is an n-dimensional F-vector space, then $V \cong F^n$. This result gives that for any positive integer n, all F-vector spaces of dimension n are isomorphic. One obtain this isomorphism by choosing a basis. This is why in undergraduate linear algebra one often focuses almost exclusively on the vector spaces F^n and matrices as the linear transformations.

The following example gives a nice application of what we have studied thus far.

Example 2.2.29. Recall $P_n(\mathbb{R})$ is the vector space of polynomials of degree less than or equal to n and $\dim_{\mathbb{R}} P_n(\mathbb{R}) = n + 1$. Set $V = P_{n-1}(\mathbb{R})$. Let $a_1, \ldots, a_k \in \mathbb{R}$ be distinct and pick $m_1, \ldots, m_k \in \mathbb{Z}_{\geq 0}$ such that $\displaystyle\sum_{j=1}^{k}(m_j+1) = n$. Our goal is to show given any real numbers $b_{1,0}, \ldots, b_{1,m_1}, \ldots, b_{k,m_k}$ there is a unique polynomial $f \in P_{n-1}(\mathbb{R})$ satisfying $f^{(j)}(a_i) = b_{i,j}$.

Define

$$\begin{aligned} T :&P_{n-1}(\mathbb{R}) \to \mathbb{R}^n \\ f(x) \mapsto &\begin{pmatrix} f(a_1) \\ \vdots \\ f^{(m_1)}(a_1) \\ \vdots \\ f(a_k) \\ \vdots \\ f^{(m_k)}(a_k) \end{pmatrix}. \end{aligned}$$

If $f \in \ker T$, then for each $i = 1, \ldots, k$ we have $f^{(j)}(a_i) = 0$ for $j = 1, \ldots, m_i$. Thus, for each i we have $(x - a_i)^{m_i+1} \mid f(x)$. Since these polynomials are relatively prime, this gives their product divides f and thus f is divisible by a polynomial of degree n. Since $f \in P_{n-1}(F)$ this implies $f = 0$. Hence $\ker T = 0$. Applying Theorem 2.2.24 we have $\mathrm{Im}\, T$ must have dimension n, i.e., $\mathrm{Im}\, T = \mathbb{R}^n$. This gives the result.

2.3 Direct sums and quotient spaces

In this section we cover two of the basic constructions in vector space theory, namely the direct sum and the quotient space. The direct sum is a way to split a vector space into subspaces that are "independent." The quotient space construction is a way to identify some vectors to be 0. We begin with the direct sum.

Definition 2.3.1. Let V be an F-vector space and $V_1, \dots, V_k$ be subspaces of V. The *sum* of $V_1, \dots, V_k$ is defined by

$$V_1 + \cdots + V_k = \{v_1 + \cdots + v_k : v_i \in V_i\}.$$

Definition 2.3.2. Let $V_1, \dots, V_k$ be subspaces of V. We say $V_1, \dots, V_k$ are *independent* if whenever $v_1 + \cdots + v_k = 0$ with $v_i \in V_i$, then $v_i = 0$ for all i.

Definition 2.3.3. Let $V_1, \dots, V_k$ be subspaces of V. We say V is the *direct sum* of $V_1, \dots, V_k$ and write

$$V = V_1 \oplus \cdots \oplus V_k$$

if the following two conditions are satisfied

1. $V = V_1 + \cdots + V_k$;
2. $V_1, \dots, V_k$ are independent.

Example 2.3.4. Set $V = F^2$, $V_1 = \{(x, 0) : x \in F\}$, $V_2 = \{(0, y) : y \in F\}$. Then we clearly have $V_1 + V_2 = \{(x, y) : x, y \in F\} = V$. Moreover, if $(x, 0) + (0, y) = (0, 0)$, then $(x, y) = (0, 0)$ and so $x = y = 0$. This gives $(x, 0) = (0, 0) = (0, y)$ and so V_1 and V_2 are independent. Thus $F^2 = V_1 \oplus V_2$.

Example 2.3.5. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of V. Set $V_i = F \cdot v_i = \{av_i : a \in F\}$. We have $V_1, \dots, V_n$ are clearly subspaces of V, and by the definition of a basis we obtain $V = V_1 \oplus \cdots \oplus V_n$.

Lemma 2.3.6. *Let V be a vector space, $V_1, \dots, V_k$ be subspaces of V. We have $V = V_1 \oplus \cdots \oplus V_k$ if and only if each $v \in V$ can be written uniquely in the form $v = v_1 + \cdots + v_k$ with $v_i \in V_i$.*

Proof. Suppose $V = V_1 \oplus \cdots \oplus V_k$. Then certainly we have $V = V_1 + \cdots + V_k$ and so given any $v \in V$ there are elements $v_i \in V_i$ so that $v = v_1 + \cdots + v_k$. The only thing to show is this expression is unique. Suppose $v = v_1 + \cdots + v_k = w_1 + \cdots + w_k$ with $v_i, w_i \in V_i$. Then $0 = (v_1 - w_1) + \cdots + (v_k - w_k)$. Since the V_i are independent we have $v_i - w_i = 0$, i.e., $v_i = w_i$ for all i.

Suppose each $v \in V$ can be written uniquely in the form $v = v_1 + \cdots + v_k$ with $v_i \in V_i$. Immediately we get $V = V_1 + \cdots + V_k$. Suppose $0 = v_1 + \cdots + v_k$ with $v_i \in V_i$. We have $0 = 0 + \cdots + 0$ as well, so by uniqueness, we get $v_i = 0$ for all i. This gives $V_1, \dots, V_k$ are independent and so $V = V_1 \oplus \cdots \oplus V_k$. □

Exercise 2.3.7. Let $V_1, \dots, V_k$ be subspaces of V. For each i let $\mathcal{B}_i$ be a basis of V_i. Set $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$. Then

1. $\mathcal{B}$ spans V if and only if $V = V_1 + \cdots + V_k$;
2. $\mathcal{B}$ is linearly independent if and only if $V_1, \dots, V_k$ are independent;
3. $\mathcal{B}$ is a basis if and only if $V = V_1 \oplus \cdots \oplus V_k$.

Given a subspace $U \subset V$, it is natural to ask if there is a subspace W so that $V = U \oplus W$. This leads us to the following definition.

Definition 2.3.8. Let V be a vector space and $U \subseteq V$ a subspace. We say $W \subseteq V$ is a *complement of* U *in* V if $V = U \oplus W$.

Lemma 2.3.9. *Let $U \subseteq V$ be a subspace. Then U has a complement.*

Proof. Let $\mathcal{A}$ be a basis of U, and extend $\mathcal{A}$ to a basis of $\mathcal{B}$ of V. Set $W = \operatorname{span}_F(\mathcal{B} - \mathcal{A})$. One checks immediately that $V = U \oplus W$. □

Exercise 2.3.10. Let $U \subset V$ be a subspace. Is the complement of U unique? If so, prove it. If not, give a counterexample.

We now turn our attention to quotient spaces. We have already seen an example of a quotient space. Namely, recall Example 2.2.16. Consider $V = F[x]$ and $W = (f(x)) := \{g \in F[x] : f|g\} = [0]$. One can check that W is a subspace of V. In that example we defined a vector space $V/W = F[x]/(f(x))$ and saw that the elements in W become 0 in the new space. This construction is the one we generalize to form quotient spaces.

Let V be a vector space and $W \subseteq V$ be a subspace. Define an equivalence relation on V as follows $v_1 \sim_W v_2$ if and only if $v_1 - v_2 \in W$. We write the equivalence classes as

$$[v_1] = \{v_2 \in V : v_1 - v_2 \in W\} = v_1 + W.$$

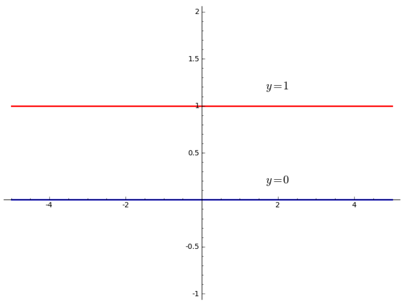
Set $V/W = \{v + W : v \in V\}$. Addition and scalar multiplication on V/W are defined as follows. Let $v_1, v_2 \in V$ and $c \in F$. Define

$$\begin{aligned}(v_1 + W) + (v_2 + W) &= (v_1 + v_2) + W;\\ c(v_1 + W) &= cv_1 + W.\end{aligned}$$

Exercise 2.3.11. Show that V/W is an F-vector space.

We call V/W the *quotient space* of V by W.

Example 2.3.12. Let $V = \mathbb{R}^2$ and $W = \{(x, 0) : x \in \mathbb{R}\}$. Clearly we have $W \subseteq V$ is a subspace. Let $(x_0, y_0) \in V$. To find $(x_0, y_0) + W$, we want all (x, y) such that $(x_0, y_0) - (x, y) \in W$. However, it is clear that $(x_0 - x, y_0 - y) \in W$ if and only if $y = y_0$. Thus, $(x_0, y_0) + W = \{(x, y_0) : x \in \mathbb{R}\}$. The graph below gives the elements $(0, 0) + W$ and $(0, 1) + W$.



One immediately sees from this that $(x, y) + W$ is not a subspace of V unless $(x, y) + W = (0, 0) + W$.

Define

$$\begin{aligned} \pi : \mathbb{R} &\to V/W \\ y_0 &\mapsto (x_0, y_0). \end{aligned}$$

It is straightforward to check this is an isomorphism, so $V/W \cong \mathbb{R}$.

Example 2.3.13. More generally, let $m, n \in \mathbb{Z}_{>0}$ with $m > n$. Consider $V = F^m$ and let W be the subspace of V spanned by $e_1, \ldots, e_n$ with $\{e_1, \ldots, e_m\}$ the standard basis. We can form the quotient space V/W. This space is isomorphic to F^{m-n} with a basis given by $\{e_{n+1} + W, \ldots, e_m + W\}$.

Example 2.3.14. Let $V = F[x]$ and let $W = (f(x))$. We saw before that the quotient space $V/W = F[x]/(f(x))$ has as a basis $\{[1], [x], \ldots, [x^{n-1}]\}$. Define $T : V/W \to P_{n-1}(F)$ by $T([x^j]) = x^j$. One can check this is an isomorphism, and so $F[x]/(x^n) \cong P_{n-1}(F)$ as F-vector spaces.

Definition 2.3.15. Let $W \subseteq V$ be a subspace. The *canonical projection map* is given by

$$\begin{aligned} \pi_W : V &\to V/W \\ v &\mapsto v + W. \end{aligned}$$

It is immediate that $\pi_W \in \operatorname{Hom}_F(V, V/W)$.

One important point to note is that when working with quotient spaces, if one defines a map from V/W to another vector space, one must always check the map is well-defined as defining the map generally involves a choice of representative for $v + W$. In other words, one must show if $v_1 + W = v_2 + W$ then $T(v_1 + W) = T(v_2 + W)$. Consider the following example.

Example 2.3.16. Let $V = \mathbb{R}^2$ and $W = \{(x, 0) : x \in \mathbb{R}\}$. We saw above the elements of the quotient space V/W are of the form $(x, y) + W$ and $(x_1, y_1) + W = (x_2, y_2) + W$ if and only if $y_1 = y_2$. Suppose we with to define a linear map $T : V/W \to \mathbb{R}$. We could try to define such a map by specifying $T((x, y) + W) = x$.

However, we know that $(x,y)+W=(x+1,y)+W$, so $x=T((x,y)+W)=T((x+1,y)+W)=x+1$. This doesn't make sense, so our map is not well-defined. The "correct" map in this situation is to send $(x,y)+W$ to y since y is fixed across the equivalence class.

The following result allows us to avoid checking the map is well-defined when it is induced from another linear map. One should look back at the examples of quotient spaces above to see how this theorem applies.

Theorem 2.3.17. *Let $T \in \operatorname{Hom}_F(V,W)$. Define*

$$\begin{aligned} \overline{T} : V/\ker T &\to W \\ v + \ker T &\mapsto T(v). \end{aligned}$$

Then $\overline{T} \in \operatorname{Hom}_F(V/\ker T, W)$. Moreover, $\overline{T}$ gives an isomorphism

$$V/\ker T \xrightarrow{\simeq} \operatorname{Im} T.$$

Proof. The first step is to show $\overline{T}$ is well-defined. Suppose $v_1+\ker T = v_2+\ker T$, i.e, $v_1 - v_2 \in \ker T$. So there exists $x \in \ker T$ such that $v_1 - v_2 = x$. We have

$$\begin{aligned} \overline{T}(v_1+\ker T) &= T(v_1) \\ &= T(v_2+x) \\ &= T(v_2)+T(x) \\ &= T(v_2) \\ &= \overline{T}(v_2+\ker T). \end{aligned}$$

Thus, $\overline{T}$ is well-defined.

The next step is to show $\overline{T}$ is linear. Let $v_1+W, v_2+W \in V/W$ and $c \in F$. We have

$$\begin{aligned} \overline{T}(v_1+\ker T+v_2+\ker T) &= \overline{T}(v_1+v_2+\ker T) \\ &= T(v_1+v_2) \\ &= T(v_1)+T(v_2) \\ &= \overline{T}(v_1+\ker T)+\overline{T}(v_2+\ker T) \end{aligned}$$

and

$$\begin{aligned} \overline{T}(c(v_1+\ker T)) &= \overline{T}(cv_1+\ker T) \\ &= T(cv_1) \\ &= cT(v_1) \\ &= c\overline{T}(v_1+\ker T). \end{aligned}$$

Thus, $\overline{T} \in \operatorname{Hom}_F(V/\ker T, W)$.

It only remains to show that $\overline{T}$ is a bijection. Let $w \in \operatorname{Im}(T)$. There exists $v \in V$ so that $T(v)=w$. Thus, $\overline{T}(v+\ker T)=T(v)=w$, so $\overline{T}$ is surjective. Now suppose $v+\ker T \in \ker \overline{T}$. Then $0=\overline{T}(v+\ker T)=T(v)$. Thus, $v \in \ker T$ which means $v+\ker T = 0+\ker T$ and so $\overline{T}$ is injective. □

Example 2.3.18. Let $V = F[x]$. Define a map $T : V \to P_2(F)$ by sending $f(x) = a_0 + a_1x + \cdots a_nx^n$ to $a_0 + a_1x + a_2x^2$. This is a surjective linear map. The kernel of this map is exactly $(x^3) = \{g(x) : x^3 | g(x)\}$. Thus, the previous result gives an isomorphism $F[x]/(x^3) \cong P_2(F)$ as F-vector spaces.

Example 2.3.19. Let $V = \mathrm{Mat}_2(F)$. Define a map $T : \mathrm{Mat}_2(F) \to F^2$ by $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} b \\ c \end{pmatrix}$. One can check this is a surjective linear map. The kernel of this map is given by

$$\ker T = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in F \right\}.$$

Thus, $\mathrm{Mat}_2(F)/\ker T$ is isomorphic to F^2 as F-vector spaces.

Theorem 2.3.20. *Let $W \subseteq V$ be a subspace. Let $\mathcal{B}_W = \{v_i\}$ be a basis for W and extend to a basis $\mathcal{B}$ for V. Set $\mathcal{B}_U = \mathcal{B} - \mathcal{B}_W = \{z_i\}$. Let $U = \mathrm{span}_F \mathcal{B}_U$, i.e., U is a complement of W in V. Then the linear map*

$$\begin{aligned} p : U &\to V/W \\ z_i &\mapsto z_i + W \end{aligned}$$

is an isomorphism. Thus, $\overline{\mathcal{B}}_U = \{z_i + W : z_i \in \mathcal{B}_U\}$ is a basis of V/W.

Proof. Note that p is linear because we defined it on a basis. It only remains to show p is an isomorphism. We will do this by showing $\overline{\mathcal{B}}_U = \{z_i + W : z_i \in \mathcal{B}_U\}$ is a basis for V/W.

Let $v + W \in V/W$. Since $v \in V$, there exists $a_i, b_j \in F$ such that $v = \sum a_iv_i + \sum b_jz_j$. We have $\sum a_iv_i \in W$, so $v - \sum b_jz_j \in W$. Thus

$$\begin{aligned} v + W &= \sum b_jz_j + W \\ &= \sum b_j(z_j + W). \end{aligned}$$

This shows that $\overline{\mathcal{B}}_U$ spans V/W.

Suppose there exists $b_j \in F$ such that $\sum b_j(z_j + W) = 0 + W$, i.e., $\sum b_j(z_j + W) \in W$. So there exists $a_i \in F$ such that $\sum b_jz_j = \sum a_iv_i$. Thus $\sum a_iv_i + \sum -b_jz_j = 0$. Since $\{v_i, z_j\}$ is a basis of V we obtain $a_i = 0 = b_j$ for all i, j. This gives $\overline{\mathcal{B}}_U$ is linearly independent and so completes the proof. □

2.4 Dual spaces

We conclude this chapter by discussing dual spaces. Throughout this section V is an F-vector space.

Definition 2.4.1. The *dual space* of V, denoted $V^\vee$, is given by $V^\vee = \mathrm{Hom}_F(V, F)$. Elements of the dual space are called *linear functionals.*

Theorem 2.4.2. *The vector space V is isomorphic to a subspace of $V^\vee$. If $\dim_F V < \infty$, then $V \cong V^\vee$.*

Proof. Let $\mathcal{B} = \{v_i\}$ be a basis of V. For each v_i, define an element $v_i^\vee$ by setting

$$v_i^\vee(v_j) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

One sees immediately that $v_i^\vee \in V^\vee$ as it is defined on a basis. Define $T \in \operatorname{Hom}_F(V, V^\vee)$ by $T(v_i) = v_i^\vee$. We claim T is an injective linear map. Let $v \in V$ and suppose $T(v) = 0$. Write $v = \sum a_i v_i$, so that $\sum a_i v_i^\vee$ is then the 0 map, i.e., for any $v' \in V$ this gives $\sum a_i v_i^\vee(v') = 0$. In particular,

$$\begin{aligned} 0 &= \sum a_i v_i^\vee(v_j) \\ &= a_j v_j^\vee(v_j) \\ &= a_j. \end{aligned}$$

Since $a_j = 0$ for all j, we have $v = 0$ and so T is injective We now use the fact that $V/\ker T \cong \operatorname{Im} T$ and the fact that $\ker T = 0$ to conclude that V is isomorphic to a subspace of W, which gives the first statement of the theorem.

Assume $\dim_F V < \infty$ so we can write $\mathcal{B} = \{v_1, \ldots, v_n\}$. Given $v^\vee \in V^\vee$, define $a_j = v^\vee(v_j)$. Set $v = \sum a_i v_i \in V$. Define $S : V^\vee \to V$ by $v^\vee \to v$. This map defines an inverse to the map T given above and thus $V \cong V^\vee$. □

Note it is not always the case that V is isomorphic to its dual. In fact, if V is infinite dimensional it is never the case that V is isomorphic to its dual. In functional analysis or topology this problem is often overcome by requiring the linear functionals to be continuous, i.e. the dual space consists of continuous linear maps from V to F. In that case one would refer to our dual as the "algebraic dual." However, we will only consider the algebraic setting here so we don't bother with saying algebraic dual.

Example 2.4.3. Let V be a vector space over a field F and let the dimension be denoted by α. Note that α is not finite by assumption, but we need to work with different cardinalities here so we must keep track of this. The cardinality of V as a set is given by $\alpha \cdot \#F = \max\{\alpha, \#F\}$. Moreover, we have V is naturally isomorphic to the set of functions from a set of cardinality α to F with finite support. We denote this space by $F^{(\alpha)}$.

The dual space of V is the set of all functions from a set of cardinality α to F, i.e., to F^α. If we set $\alpha' = \dim_F V^\vee$, we wish to show $\alpha' > \alpha$. As above, the cardinality of $V^\vee$ as a set is $\max\{\alpha', \#F\}$.

Let $\mathcal{A} = \{v_i\}$ be a countable linear independent subset of V and extend it to a basis of V. For each nonzero $c \in F$ define $f_c : V \to F$ by $f_c(v_i) = c^i$ for $v_i \in \mathcal{A}$ and 0 for the other elements in the basis. One can show that $\{f_c\}$ is linearly independent, so $\alpha' \geq \#F$. Thus, we have $\#V^\vee = \alpha' \#F = \max\{\alpha', \#F\} = \alpha'$. However, we also have $\#V^\vee = \#F^\alpha$. Since $\alpha < \#F^\alpha$ because $\#F \geq 2$, we have $\alpha' = \#F^\alpha > \alpha$ as desired.

One should note that in the finite dimensional case when we have $V \cong V^\vee$, the isomorphism depends upon the choice of a basis. This means that while V is isomorphic to its dual, the isomorphism is non-canonical. In particular, there is no "preferred" isomorphism between V and its dual. Studying the possible isomorphisms that arise between V and its dual is interesting in its own right. We will return to this problem in Chapter 6.

Definition 2.4.4. Let V be a finite dimensional F-vector space and let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of V. The *dual basis of* $V^\vee$ *with respect to* $\mathcal{B}$ is given by $\mathcal{B}^\vee = \{v_1^\vee, \dots, v_n^\vee\}$.

If V is finite dimensional then we have $V \cong V^\vee \cong (V^\vee)^\vee$, i.e., $V \cong (V^\vee)^\vee$. The major difference is that while there is no canonical isomorphism between V and $V^\vee$, there is a canonical isomorphism $V \cong (V^\vee)^\vee$! Note that in proof below we construct the map from V to $(V^\vee)^\vee$ without choosing any basis. This is what makes the map canonical. We do use a basis in proving injectivity, but the map does not depend on this basis so it does not matter.

Proposition 2.4.5. *There is a canonical injective linear map from* V *to* $(V^\vee)^\vee$*. If* $\dim_F V < \infty$*, then this is an isomorphism.*

Proof. Let $v \in V$. Define $\mathrm{eval}_v : V^\vee \to F$ by sending $\varphi \in \mathrm{Hom}_F(V, F) = V^\vee$ to $\mathrm{eval}_v(\varphi) = \varphi(v)$. One must check that eval_v is a linear map. To see this, let $\varphi, \psi \in V^\vee$ and $c \in F$. We have

$$\begin{aligned}\mathrm{eval}_v(c\varphi + \psi) &= (c\varphi + \psi)(v)\\ &= c\varphi(v) + \psi(v)\\ &= c\,\mathrm{eval}_v(\varphi) + \mathrm{eval}_v(\psi).\end{aligned}$$

Thus, for each $v \in V$ we obtain a map $\mathrm{eval}_v \in \mathrm{Hom}_F(V^\vee, F)$. This allows us to define a well-defined map

$$\begin{aligned}\Phi : V &\to \mathrm{Hom}_F(V^\vee, F) = (V^\vee)^\vee\\ v &\mapsto \mathrm{eval}_v : \varphi \mapsto \varphi(v).\end{aligned}$$

We claim that Φ is a linear map. Since Φ maps into a space of maps, we check equality by checking the maps agree on each element, i.e., for $v, w \in V$ and $c \in F$ we want to show that for each $\varphi \in \mathrm{Hom}_F(V^\vee, F)$ that $\Phi(cv+w)(\varphi) = c\Phi(v)(\varphi) + \Phi(w)(\varphi)$. Observe we have

$$\begin{aligned}\Phi(cv + w)(\varphi) &= \mathrm{eval}_{cv+w}(\varphi)\\ &= \varphi(cv + w)\\ &= c\varphi(v) + \varphi(w)\\ &= c\Phi(v)(\varphi) + \Phi(w)(\varphi).\end{aligned}$$

Thus, we have $\Phi \in \mathrm{Hom}_F(V, (V^\vee)^\vee)$.

It remains to show that Φ is injective. Let $v \in V$, $v \neq 0$. Let $\mathcal{B}$ be a basis containing v. This is possible because we can start with the set $\{v\}$ and complete it to a basis. Note $v^\vee \in V^\vee$ and $\mathrm{eval}_v(v^\vee) = v^\vee(v) = 1$. Moreover, for any $w \in \mathcal{B}$ with $w \neq v$ we have $\mathrm{eval}_w(v^\vee) = v^\vee(w) = 0$. Thus, we have $\Phi(v) = \mathrm{eval}_v$ is not the 0 map, so $\ker \Phi = \{0\}$, i.e., Φ is injective.

If $\dim_F V < \infty$, we have Φ is an isomorphism because $\dim_F V = \dim_F V^\vee$ since they are isomorphic, so $\dim_F V = \dim_F V^\vee = \dim_F (V^\vee)^\vee$. □

Let $T \in \mathrm{Hom}_F(V, W)$. We obtain a natural map $T^\vee \in \mathrm{Hom}_F(W^\vee, V^\vee)$ as follows. Let $\varphi \in W^\vee$, i.e., $\varphi : W \to F$. To obtain a map $T^\vee(\varphi) : V \to F$, we simply compose the maps, namely, $T^\vee(\varphi) = \varphi \circ T$. It is easy to check this is a linear map. In particular, we have the following result.

Proposition 2.4.6. *Let $T \in \mathrm{Hom}_F(V, W)$. The map $T^\vee$ defined by $T^\vee(\varphi) = \varphi \circ T$ is a linear map from $W^\vee$ to $V^\vee$.*

We will see in the next chapter how the dual map gives the proper definition of a transpose.

2.5 Basics of module theory

The theory of vector spaces we have been studying so far is just a special case of the theory of modules. In spirit modules are just vector spaces where one considers the scalars to be in a ring instead of restricting them to be in a field. However, as we will see, restricting scalars to be in a field allows many nice results that are not available for general modules.

As usual, we assume all our rings have an identity element.

Definition 2.5.1. Let R be a ring. A *left R-module* is an abelian group $(M, +)$ along with a map $R \times M \to M$ denoted $(r, m) \mapsto rm$ satisfying

1. $(r_1 + r_2)m = r_1 m + r_2 m$ for all $r_1, r_2 \in R$, $m \in M$;
2. $r(m_1 + m_2) = rm_1 + rm_2$ for all $r \in R$, $m_1, m_2 \in M$;
3. $(r_1 r_2)m = r_1(r_2 m)$ for all $r_1, r_2 \in R$, $m \in M$;
4. $1_R m = m$ for all $m \in M$.

One can also define a right R-module M by acting on the right by the scalars. Moreover, given rings R and S one can define an (R, S)-bimodule by acting on the left by R and the right by S. For now we will work only with left R-modules and refer to these just as modules for this section. Morever, if R is a commutative ring and M is a left R-module, then M is a right R-module as well by setting $m \cdot r = rm$. (Check this does not work if R is not commutative!) Note that if we take R to be a field this is exactly the definition of a vector space.

Example 2.5.2. Let R be a ring and set $M = R^n = \{(r_1, \dots, r_n) : r_i \in R\}$. We have M is an R-module via componentwise addition and scalar multiplication.

Example 2.5.3. Let $M = \mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$. This is a $\mathbb{Z}$-module via the usual addition and $n(a + bi) = na + nbi$.

Example 2.5.4. Let G be any abelian group. We have that G is a $\mathbb{Z}$-module with the scalar multiplication defined by $ng = g + \cdots + g$ where there are n copies of g if $n > 0$. If $n = 0$ we set $ng = e_G$. If $n < 0$, we set $ng = -g - \cdots - g$ where there are $-n$ copies of g. Conversely, any $\mathbb{Z}$-module is clearly an abelian group. Thus we have that abelian groups and $\mathbb{Z}$-modules are the same objects.

Example 2.5.5. Let M be any abelian group and write $\mathrm{End}_{\mathrm{grp}}(M)$ for the set of group homomorphisms from M to M. This set is a ring where addition is given point-wise and multiplication is given by composition, i.e., $(f + g)(m) = f(m) + g(m)$ and $(f \cdot g)(m) = f(g(m))$. One should check this satisfies the ring axioms as an exercise. Now suppose we have a ring R and a ring homomorphism $\phi : R \to \mathrm{End}_{\mathrm{grp}}(M)$ that sends 1_R to the identity map in $\mathrm{End}_{\mathrm{grp}}(M)$. Set $rm = \phi(r)(m)$. We claim that this makes M into an R-module. Let $r_1, r_2 \in R$ and $m_1, m_2 \in M$. We have

1.
$$\begin{aligned}(r_1 + r_2)m &= \phi(r_1 + r_2)(m)\\ &= (\phi(r_1) + \phi(r_2))(m)\\ &= \phi(r_1)(m) + \phi(r_2)(m)\\ &= r_1m + r_2m;\end{aligned}$$

2.
$$\begin{aligned}r_1(m_1 + m_2) &= \phi(r_1)(m_1 + m_2)\\ &= \phi(r_1)(m_1) + \phi(r_1)(m_2)\\ &= r_1m_1 + r_1m_2;\end{aligned}$$

3.
$$\begin{aligned}(r_1r_2)m_1 &= \phi(r_1r_2)m_1\\ &= \phi(r_1)(\phi(r_2))m_1)\\ &= \phi(r_1)(r_2m_1)\\ &= r_1(r_2m_1).\end{aligned}$$

This gives all of the axioms, so we have M is an R-module. Conversely, now assume we are given an R-module M. We obtain a ring homomorphism $\phi : R \to \mathrm{End}_{\mathrm{grp}}(M)$ by setting $\phi(r)(m) = rm$. You should check this is a ring homomorphism as an exercise. Combining these two results, we see an R-module is nothing more than an abelian group M along with a ring homomorphism $R \to \mathrm{End}_{\mathrm{grp}}(M)$.

Definition 2.5.6. Let M be an R-module. Let $N \subset M$. We say N is an *R-submodule of M* if is is closed under scalar multiplication by R and it is a subgroup.

Exercise 2.5.7. Let M be an R-module. Show a subset $N \subset M$ is a submodule if and only if it is nonempty and satisfies $x + ry \in N$ for every $x, y \in N$, $r \in R$.

The next example is the most important example for this course of a module that is not a vector space.

Example 2.5.8. Let F be a field and V an F-vector space. Let $R = F[x]$ and $T \in \mathrm{Hom}_F(V, V)$. We use the linear map T to make V into an $F[x]$-module. Let $f(x) = a_n x^n + \cdots a_1 x + a_0 \in F[x]$ and $v \in V$. We define

$$f(x)v = (a_n T^n + \cdots + a_1 T + a_0)v$$

where $T^n = T \circ \cdots \circ T$ with n-copies of T. One can now easily check this makes V into an $F[x]$-module. Note that the module structure on V is very dependent on the choice of T!

Conversely, suppose we have an $F[x]$-module V. Since $F[x]$ acts on V, certainly one has F also acts on V by just restricting the action. Thus, we obtain that V is an F-vector space. The action of $F[x]$ also gives a linear transformation $T \in \mathrm{Hom}_F(V, V)$ by setting $T(v) = xv$. Thus, we have a bijection between $F[x]$-modules and pairs (V, T) where V is an F-vector space and $T \in \mathrm{Hom}_F(V, V)$.

It is also natural to ask about what submodules look like in this case. Let $W \subset V$ be a subspace and let $T \in \mathrm{Hom}_F(V, V)$. Recall we say W is *T-invariant* if $T(W) \subset W$. If W is an $F[x]$-submodule of V, then $xW \subset W$. In particular, this means $T(W) \subset W$ and so W is T-stable. On the other hand, if W is a T-stable subspace of V, then $T^n(W) \subset W$ for each n and so $f(x)W \subset W$ for each $f(x) \in F[x]$. Thus, one has $F[x]$-submodules of V are exactly the T-invariant subspaces of V.

Definition 2.5.9. Let M and N be R-modules.

1. A map $\phi : M \to N$ is an *R-module homomorphism* or an *R-linear map* if

   - $\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2)$ for all $m_1, m_2 \in M$;
   - $\phi(rm) = r\phi(m)$ for all $r \in R$, $m \in M$.

   The set of all R-linear maps is denoted $\mathrm{Hom}_R(M, N)$.

2. We say $\phi \in \mathrm{Hom}_R(M, N)$ is an *isomorphism of R-modules* if ϕ is bijective. We write $M \cong N$ if there is an isomorphism from M to N and say M and N are *isomorphic*.

3. Let $\phi \in \mathrm{Hom}_R(M, N)$. The *kernel of ϕ* is given by

   $$\ker \phi = \{m \in M : \phi(m) = 0_N\}.$$

   The image of ϕ is given by

   $$\mathrm{Im}\, \phi = \{\phi(m) : m \in M\}.$$

Exercise 2.5.10. Show that $\ker \phi$ is a submodule of M and $\operatorname{Im} \phi$ is a submodule of N.

Exercise 2.5.11. Let M and N be $\mathbb{Z}$-modules. Show that $\operatorname{Hom}_{\text{grp}}(M, N) = \operatorname{Hom}_{\mathbb{Z}}(M, N)$.

Just as for vector spaces, one can define quotient modules. Given $N \subset M$ an R-submodule, we set

$$M/N - \{m + N : m \in M\}.$$

One has addition and scalar multiplication on M/N just as for vector spaces. One also obtains via the same proof the following isomorphism theorem.

Theorem 2.5.12. *Let* $\phi \in \operatorname{Hom}_R(M, N)$*. Then one has*

$$M/\ker \phi \cong \operatorname{Im} \phi.$$

One of the main features of vector spaces is they have a basis. Unfortunately one does not have this nice property for general modules.

Definition 2.5.13. We say an R-module M is *generated by* $\mathcal{B} = \{x_i\}$ if every element $x \in M$ can be written as $x = \sum r_i x_i$ where $r_i \in R$, $r_i = 0$ for all but finitely many i. We say M is *finitely generated* if one can choose $\mathcal{B}$ to be a finite set. We say $\mathcal{B}$ is a basis for M if M is generated by $\mathcal{B}$ and the elements of $\mathcal{B}$ are linearly independent over R.

Example 2.5.14. Consider the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z} = \{m + n\mathbb{Z} : m \in \mathbb{Z}\}$. Suppose $\mathcal{B}$ is a basis of $\mathbb{Z}/n\mathbb{Z}$ and let $x_1 \in \mathcal{B}$. Then we have $nx_1 = 0$, but $n \neq 0$ in $\mathbb{Z}$. Thus, we cannot have x_1 in a basis. However, x_1 was arbitrary so it must be $\mathcal{B}$ cannot exist.

Suppose one has an R-module M that is finitely generated; can one conclude that any submodule N of M is finitely generated as well?

Example 2.5.15. Let $F[x_1, x_2, \dots]$ be considered as an $F[x_1, x_2, \dots]$-module. Clearly we have 1 is a basis of $F[x_1, x_2, \dots]$ over $F[x_1, x_2, \dots]$. Consider $N = \langle x_1, x_2, \dots, \rangle$. Then N is a submodule of $F[x_1, x_2, \dots]$. However, N is not finitely generated over $F[x_1, x_2, \dots]$ as it does not contain 1 and the elements x_i are algebraically independent.

This two examples show one has to be careful when working with general modules. Fortunately, our applications will involve particulary nice modules where things work out much nicer.

Definition 2.5.16. Let M be an R-module that has a basis $\mathcal{B}$. We say M is a *free module.* If $\#\mathcal{B} = n$ we say M is a *free module of rank* n*.*

Free modules behave in essentially the same way as a vector space does. If M is a free R-module of rank n one has a non-canonical isomorphism $M \cong R^n$. Given a vector space V over a field F, V is necessarily a free F-module so this concept directly generalizes vector spaces. Unfortunately, for our purposes it isn't enough to study just free modules. We will also need to consider finitely generated modules that contain torsion.

Definition 2.5.17. Let M be an R-module. We say $x \in M$ is a *torsion element* if there exists a nonzero element $r \in R$ so that $rm = 0$. The collection of all torsion elements in M is denoted $\mathrm{Tor}_R(M)$.

Exercise 2.5.18. Show that $\mathrm{Tor}_R(M)$ is a submodule of M.

Example 2.5.19. Let $R = F$ and V be an F-vector space. Then $\mathrm{Tor}_R(V) = 0$.

Example 2.5.20. Let $M = (\mathbb{Z}/n\mathbb{Z}) \oplus \mathbb{Z}^r$ considered as a $\mathbb{Z}$-module. We have $\mathrm{Tor}_{\mathbb{Z}}(M) \cong \mathbb{Z}/n\mathbb{Z}$.

While modules can behave much more wildly than vector spaces, the situation of interest to us is primarily in the case that R is a principal ideal domain and M is a finitely generated R-module. In that case, one has the following key result. We include a proof for completeness, but this result can be taken on faith for this course without losing much. This theorem is the main input needed into the fundamental theorem of finitely generated modules over a principal ideal domain (Theorem 4.6.2), which gives a quick and easy proof of both the rational and Jordan canonical form of a matrix.

Theorem 2.5.21. *Let R be a principal ideal domain and M a free R-module of finite rank n. Let $N \subset M$ be a submodule. Then one has:*

1. *N is a free module of rank $m \leq n$;*
2. *there is a basis $z_1, \ldots, z_n$ of M and nonzero elements $a_1, \ldots, a_m \in R$ with $a_m \mid a_{m-1} \mid \cdots \mid a_1$ so that $a_1 z_1, \ldots, a_m z_m$ is a basis of N.*

Proof. If $N = 0$ the result holds trivially so we assume $N \neq 0$. Let $\varphi \in \mathrm{Hom}_R(M, R)$. Then we have $\varphi(N)$ is a submodule of R, i.e., an ideal of R, and since we are assuming R is a principal ideal domain we have $\varphi(N) = b_\varphi R$ for some $b_\varphi \in R$. Let

$$\Sigma = \{b_\varphi R : \varphi \in \mathrm{Hom}_R(M, R)\}.$$

Since $0 \in \Sigma$ via the map sending everything to 0_R, we have $\Sigma \neq$. This gives Σ as a nonempty collection of ideals, which can be partially ordered via inclusion, so one obtains a maximal element, i.e., there exists $\psi \in \mathrm{Hom}_R(M, R)$ so that $\psi(N) = b_\psi R$ is not properly contained in any other element of Σ. Set $b_1 = b_\psi$ and let $y \in N$ so that $\psi(y) = b_1$. We claim that $b_1 \neq 0$. Let $x_1, \ldots, x_n$ be a basis of M and defined $\pi_i \in \mathrm{Hom}_R(M, R)$ via $\pi_i(c_1 x_1 + \cdots c_n x_n) = c_i$. Since $N \neq 0$ there is an i so that $\pi_i(N) \neq 0$. Thus, since $b_\psi R$ is maximal, we must have $b_1 = b_\psi \neq 0$.

Our next step is to show $b_1 \mid \varphi(y)$ for every $\varphi \in \mathrm{Hom}_R(M, R)$. Let $d = \gcd(b_1, \varphi(y))$. We have $d \mid b_1$ and $d \mid \varphi(y)$ in R, so there exists $r_1, r_2 \in R$ with $d = r_1 b_1 + r_2 \varphi(y)$. Set $\eta = r_1 \psi + r_2 \varphi \in \mathrm{Hom}_R(M, R)$. Then we have $\eta(y) = r_1 b_1 + r_2 \varphi(y) = d$, and so $d \in \eta(N)$ which gives $dR \subset \eta(N)$. However, since $d \mid b_1$ we have $b_1 R \subset dR \subset \eta(N)$. Since $b_1 R$ is maximal, this gives $b_1 R = \eta(N)$ and so $b_1 R = dR$. This gives the result that $b_1 \mid \varphi(y)$ for all $\varphi \in \mathrm{Hom}_R(M, R)$.

We apply this to the maps π_i to see that $b_1 \mid \pi_i(y)$ for each i. Write $\pi_i(y) = b_1 c_i$ for some $c_i \in R$ for each $1 \leq i \leq n$. Set

$$y_1 = \sum_{i=1}^{n} c_i x_i.$$

Observe we have $b_1 y_1 = y$ by the definition of the c_i. However, this gives $b_1 = \psi(y) = \psi(b_1 y_1) = b_1 \psi(y_1)$. Since we have R is an integral domain, this gives $\psi(y_1) = 1$.

Our next step is to show that y_1 can be taken as a basis element of M and $b_1 y_1$ can be taken as a basis element for N. This is equivalent to checking

(1) $M = Ry_1 \oplus \ker(\psi)$

(2) $N = Rb_1 y_1 \oplus (\ker(\psi) \cap N)$.

We begin by showing (1). Let $x \in M$ and write $x = \psi(x)y_1 + (x - \psi(x)y_1)$. Observe we have

$$\begin{aligned} \psi(x - \psi(x)y_1) &= \psi(x) - \psi(x)\psi(y_1) \\ &= \psi(x) - \psi(x) \cdot 1 \\ &= 0. \end{aligned}$$

Thus, we have $x - \psi(x)y_1 \in \ker(\psi)$. This gives $M = Ry_1 + \ker(\psi)$. Suppose we have $ry_1 \in \ker(\psi)$ for some $r \in R$. Then we have

$$\begin{aligned} 0 &= \psi(ry_1) \\ &= r\psi(y_1) \\ &= r. \end{aligned}$$

Thus, $Ry_1 \cap \ker(\psi) = 0$ and so the sum is direct as claimed.

We now prove (2). Since b_1 is a generator for $\psi(N)$, we have $b_1 \mid \psi(x')$ for any $x' \in N$. Let $x' \in N$ and write $\psi(x') = c_{x'} b_1$ for some $c_{x'} \in R$. Then, as above, we have

$$\begin{aligned} x' &= \psi(x')y_1 + (x' - \psi(x')y_1) \\ &= c_{x'} b_1 y_1 + (x' - c_{x'} b_1 y_1) \end{aligned}$$

where $x' - c_{x'} b_1 y_1 \in \ker(\psi) \cap N$. Thus, we obtain immediately that $N = Rb_1 y_1 + (\ker(\psi) \cap N)$. It remains to show the sum is direct. However, this follows immediately from the proof of (1) because this is a special case of that sum.

It is now possible to prove N is free of rank $m \leq n$ by induction. If $m = 0$ then N is a torsion module. However, free modules are torsion free and since M is free, the torsion subgroup of M is 0 and so in this case $N = 0$. Now assume $m > 0$. We have

$$N = Rb_1 y_1 \oplus (N \cap \ker(\psi)).$$

This gives the rank of $N \cap \ker(\psi)$ as $m-1$. Thus, we apply the induction hypothesis to $N \cap \ker(\psi)$ to see this is free of rank $m-1$. Thus, adjoining $b_1 y_1$ to any basis of $N \cap \ker(\psi)$ we have a basis of N with m elements, so N is free of rank m.

It only remains to prove the second statement of the theorem that a nice basis can be chosen. We proceed by induction on n, the rank of M. Applying the previous paragraph to this we obtain

$$M = Rb_1 \oplus \ker(\psi)$$

with $\ker(\psi)$ free of rank $n-1$. We now apply the induction hypothesis to the free module $\ker(\psi)$ and its submodule $\ker(\psi) \cap N$. Thus, we obtain a basis $y_2, \ldots, y_n$ of $\ker(\psi)$ and $b_2, \ldots, b_m \in R$ with $b_m \mid b_{m-1} \mid \cdots \mid b_2$ so that $b_2 y_2, \ldots, b_m y_m$ is a basis of $\ker(\psi) \cap N$. We now use the sums are direct to get $y_1, \ldots, y_n$ is a basis of M and $b_1 y_2, b_2 y_2, \ldots, b_m y_m$ is a basis of $\ker(\psi) \cap N$. It remains to relate b_1 to the b_i. Set $a_m = b_1, a_{m-1} = b_m, a_{m-2} = b_{m-1}, \ldots, a_1 = b_2$ and $z_m = y_1, z_{m-1} = y_m, \ldots, z_1 = y_2$. Then we have $a_{m-1} \mid a_{m-2} \mid \cdots \mid a_1$ and it only remains to show $a_m \mid a_{m-1}$. Define $\varphi \in \mathrm{Hom}_R(M, R)$ by $\varphi(z_m) = \varphi(z_{m-1}) = 1$ and $\varphi(z_j) = 0$ for $j \leq m-1$. We have $a_m = \varphi(a_m z_m)$, so $a_m \in \varphi(N)$; thus $a_m R \subset \varphi(N)$. However, we know $a_m R$ is maximal in Σ, so it must be that $a_m R = \varphi(N)$. However, we also have $a_{m-1} = \varphi(a_{m-1} z_{m-1}) \in \varphi(N)$, so we must have $a_{m-1} \in a_m R$, i.e., $a_m \mid a_{m-1}$. This gives the result. □

2.6 Problems

For these problems F is assumed to be a field.

1. Define

$$\mathfrak{sl}_n(\mathbb{Q}) = \left\{ X = (x_{i,j}) \in \mathrm{Mat}_n(\mathbb{Q}) : \mathrm{Tr}(X) = \sum_{i=1}^{n} x_{i,i} = 0 \right\}.$$

Show that $\mathfrak{sl}_n(\mathbb{Q})$ is a $\mathbb{Q}$-vector space.

2. Consider the vector space F^3. Determine, and justify your answer, whether each of the following are subspaces of F^3:

(a) $W_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 + 2x_2 + 3x_3 = 0 \right\}$

(b) $W_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 x_2 x_3 = 0 \right\}$

(c) $W_3 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 = 5x_3 \right\}$.

3. Let V be an F-vector space.

(a) Prove that an arbitrary intersection of subspaces of V is again a subspace of V.

(b) Prove that the union of two subspace of V is a subspace of V if and only if one of the subspaces is contained in the other.

4. Let $T \in \mathrm{Hom}_F(F, F)$. Prove there exists $\alpha \in F$ so that $T(v) = \alpha v$ for every $v \in F$.

5. Let U, V, and W be F-vector spaces. Let $S \in \mathrm{Hom}_F(U, V)$ and $T \in \mathrm{Hom}_F(V, W)$. Prove that $T \circ S \in \mathrm{Hom}_F(U, W)$.

6. Let V be an F-vector space. Prove that if $\{v_1, \ldots, v_n\}$ is linearly independent in V, then so is the set $\{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n\}$.

7. Let V be the subspace of $\mathbb{R}^5$ defined by

$$V = \{(x_1, x_2, \dots, x_5) \in \mathbb{R}^5 : x_1 = 4x_4, x_2 = 5x_5\}.$$

Find a basis for V.

8. Prove that there does not exist a $T \in \mathrm{Hom}_F(F^5, F^2)$ so that

$$\ker(T) = \{(x_1, x_2, \dots, x_5) \in F^5 : x_1 = x_2 \text{ and } x_3 = x_4 = x_5\}.$$

9. Let V be a finite dimensional vector space and $T \in \mathrm{Hom}_F(V, V)$ with $T^2 = T$.

   (a) Prove that $\mathrm{Im}(T) \cap \ker(T) = \{0\}$.

   (b) Prove that $V = \mathrm{Im}(T) \oplus \ker(T)$.

   (c) Let $V = F^n$. Prove that there is a basis of V such that the matrix of T with respect to this basis is a diagonal matrix whose entries are all 0 or 1.

10. Let $T \in \mathrm{Hom}_F(V, F)$. Prove that if $v \in V$ is not in $\ker(T)$, then

$$V = \ker(T) \oplus \{cv : c \in F\}.$$

11. Let V_1, V_2 be subspaces of the finite dimensional vector space V. Prove

$$\dim_F(V_1 + V_2) = \dim_F(V_1) + \dim_F(V_2) - \dim_F(V_1 \cap V_2).$$

12. Suppose that V and W are both 5-dimensional $\mathbb{R}$-subspaces of $\mathbb{R}^9$. Prove that $V \cap W \neq \{0\}$.

13. Let p be a prime and V a dimension n vector space over $\mathbb{F}_p$. Show there are

$$(p^n - 1)(p^n - p)(p^n - p^2) \cdots (p^n - p^{n-1})$$

distinct bases of V.

14. Let V be an F-vector space of dimension n. Let $T \in \mathrm{Hom}_F(V, V)$ so that $T^2 = 0$. Prove that the image of T is contained in the kernel of T and hence the dimension of the image of T is at most $n/2$.

15. Let W be a subspace of a finite dimensional vector space V. Let $T \in \mathrm{Hom}_F(V, V)$ so that $T(W) \subset W$. Show that T induces a linear transformation $\overline{T} \in \mathrm{Hom}_F(V/W, V/W)$. Prove that T is nonsingular (i.e., injective) on V if and only if T restricted to W and $\overline{T}$ on V/W are both nonsingular.

16. Let $T \in \operatorname{Hom}_F(V, V)$.

    (a) Give an example to show that one does not always have $V \cong \ker(T) \oplus \operatorname{Im}(T)$.

    (b) Show that $\ker(T^j) \subset \ker(T^{j+1})$ for all $j \geq 1$. Prove that this sequence stabilizes, i.e., there exists $m \geq 1$ so that $\ker(T^{m+j}) = \ker(T^m)$ for all $j \geq 1$. The subspace $\ker(T^m)$ is called the *eventual kernel* and denoted $\ker(T^\infty)$.

    (c) Show that $\operatorname{Im}(T^j) \supset \operatorname{Im}(T^{j+1})$ for all $j \geq 1$. Prove that this sequence stabilizes, i.e., there exists $m \geq 1$ so that $\operatorname{Im}(T^{m+j}) = \operatorname{Im}(T^m)$ for all $j \geq 1$. The subspace $\operatorname{Im}(T^m)$ is called the *eventual image* and denoted $\operatorname{Im}(T^\infty)$.

    (d) Prove that $V \cong \ker(T^\infty) \oplus \operatorname{Im}(T^\infty)$.

Chapter 3

Choosing coordinates

This chapter will make the connection between the more abstract version of vector space and linear transformation given in Chapter 2 and the material given in an undergraduate linear algebra class. Throughout this chapter all vector spaces are assumed to be finite dimensional.

3.1 Linear transformations and matrices

Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for V. This choice of basis gives an isomorphism between V and F^n. Namely, for $v \in V$ if we write $v = \sum a_i v_i$ for some $a_i \in F$, we have an isomorphism $T_{\mathcal{B}} : V \to F^n$ given by $v \mapsto \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. When we identify V with F^n via this map, we will write $[v]_{\mathcal{B}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. We refer to this as *choosing coordinates on V*.

Example 3.1.1. Let $V = \mathfrak{sl}_2(\mathbb{C})$. Recall a basis for this vector space is

$$\mathcal{B} = \left\{ v_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Let $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ be an element of V. Observe we have $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = bv_1 + cv_2 + av_3$. Thus,

$$\left[\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} b \\ c \\ a \end{pmatrix}.$$

Example 3.1.2. Let $V = P_2(\mathbb{R})$. Recall a basis for V is given by $\mathcal{B} = \{1, x, x^2\}$. Let $f = a + bx + cx^2$. Then

$$[f]_{\mathcal{B}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Example 3.1.3. Let $V = P_2(\mathbb{R})$. One can easily check that $\mathcal{C} = \{1, (x-1), (x-1)^2\}$ is a basis for V. Let $f = a + bx + cx^2$. We can write f in terms of $\mathcal{C}$ as

$$f = (a+b+c) + (b+2c)(x-1) + c(x-1)^2.$$

Thus, we have

$$[f]_{\mathcal{C}} = \begin{pmatrix} a+b+c \\ b+2c \\ c \end{pmatrix}.$$

Let $T \in \mathrm{Hom}_F(V, W)$. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for V and $\mathcal{C} = \{w_1, \dots, w_m\}$ be a basis for W. Recall that we have $W \cong F^m$ via the map $Q(w) = [w]_{\mathcal{C}}$ and $V \cong F^n$ via the map $P(v) = [v]_{\mathcal{B}}$. Furthermore, recall that any linear transformation from F^n to F^m is given by a matrix $A \in \mathrm{Mat}_{m,n}(F)$. Thus, we have the following diagram:

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "P"'] & W \arrow[d, "Q"] \\
F^n \arrow[r, dotted, "A"] & F^m
\end{tikzcd}$$

Thus, we have a unique matrix $A \in \mathrm{Mat}_{m,n}(F)$ given by $A = Q \circ T \circ P^{-1}$. Write $A = [T]_{\mathcal{B}}^{\mathcal{C}}$, i.e., A is the matrix that gives the map T when one chooses $\mathcal{B}$ as the coordinates on V and $\mathcal{C}$ as the coordinates on W. In particular, $[T]_{\mathcal{B}}^{\mathcal{C}}$ is the unique matrix that satisfies $[T]_{\mathcal{B}}^{\mathcal{C}}[v]_{\mathcal{B}} = [T(v)]_{\mathcal{C}}$.

One can easily compute $[T]_{\mathcal{B}}^{\mathcal{C}}$. Since $\mathcal{C}$ is a basis for W, there are scalars $a_{ij} \in F$ so that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i.$$

Observe that

$$[T(v_j)]_{\mathcal{C}} = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

We also have $[v_j]_{\mathcal{B}} = e_j$, so $[T]_{\mathcal{B}}^{\mathcal{C}}[v_j]_{\mathcal{B}}$ is exactly the jth column of $[T]_{\mathcal{B}}^{\mathcal{C}}$. Thus, the matrix $[T]_{\mathcal{B}}^{\mathcal{C}}$ is given by

$$\begin{aligned} [T]_{\mathcal{B}}^{\mathcal{C}} &= (a_{ij}) \\ &= \left([T(v_1)]_{\mathcal{C}} | \cdots | [T(v_n]_{\mathcal{C}})\right). \end{aligned}$$

Example 3.1.4. Let $V = P_3(\mathbb{R})$. Define $T \in \mathrm{Hom}_{\mathbb{R}}(V, V)$ by $T(f(x)) = f'(x)$. Let $\mathcal{B} = \{1, x, x^2, x^3\}$. For $f(x) = a + bx + cx^2 + dx^3$, we have $T(f(x)) = b + 2cx + 3dx^2$. In particular, $T(1) = 0, T(x) = 1, T(x^2) = 2x$, and $T(x^3) = 3x^2$. The matrix for T with respect to $\mathcal{B}$ is given by

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example 3.1.5. Let $V = \mathfrak{sl}_2(\mathbb{C})$ and $W = \mathbb{C}^4$. We pick the standard basis

$$\mathcal{B} = \left\{ v_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, v_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

for $\mathfrak{sl}_2(\mathbb{C})$. Let

$$\mathcal{C} = \left\{ w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, w_2 = \begin{pmatrix} 0 \\ i \\ 0 \\ 0 \end{pmatrix}, w_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, w_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

It is easy to check that $\mathcal{C}$ is a basis for W. Define $T \in \mathrm{Hom}_F(V, W)$ by

$$T(v_1) = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T(v_2) = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

$$T(v_3) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

From this it is easy to check that

$$\begin{aligned} T(v_1) &= 2w_1 - w_3 \\ T(v_2) &= -3iw_2 + w_4 \\ T(v_3) &= w_4. \end{aligned}$$

Thus, the matrix for T with respect to $\mathcal{B}$ and $\mathcal{C}$ is

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3i & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Exercise 3.1.6. Let $\mathcal{A}$ is a basis of U, $\mathcal{B}$ a basis of V, and $\mathcal{C}$ a basis of W. If $S \in \mathrm{Hom}_F(U,V)$ and $T \in \mathrm{Hom}_F(V,W)$, show that

$$[T \circ S]_{\mathcal{A}}^{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}}[S]_{\mathcal{A}}^{\mathcal{B}}.$$

Let $T\,\mathrm{Hom}_F(V,V)$ and $\mathcal{B}$ a basis of V. To save notation we will write $[T]_{\mathcal{B}}$ for the matrix $[T]_{\mathcal{B}}^{\mathcal{B}}$.

As one learns in undergraduate linear algebra, it can often be the case that one has information about V given in terms of a basis $\mathcal{B}$, but it would be more useful if it were given in terms of a basis $\mathcal{B}'$. We now recall how to change from $\mathcal{B}$ to $\mathcal{B}'$. We can recover this by specializing the situation we just studied. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ and $\mathcal{B}' = \{v_1', \dots, v_n'\}$. Define $T : V \to F^n$ by $T(v) = [v]_{\mathcal{B}}$ and $S : V \to F^n$ by $S(v) = [v]_{\mathcal{B}'}$. We have the following diagram:

$$\begin{tikzcd}
V \arrow[r, "id"] \arrow[d, "T"'] & W \arrow[d, "S"] \\
F^n \arrow[r, dotted, "A"] & F^n
\end{tikzcd}$$

Applying our previous results we see $[v]_{\mathcal{B}'} = (S \circ \mathrm{id} \circ T^{-1})([v]_{\mathcal{B}}) = (S \circ T^{-1})([v]_{\mathcal{B}})$. The *change of basis matrix* is $[\mathrm{id}]_{\mathcal{B}}^{\mathcal{B}'}$.

Exercise 3.1.7. Let $\mathcal{B} = \{v_1, \dots, v_n\}$. Prove the change of basis matrix $[\mathrm{id}]_{\mathcal{B}}^{\mathcal{B}'}$ is given by $([v_1]_{\mathcal{B}'}|\cdots|[v_n]_{\mathcal{B}'})$

Example 3.1.8. Let $V = \mathbb{Q}^2$. It is elementary to check that $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ and $\mathcal{B}' = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 7 \end{pmatrix} \right\}$ both give bases of V. To compute the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$, we expand the elements of $\mathcal{B}$ in terms of the basis $\mathcal{B}'$. For example, we want to find $a, b \in \mathbb{Q}$ so that

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = a \begin{pmatrix} 2 \\ 3 \end{pmatrix} + b \begin{pmatrix} 5 \\ 7 \end{pmatrix}.$$

This leads to the system of linear equations

$$\begin{aligned} 1 &= 2a + 5b \\ -1 &= 3a + 7b. \end{aligned}$$

One solves these by writing expressing them as the matrix $\begin{pmatrix} 2 & 5 & 1 \\ 3 & 7 & -1 \end{pmatrix}$ and using Gaussian elimination to reduce this matrix to $\begin{pmatrix} 1 & 0 & -12 \\ 0 & 1 & 5 \end{pmatrix}$.

Thus, $a = -12$ and $b = 5$. One now performs the same operation on the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to obtain

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = -2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 1 \begin{pmatrix} 5 \\ 7 \end{pmatrix}.$$

Thus, the change of basis matrix is given by $\begin{pmatrix} -12 & -2 \\ 5 & 1 \end{pmatrix}$.

Example 3.1.9. Consider $V = P_2(F)$. Let $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{B}' = \{1, x-2, (x-2)^2\}$ be bases of V. We calculate the change of basis matrix. We have

$$\begin{aligned} [1]_{\mathcal{B}'} &= 1, \\ [x]_{\mathcal{B}'} &= 1 \cdot (x-2) + 2 \cdot 1, \\ [x]_{\mathcal{B}'} &= 1 \cdot (x_2)^2 + 4 \cdot (x-2) + 4 \cdot 1. \end{aligned}$$

Thus, the change of basis matrix is given by $A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$.

Exercise 3.1.10. Let $V = P_3(F)$. Define

$$x^{(i)} = x(x-1)(x-2)\cdots(x-i+1).$$

In particular, $x^{(0)} = 1, x^{(1)} = x, x^{(2)} = x(x-1)$ and $x^{(3)} = x(x-1)(x-2)$. Set $\mathcal{B} = \{1, x, x^{(2)}, x^{(3)}\}$ and $\mathcal{B}' = \{1, x, x^2, x^3\}$.

(a) Show that $\mathcal{B}$ is a basis for V.

(b) Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$.

This gives the language to allow one to translate the results we prove in these notes from the language of vector spaces and linear transformations to the language of F^n and matrices. Many of the familiar results from undergraduate linear algebra will be proven in the problems at the end of the chapter. We conclude this section with a familiar example from undergraduate linear algebra. One should work the exercises to gain a better understanding of the theory behind the calculations.

Example 3.1.11. Consider the matrix

$$A = \begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -1 & 1 \end{pmatrix}.$$

We wish to find a basis for the kernel and image of this matrix. To find this, we will compute the reduced row echelon form of the matrix. The first thing to recall is that doing row operations corresponds to changing the basis of the domain and doing column operations corresponds to changing the basis of the range space. The reduced row echelon form of this matrix is

$$B = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To find a basis for the kernel, we consider the augmented matrix

$$(B|0) = \begin{pmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In terms of equations, if the variables are x_1, x_2, x_3, this gives the equations $x_1 + 1/2x_3 = 0$ and $x_2 = 0$. Thus, a basis element for the kernel is given by $\begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}$. We know from the exercises that the basis of the image consists of the columns of A that correspond to the columns of B containing the pivots, so the first and second columns. Thus, a basis for the image is given by $\left\{ \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 4 \\ -1 \end{pmatrix} \right\}$.

3.2 Transpose of a matrix via the dual map

Recall at the end of last chapter we introduced a dual map. Namely, given a linear map $T \in \mathrm{Hom}_F(V, W)$, we defined a map $T^\vee : W^\vee \to V^\vee$ given by $T(\varphi) = \varphi \circ T$. We also remarked at that point that this map could be used to properly define a transpose. This section deals with this construction.

Given a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of V and a basis $\mathcal{C} = \{w_1, \dots, w_m\}$ of W we have dual bases $\mathcal{B}^\vee$ of $V^\vee$ and $C^\vee$ of $W^\vee$. The previous section gave an associated matrix to any linear transformation, so we have a matrix $[T^\vee]_{\mathcal{C}^\vee}^{\mathcal{B}^\vee} \in \mathrm{Mat}_{n,m}(F)$.

Definition 3.2.1. Let $T \in \mathrm{Hom}_F(V, W)$, $\mathcal{B}$ a basis of V, $\mathcal{C}$ a basis of W, and set $A = [T]_{\mathcal{B}}^{\mathcal{C}}$. The *transpose of* A, denoted ${}^tA$, is given by ${}^tA = [T^\vee]_{\mathcal{C}^\vee}^{\mathcal{B}^\vee}$.

For this definition to be of interest we need to show it agrees with the definition of a transpose given in undergraduate linear algebra.

Lemma 3.2.2. *Let* $A = (a_{ij}) \in \mathrm{Mat}_{m,n}(F)$*. Then* ${}^tA = (b_{ij}) \in \mathrm{Mat}_{n,m}(F)$ *with* $b_{ij} = a_{ji}$.

Proof. Let $\mathcal{E}_n = \{e_1, \dots, e_n\}$ be the standard basis of F^n and $\mathcal{F}_m = \{f_1, \dots, f_m\}$ the standard basis for F^m. Let $\mathcal{E}_n^\vee$ and $\mathcal{F}_m^\vee$ be the dual bases. Let T be the linear map associated to A, i.e., $[T]_{\mathcal{E}_n}^{\mathcal{F}_m} = A$. In particular, we have

$$T(e_i) = \sum_{k=1}^{m} a_{ki} f_k.$$

We also have that $[T^\vee]_{\mathcal{F}_m^\vee}^{\mathcal{E}_n^\vee}$ is a matrix $B = (b_{ij}) \in \mathrm{Mat}_{n,m}(F)$ where the entries of B are given by

$$T^\vee(f_j^\vee) = \sum_{k=1}^{n} b_{kj} e_k^\vee.$$

If we apply $f_j^\vee$ to the first sum we see

$$\begin{aligned} f_j^\vee(T(e_i)) &= \sum_{k=1}^{m} a_{ki} f_j^\vee(f_k) \\ &= a_{ji}. \end{aligned}$$

If we evaluate the second sum at e_i we have

$$\begin{aligned} T^\vee(f_j^\vee)(e_i) &= \sum_{k=1}^{n} b_{kj} e_k^\vee(e_i) \\ &= b_{ij}. \end{aligned}$$

We now use the definition of the map $T^\vee$ to see that $f_j^\vee T(e_i) = T^\vee(f_j^\vee)(e_i)$, and so $a_{ji} = b_{ij}$, as desired. □

Exercise 3.2.3. Let $A_1, A_2 \in \mathrm{Mat}_{m,n}(F)$. Use the definition given above to show that ${}^t(A_1 + A_2) = {}^tA_1 + {}^tA_2$.

Exercise 3.2.4. Let $A \in \mathrm{Mat}_{m,n}(F)$ and $c \in F$. Use the definition given above to show that ${}^t(cA) = c\,{}^tA$.

We can use our definition of the transpose to give a very simple proof of the following fact.

Lemma 3.2.5. *Let $A \in \mathrm{Mat}_{m,n}(F)$ and $B \in \mathrm{Mat}_{p,m}(F)$. Then ${}^t(BA) = {}^tA\,{}^tB$.*

Proof. Write $\mathcal{E}_m$ be the standard basis on F^m, and likewise for $\mathcal{E}_n$ and $\mathcal{E}_p$. Let S be the multiplication by A map and T the multiplication by B map so that $A = [S]_{\mathcal{E}_n}^{\mathcal{E}_m}$ and $B = [T]_{\mathcal{E}_m}^{\mathcal{E}_p}$. We also have $BA = [T \circ S]_{\mathcal{E}_n}^{\mathcal{E}_p}$. We now have

$$\begin{aligned} {}^t(BA) &= [(T\circ S)^\vee]_{\mathcal{E}_p^\vee}^{\mathcal{E}_n^\vee} \\ &= [S^\vee \circ T^\vee]_{\mathcal{E}_p^\vee}^{\mathcal{E}_n^\vee} \\ &= [S^\vee]_{\mathcal{E}_m^\vee}^{\mathcal{E}_n^\vee}[T^\vee]_{\mathcal{E}_p^\vee}^{\mathcal{E}_m^\vee} \\ &= {}^tA\,{}^tB, \end{aligned}$$

as claimed. □

We also get the transpose of the inverse of a matrix very easily.

Lemma 3.2.6. *Let $A \in \mathrm{GL}_n(F)$. Then ${}^t(A^{-1}) = ({}^tA)^{-1}$.*

Proof. The strategy of proof is to show that ${}^t(A^{-1})$ satisfies the conditions of being an inverse of ${}^tA$, i.e., ${}^tA\,{}^t(A^{-1}) = 1_n = {}^t(A^{-1})\,{}^tA$. We then use that inverses in a group are unique, so it must be that ${}^t(A^{-1}) = ({}^tA)^{-1}$.

Let $\mathcal{E}_n$ be the standard basis of F^n and let T be the multiplication by A map, i.e., $A = [T]_{\mathcal{E}_n}^{\mathcal{E}_n}$. By assumption we have A is invertible, so T^{-1} exists and $A^{-1} = [T^{-1}]_{\mathcal{E}_n}^{\mathcal{E}_n}$. Write id for the identity map and we continue to denote the identity matrix by 1_n. Observe we have

$$\begin{aligned} 1_n &= [\mathrm{id}^\vee]_{\mathcal{E}_n}^{\mathcal{E}_n} \\ &= [(T^{-1}\circ T)^\vee]_{\mathcal{E}_n^\vee}^{\mathcal{E}_n^\vee} \\ &= [T^\vee \circ (T^{-1})^\vee]_{\mathcal{E}_n^\vee}^{\mathcal{E}_n^\vee} \\ &= [T^\vee]_{\mathcal{E}_n^\vee}^{\mathcal{E}_n^\vee}[(T^{-1})^\vee]_{\mathcal{E}_n^\vee}^{\mathcal{E}_n^\vee} \\ &= {}^tA\,{}^t(A^{-1}). \end{aligned}$$

Similarly one has $1_n = {}^t(A^{-1})\,{}^tA$. As noted above, the uniqueness of inverses in $\mathrm{GL}_2(F)$ completes the proof. □

3.3 Problems

For all of these problems V is a finite dimensional F-vector space.

(a) Let $V = P_n(F)$. Let $\mathcal{B} = \{1, x, \ldots, x^n\}$ be a basis of $P_n(F)$. Let $\lambda \in F$ and set $\mathcal{C} = \{1, x - \lambda, \ldots, (x - \lambda)^{n-1}, (x - \lambda)^n\}$. Define a linear transformation $T \in \mathrm{Hom}_F(V, V)$ by defining $T(x^j) = (x - \lambda)^j$. Determine the matrix of this linear transformation. Use this to conclude that $\mathcal{C}$ is also a basis of $P_n(F)$.

(b) Let $V = P_5(\mathbb{Q})$ and let $\mathcal{B} = \{1, x, \ldots, x^5\}$. Prove that the following are elements of V^* and express them as linear combinations of the dual basis:

(a) $\phi : V \to \mathbb{Q}$ defined by $\phi(p(x)) = \int_0^1 t^2 p(t)dt$.

(b) $\phi : V \to \mathbb{Q}$ defined by $\phi(p(x)) = p'(5)$ where $p'(x)$ denotes the derivative of $p(x)$.

(c) Let V be a vector space over F and let $T \in \mathrm{Hom}_F(V, V)$. A nonzero element $v \in V$ satisfying $T(v) = \lambda v$ for some $\lambda \in F$ is called an eigenvector of T with eigenvalue λ.

(a) Prove that for any fixed $\lambda \in F$ the collection of eigenvectors of T with eigenvalue λ together with 0 forms a subspace of V.

(b) Prove that if V has a basis $\mathcal{B}$ consisting of eigenvectors for T then $[T]_{\mathcal{B}}^{\mathcal{B}}$ is a diagonal matrix with the eigenvalues of T as diagonal entries.

(d) Let $A, B \in \mathrm{Mat}_n(F)$. We say A and B are *similar* if there exists $T \in \mathrm{Hom}_F(V, V)$ for some n-dimensional F-vector space V so that $A = [T]_{\mathcal{B}}$ and $B = [T]_{\mathcal{C}}$ for bases $\mathcal{B}$ and $\mathcal{C}$ of V.

(a) Show that if A and B are similar, there exists $P \in \mathrm{GL}_n(F)$ so that $A = PBP^{-1}$. Conversely, if there exists $P \in \mathrm{GL}_n(F)$ so that $A = PBP^{-1}$, show that A is similar to B.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined so that $T = T_A$ where $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 1 & -2 & 4 \end{pmatrix}$. Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ be a basis of $\mathbb{R}^3$. First, calculate $[T]_{\mathcal{B}}$ directly. Then find P so that $[T]_{\mathcal{B}} = PAP^{-1}$.

(e) Let $T \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^4, \mathbb{R}^4)$ be the linear transformation given by the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 1 & -1 \\ -1 & 1 & 0 & -3 \\ 1 & -2 & -1 & 1 \end{pmatrix}$$

with respect to the standard basis $\mathcal{E}_4$. Determine a basis for the image and kernel of T.

(f) Let $T \in \mathrm{Hom}_F(P_7(F), P_7(F))$ be defined by $T(f) = f'$ where f' denotes the usual derivative of a polynomial $f \in P_7(F)$. For each of the fields below, determine a basis for the image and kernel of T:

(a) $F = \mathbb{R}$

(b) $F = \mathbb{F}_3$.

(g) Let V and W be F vector spaces of dimensions n and m respectively. Let $A \in \mathrm{Mat}_{m,n}(F)$ be a matrix representing a linear transformation T from V to W with respect to bases $\mathcal{B}_1$ for V and $\mathcal{C}_1$ for W. Suppose B is the matrix for T with respect to the bases $\mathcal{B}_2$ for V and $\mathcal{C}_2$ for W. Let id_V denote the identity map on V and id_W denote the identity map on W. Set $P = [\mathrm{id}_V]_{\mathcal{B}_2}^{\mathcal{B}_1}$ and $Q = [\mathrm{id}_W]_{\mathcal{C}_2}^{\mathcal{C}_1}$. Prove that $Q^{-1} = [\mathrm{id}_W]_{\mathcal{C}_1}^{\mathcal{C}_2}$ and that $Q^{-1}AP = B$.

The next problems recall Gaussian elimination. First we recall the set-up from undergraduate linear algebra.

Consider a system of equations

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= c_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= c_m \end{aligned}$$

for unknowns $x_1, \ldots, x_n$ and scalars a_{ij}, c_i. We have a coefficient matrix $A = (a_{ij}) \in \mathrm{Mat}_{m,n}(F)$, and an augmented matrix $(A|C) \in \mathrm{Mat}_{m,(n+1)}(F)$ where we add the column vector given by the c_i's on the right side. Note the solutions to the equations above are not altered if we perform the following operations:

(i) interchange any two equations

(ii) add a multiple of one equation to another

(iii) multiply any equation by a nonzero element of F.

In terms a the matrix these correspond to the elementary row operations given by

(r1) interchange any two rows
(r2) add a multiple of one row to another
(r3) multiply any row by a nonzero element of F.

A matrix A that can be transformed into a matrix B by a series of elementary row operations is said to be row reduced to B.

(h) Describe the elementary row operations in terms of matrices. In particular, explain what it is doing on a basis. You can do this separately for each elementary operation.

We say $A \sim B$ if A can be row reduced to B.

(i) Prove that $\sim$ is an equivalence relation. Prove that if $A \sim B$ then the row rank of A is the same as the row rank of B.

An m by n matrix is said to be in reduced row echelon form if

i. the first nonzero entry a_{i,j_i} in row i is 1 and all other entries in the corresponding j_ith column are 0 and
ii. $j_1 < j_2 < \cdots < j_r$ where r is the number of nonzero rows.

An augmented matrix $(A|C)$ is said to be in reduced row echelon form if the coefficient matrix A is in reduced row echelon form. For example, the following matrix is in reduced row echelon form:

$$A = \begin{pmatrix} 1 & 0 & 5 & 7 & 0 & 3 \\ 0 & 1 & -1 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The first nonzero entry in any given row of a reduced row echelon matrix is referred to as a pivotal element. The columns containing pivotal elements are referred to as pivotal columns.

(j) Prove by induction that any augmented matrix can be put in reduced row echelon form by a series of elementary row operations.

(k) Let A and B be two matrices in reduced row echelon form. Prove that if A and B are row equivalent, then $A = B$.

(l) Prove that the row rank of a matrix in reduced row echelon form is the number of nonzero rows.

(m) Find the reduced row echelon form of the matrix

$$A = \begin{pmatrix} 1 & 1 & 4 & 8 & 0 & -1 \\ 1 & 2 & 3 & 9 & 0 & -5 \\ 0 & -2 & 2 & -2 & 1 & 14 \\ 1 & 4 & 1 & 11 & 0 & -13 \end{pmatrix}.$$

14. Use what you have done above to find solutions of the system of equations

$$\begin{aligned} x - 2y + z &= 5 \\ x - 4y + 6z &= 10 \\ 4x - 11y + 11z &= 12. \end{aligned}$$

(n) Let V be an n-dimensional F vector space with basis $\mathcal{E} = \{e_1, \ldots, e_n\}$ and let W be an m-dimensional F vector space with basis $\mathcal{F} = \{f_1, \ldots, f_m\}$. Let $T \in \mathrm{Hom}_F(V, W)$ with $[T]_{\mathcal{E}}^{\mathcal{F}} = (a_{ij})$. Let A' be the reduced row echelon form of A.

(a) Prove that the image $T(V)$ has dimension r where r is the number of nonzero rows of A' and that a basis for $T(V)$ is given by the vectors $T(e_{j_i})$, i.e., the columns of A corresponding to the pivotal columns of A' give the coordinates of a basis for the image of T.
The elements in the kernel of T are the vectors in V whose coordinates $(x_1, \ldots, x_n)$ with respect to $\mathcal{E}$ satisfy the equation

$$A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0$$

and the solutions $x_1, \ldots, x_n$ to this system of linear equations are determined by the matrix A'.

(b) Prove that T is injective if and only if A' has n nonzero rows.

(c) By (a) the kernel of T is nontrivial if and only if A' has nonpivotal columns. Show that each of the variables $x_1, \ldots, x_n$ above corresponding to the nonpivotal columns of A' can be prescribed arbitrarily and the values of the remaining variables are then uniquely determined to give an element $x_1e_1 + \cdots + x_ne_n$ in the kernel of T. In particular, show that the coordinates of a basis for the kernel are obtained by successively setting one nonpivotal variable equal to 1 and all other nonpivotal variables to 0 and solving for the remaining pivotal variables. Conclude that the kernel of T has dimension $n - r$ where r is the rank of A.

(d) Give a basis for the image and kernel of T if the matrix associated to T with respect to the standard bases is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 4 \\ -2 & -4 & 0 & 0 & 2 \\ 1 & 2 & 0 & 1 & -2 \\ 1 & 2 & 0 & 0 & -1 \end{pmatrix}$$

Chapter 4

Structure Theorems for Linear Transformations

Let $T \in \operatorname{Hom}_F(V, V)$. Let $\mathcal{B}$ be a basis for V, where $\dim_F V < \infty$, then we have a matrix $[T]_{\mathcal{B}}$. Our goal is to pick this basis so that $[T]_{\mathcal{B}}$ is as nice as possible.

Throughout this chapter we will always take V to be a finite dimensional vector space.

Though we do not formally introduce determinants until Chapter 7, we use determinants throughout this chapter. Here one should just treat the determinant as it was used in undergraduate linear algebra, namely, in terms of the cofactor expansion.

4.1 Invariant subspaces

In this section we will define some of the basic objects needed throughout the remainder of the chapter. However, we begin with an example.

Example 4.1.1. Let V be a 2-dimensional vector space with basis $\{v_1, v_2\}$. Let $T \in \operatorname{Hom}_F(V, V)$ such that $T(v_1) = v_1 + 3v_2$ and $T(v_2) = 2v_1 + 4v_2$. Thus,

$$[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

It is natural to ask if there is a basis $\mathcal{C}$ so that $[T]_{\mathcal{C}}$ is a diagonal matrix. We will see that if $F = \mathbb{Q}$, there is no such basis $\mathcal{C}$. However if $\mathbb{Q}(\sqrt{33}) \subseteq F$, then there is a basis $\mathcal{C}$ so that

$$[T]_{\mathcal{C}} = \begin{pmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5-\sqrt{33}}{2} \end{pmatrix}.$$

We can also consider the question over finite fields. For instance, $F = \mathbb{F}_3$, there is a basis $\mathcal{C}$ so that

$$[T]_{\mathcal{C}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

but if $F = \mathbb{F}_5$ the matrix cannot be diagonalized. The results of this chapter will make answering such a question routine.

Let V be an F-vector space with $\dim_F V = n$ and let $T \in \operatorname{Hom}_F(V, V)$. Let $f(x) \in F[x]$. We will use throughout this chapter the notation $f(T)$. If $f(x) = a_m x^m + \cdots + a_1 x + a_0$, we will view $f(T)$ as the linear map $a_m T^m + \cdots + a_1 T + a_0$ where $T^m = T \circ T \circ \cdots \circ T$ with m copies of T.

Theorem 4.1.2. *Let $v \in V$ with $v \neq 0$. There is a unique nonzero monic polynomial $m_{T,v}(x) \in F[x]$ of lowest degree so that $m_{T,v}(T)(v) = 0$. Moreover, $\deg m_{T,v}(x) \leq \dim_F V$.*

Proof. Consider the set $\{v, T(v), \ldots, T^n(v)\}$. There are $n+1$ vectors in this set so they must be linearly dependent, i.e., there exists $a_0, \ldots, a_n \in F$ such that $a_n T^n(v) + \cdots + a_1 T(v) + a_0 v = 0$. Set $p(x) = a_n x^n + \cdots + a_1 x + a_0$. We have that $p(T)(v) = 0$.

Consider the subset $I_v \subseteq F[x]$ given by $I_v = \{f(x) : f(T)(v) = 0\}$. Since $p(x) \in I_v$ we have $I_v \neq \emptyset$. Pick $\widetilde{f}(x) \in I_v$ nonzero of minimal degree and write $\widetilde{f}(x) = a_m x^m + \cdots + a_1 x + a_0$ with $a_m \neq 0$, $m \leq n$. Set $f(x) = \frac{1}{a_m} \widetilde{f}(x) \in I_v$. This is a monic polynomial of minimum degree in I_v.

It remains to show that $f(x)$ is unique. Suppose $g(x) \in I_v$ with $\deg g = m$ and g monic. Write $f(x) = q(x)g(x) + r(x)$ for $q(x), r(x) \in F[x]$ with $r(x) = 0$ or $\deg r(x) < m$. Rewriting this gives $r(x) = f(x) - q(x)g(x)$. Observe $r(T)(v) = f(T)(v) - q(T)(v)g(T)(v) = 0$ and so $r(x) \in I_v$. However, this contradicts f having minimal degree in I_v unless $r(x) = 0$. Thus, $f(x) = q(x)g(x)$. Now observe that since $\deg f = m = \deg g$, we must have $\deg q = 0$. This gives $q(x) \in F$. Now we use that f and g are monic to see $q = 1$ and so $f = g$. This gives the result. □

One should note the previous proof amounted to showing that $F[x]$ is a principal ideal domain. Since we are not assuming that level of abstract algebra, the proof is necessary. We will use this type of argument repeatedly so it is important to understand it at this point.

Definition 4.1.3. The polynomial $m_{T,v}(x)$ is referred to as the T-*annihilator of* v.

Corollary 4.1.4. *Let $v \in V$ and $T \in \operatorname{Hom}_F(V, V)$. If $f(x) \in F[x]$ satisfies $f(T)(v) = 0$, then $m_{T,v}(x) \mid f(x)$.*

Proof. We showed this in the course of proving the previous theorem, but we repeat the argument here as this fact will be extremely important in what follows. Let $f(x) \in F[x]$ so that $f(T)(v) = 0$. Using the division algorithm we can write

$$f(x) = q(x)m_{T,v}(x) + r(x)$$

with $q, r \in F[x]$ and $r(x) = 0$ or $\deg r < \deg m_{T,v}$. We have

$$\begin{aligned} 0 &= f(T)(v) \\ &= q(T)m_{T,v}(T)(v) + r(T)(v) \\ &= r(T)(v). \end{aligned}$$

This contradicts the minimality of the degree of $m_{T,v}(x)$ unless $r(x) = 0$. Thus, we have the result. □

Example 4.1.5. Let $V = \mathbb{R}^n$ and $\mathcal{E}_n = \{e_1, \dots, e_n\}$ be the standard basis of V. Define $T : V \to V$ by $T(e_j) = e_{j-1}$ for $2 \leq j \leq m$ and $T(e_1) = 0$. We calculate $m_{T,e_j}(x)$ for each j.

Observe that if we set $f_1(x) = x$ then $f_1(T)(e_1) = T(e_1) = 0$. Moreover, if $g(x) = a$ a constant, then $g(T)(e_1) \neq 0$. Thus, using the above corollary we have that $m_{T,e_1}(x)$ is a monic polynomial of degree at least one that divides $f_1(x) = x$, i.e., $m_{T,e_1}(x) = x$.

Next we calculate $m_{T,e_2}(x)$. Observe that if we set $f_2(x) = x^2$, then $f_2(T)(e_2) = T^2(e_2) = T(e_1) = 0$. Thus, $m_{T,e_2} \mid f_2$. This gives that $m_{T,e_2}(x) = 1, x$, or x^2 since it must be monic. Since $T(e_2) = e_1$, it must be that $m_{T,e_2}(x) = x^2$. Similarly, one obtains that $m_{T,e_j}(x) = x^j$ for $2 \leq j \leq m$.

Example 4.1.6. We now return to Example 4.1.1 and adopt the notation used there. We find $m_{T,v_1}(x)$ and leave the calculation of $m_{T,v_2}(x)$ as an exercise. We have $T(v_1) = v_1 + 3v_2$ and $T^2(v_1) = T(v_1) + 3T(v_2) = v_1 + 3v_2 + 3(2v_1 + 4v_2) = 7v_1 + 15v_2$. Since V has dimension 2 we know there exists $b, c \in F$ such that $T^2(v_1) + bT(v_1) + cv_1 = 0$. Finding b and c amounts to solving a system of linear equations: we obtain $b = -5, c = -2$. So $f_1(x) = x^2 - 5x - 2$ satisfies $f(T)(v_1) = 0$. Whether this is $m_{T,v_1}(x)$ depends on the field F and whether we can factor over F. For example, over $\mathbb{Q}$ this is an irreducible polynomial and so it must be that $m_{T,v_1}(x) = x^2 - 5x - 2$ over $\mathbb{Q}$. In fact, we will later see this means that $m_{T,v_1}(x) = x^2 - 5x - 2$ over any field F that contains $\mathbb{Q}$. One other thing to observe is that the roots of this polynomial are $\frac{5}{2} \pm \frac{\sqrt{33}}{2}$.

Exercise 4.1.7. Redo the previous example with $F = \mathbb{F}_3$ this time.

Though the annihilating polynomials are useful, they only tell us about the linear map element by element. What we would really like it is a polynomial that tells us about the overall linear map. The minimal polynomial provides just such a polynomial.

Theorem 4.1.8. *Let* $\dim_F V = n$. *There is a unique monic polynomial* $m_T(x) \in F[x]$ *of lowest degree so that* $m_T(T)(v) = 0$ *for all* $v \in V$. *Furthermore,* $\deg m_T(x) \leq n^2$.

Proof. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for V. Let $m_{T,v_i}(x)$ be the annihilating polynomial for each i. Set $m_T(x) = \operatorname{lcm}_i m_{T,v_i}(x)$. Note that $m_T(x)$ is monic and $m_T(T)(v_i) = 0$ for each $1 \leq i \leq n$. From this it is easy to show that $m_T(T)(v) = 0$ for all $v \in V$. Since each $m_{T,v_i}(x)$ has degree n and there are n polynomials, we must have $\deg m_T(x) \leq \sum_{i=1}^{n} n = n^2$. It remains to show $m_T(x)$ is unique. Suppose there exists $r(x) \in F[x]$ with $r(T)(v) = 0$ for all $v \in V$ but $m_{T,v_j}(x) \nmid r(x)$ for some j. This is a contradiction because if $m_{T,v_j}(x) \nmid r(x)$, then $r(T)(v_j) \neq 0$. Thus, by the definition of least common multiple we must have $m_T(x) \mid r(x)$ and so $m_T(x)$ is the unique monic polynomial of minimal degree satisfying $m_T(T)(v) = 0$ for all $v \in V$. □

We note the following corollary that was shown in the process of proving the last result.

Corollary 4.1.9. *Let* $T \in \operatorname{Hom}_F(V, V)$ *and suppose* $f(x) \in F[x]$ *with* $f(T)(v) = 0$ *for all* $v \in V$. *Then* $m_T(x) \mid f(x)$.

The bound on the degree of $m_T(x)$ given above is far from optimal. In fact, we will see shortly that $\deg m_T(x) \leq n$.

Definition 4.1.10. The polynomial $m_T(x)$ is called the *minimal polynomial of* T.

Example 4.1.11. We once again return to Example 4.1.1. Let $F = \mathbb{Q}$. We saw $m_{T,v_1}(x) = x^2 - 5x - 2$ and one saw in the exercise that $m_{T,v_2}(x) = x^2 - 5x - 2$. Thus,

$$\begin{aligned} m_T(x) &= \operatorname{lcm}(x^2 - 5x - 2, x^2 - 5x - 2) \\ &= x^2 - 5x - 2. \end{aligned}$$

Example 4.1.12. Let $V = \mathbb{Q}^3$ and let $\mathcal{E}_3 = \{e_1, e_2, e_3\}$ be the standard basis. Let T be given by the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}$. One can calculate from this that $m_{T,e_1}(x) = x - 1$, $m_{T,e_2} = (x-1)^2$, $m_{T,e_3} = (x-1)^2(x+1)$. Thus,

$$\begin{aligned} m_T(x) &= \operatorname{lcm}((x-1), (x-1)^2, (x-1)^2(x+1)) \\ &= (x-1)^2(x+1). \end{aligned}$$

The following result is the first step in showing that one can realize the minimal polynomial of any linear map as the annihilator of an element of the vector space.

Lemma 4.1.13. *Let $T \in \operatorname{Hom}_F(V, V)$. Let $v_1, \ldots, v_k \in V$ and set $p_i(x) = m_{T,v_i}(x)$. Suppose the $p_i(x)$ are pairwise relatively prime. Set $v = v_1 + \cdots + v_k$. Then $m_{T,v}(x) = p_1(x) \cdots p_k(x)$.*

Proof. We prove the case $k = 2$. The general case follows by induction and is left as an exercise. Let $p_1(x)$ and $p_2(x)$ be as in the statement of the lemma. The fact that they are relatively prime gives polynomials $q_1(x), q_2(x) \in F[x]$ such that $p_1(x)q_1(x) + p_2(x)q_2(x) = 1$. Thus, we have

$$\begin{aligned} v &= 1 \cdot v \\ &= (p_1(T)q_1(T) + p_2(T)q_2(T))v \\ &= q_1(T)p_1(T)(v) + q_2(T)p_2(T)(v) \\ &= q_1(T)p_1(T)(v_1) + q_1(T)p_1(T)(v_2) + q_2(T)p_2(T)(v_1) + q_2(T)p_2(T)(v_2) \\ &= q_1(T)p_1(T)(v_2) + q_2(T)p_2(T)(v_1) \end{aligned}$$

where we have used $p_1(T)(v_1) = 0$ and $p_2(T)(v_2) = 0$. Set $w_1 = q_2(T)p_2(T)(v_1)$ and $w_2 = q_1(T)p_1(T)(v_2)$ so that $v = w_1 + w_2$.

Observe that

$$\begin{aligned} p_1(T)(w_1) &= p_1(T)q_2(T)p_2(T)(v_1) \\ &= p_2(T)q_2(T)p_1(T)(v_1) \\ &= 0. \end{aligned}$$

Thus, $w_1 \in \ker p_1(T)$. Similarly, $w_2 \in \ker p_2(T)$.

Let $r(x) \in F[x]$ such that $r(T)(v) = 0$. Recall that $v = w_1 + w_2$. Since $w_2 \in \ker p_2(T)$ we have

$$\begin{aligned} p_2(T)(v) &= p_2(T)(w_1 + w_2) \\ &= p_2(T)(w_1). \end{aligned}$$

Thus,

$$\begin{aligned} 0 &= p_2(T)q_2(T)r(T)(v) \\ &= r(T)q_2(T)p_2(T)(v) \\ &= r(T)q_2(T)p_2(T)(w_1). \end{aligned}$$

Moreover, we have

$$0 = r(T)p_1(T)q_1(T)(w_1)$$

since $w_1 \in \ker p_1(T)$. Thus,

$$0 = r(T)(p_1(T)q_1(T) + p_2(T)q_2(T))(w_1).$$

Using that $p_1(T)q_1(T) + p_2(T)q_2(T) = 1$, we obtain $0 = r(T)(w_1)$. Thus, we have $0 = r(T)(w_1) = r(T)(p_2(T)q_2(T)(v_1))$. Since $p_1(x)$ is the annihilating polynomial of v_1, we obtain $p_1(x)|r(x)p_2(x)q_2(x)$. However, since $p_1(x)q_1(x) + p_2(x)q_2(x) = 1$ we have that p_1 is relatively prime to $p_2(x)q_2(x)$, so $p_1(x)|r(x)$. A similar argument shows $p_2(x)|r(x)$. Since $p_1(x), p_2(x)$ are relatively prime, $p_1(x)p_2(x)|r(x)$. Observe that

$$\begin{aligned} p_1(T)p_2(T)(v) &= p_1(T)p_2(T)(v_1) + p_1(T)p_2(T)(v_2) \\ &= 0. \end{aligned}$$

Thus $p_1(x)p_2(x)$ is a monic polynomial and for any $r(x) \in F[x]$ so that $r(T)(v) = 0$, we have $p_1(x)p_2(x) \mid r(x)$. This is exactly what it means for $m_{T,v}(x) = p_1(x)p_2(x)$. □

Theorem 4.1.14. *Let $T \in \mathrm{Hom}_F(V, V)$. There exists $v \in V$ such that $m_{T,v}(x) = m_T(x)$.*

Proof. Let $v_1, \dots, v_n$ be a basis of V so that $m_T(x) = \mathrm{lcm}_i\, m_{T,v_i}(x)$. Factor $m_T(x)$ into irreducible factors $p_1(x)^{e_1} \cdots p_k(x)^{e_k}$ with $e_i \geq 1$ and the $p_i(x)$ pairwise relatively prime. For each $1 \leq j \leq n$ there exists $i_j \in \{1, \dots, k\}$ and $q_{i_j}(x) \in F[x]$ such that $m_{T,v_j}(x) = p_{i_j}(x)^{e_{i_j}} q_{i_j}(x)$. Set $u_{i_j} = q_{i_j}(T)(v_j)$. Then $m_{T,u_{i_j}}(x) = p_{i_j}(x)^{e_{i_j}}$. Thus, if we set $v = \sum u_{i_j}$, then Lemma 4.1.13 gives

$$\begin{aligned} m_{T,v}(x) &= p_1(x)^{e_1} \cdots p_k(x)^{e_k} \\ &= m_T(x). \end{aligned}$$

□

This result gives us the desired bound on the degree of $m_T(x)$.

Corollary 4.1.15. *For $T \in \mathrm{Hom}_F(V, V)$ we have $\deg m_T(x) \leq n$.*

Proof. Since there exists $v \in V$ so that $m_T(x) = m_{T,v}(x)$ and we know $\deg m_{T,v}(x) \leq n$, this gives the result. □

Definition 4.1.16. Let $A \in \mathrm{Mat}_n(F)$. We define the *characteristic polynomial* of A as $c_A(x) = \det(x1_n - A) \in F[x]$.

One must be very careful with what is meant here. As we have seen above, we will often be interested in evaluating a polynomial at a linear map or a matrix. Without the more rigorous treatment, we have to be careful what we mean by this. For example, the Cayley-Hamilton Theorem (see Theorem 4.1.30) says that $c_A(A) = 0$. At first glance this seems trivial, but that is only the case if you misinterpret what is meant by $c_A(A)$. What this actually means is form the polynomial $c_A(x)$ and then replace the x's with A's. One easy way to see the difference is that $c_A(B)$ is a matrix for any matrix B, but if you evaluated $B1_n - A$ and then took the determinant this would give a scalar. Note that a more rigorous treatment of the characteristic polynomial in the appendix. For a first reading of the material the more rigorous treatment can certainly be skipped.

Recall that we say matrices $A, B \in \mathrm{Mat}_n(F)$ are *similar* if there exists $P \in \mathrm{GL}_n(F)$ so that $A = PBP^{-1}$.

Lemma 4.1.17. *Let $A, B \in \mathrm{Mat}_n(F)$ be similar matrices. Then $c_A(x) = c_B(x)$.*

Proof. Let $P \in \mathrm{GL}_n(F)$ such that $A = PBP^{-1}$. Then we have

$$\begin{aligned} c_A(x) &= \det(xI_n - A) \\ &= \det(xI_n - PBP^{-1}) \\ &= \det(PxI_nP^{-1} - PBP^{-1}) \\ &= \det(P(xI_n - B)P^{-1}) \\ &= \det P \det(xI_n - B) \det P^{-1} \\ &= \det(xI_n - B) \\ &= c_B(x). \end{aligned}$$

□

We can use this result to define the characteristic polynomial of a linear map as well. Given $T \in \mathrm{Hom}_F(V, V)$, we define $c_T(x) = c_{[T]_{\mathcal{B}}}(x)$ for any basis $\mathcal{B}$. Note this is well-defined because choosing a different basis gives a similar matrix, which does not affect the characteristic polynomial. If $\dim_F V = n$, then $\deg c_T(x) = n$ and $c_T(x)$ is monic.

Definition 4.1.18. Let $f(x) = x^n + a_{n-1}x^{n-1} \cdots + a_1x + a_0 \in F[x]$. The *companion matrix* of f is given by:

$$C(f(x)) = \begin{pmatrix} -a_{n-1} & 1 & 0 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The companion matrix will be extremely important when we study rational canonical form.

Lemma 4.1.19. *Let $f(x) = x^n + a_{n-1}x^{n-1} \cdots + a_1x + a_0$. Set $A = C(f(x))$. Then $c_A(x) = f(x)$.*

Proof. Observe we have

$$xI_n - A = \begin{pmatrix} x + a_{n-1} & -1 & 0 & \cdots & 0 \\ a_{n-2} & x & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_1 & 0 & 0 & x & -1 \\ a_0 & 0 & 0 & 0 & x \end{pmatrix}$$

We prove the result by induction on n. First, suppose $n = 1$. Then we have $f(x) = x + a_0$ and $A = (a_0)$. So $xI_n - A = x + a_0$ and thus $c_A(x) = \det(x + a_0) = x + a_0 = f(x)$ as claimed. Now suppose the result is true for all $n \leq k - 1$. We show the result is true for $n = k$. We have

$$
\begin{aligned}
c_A(x) &= \det(x1_k - A)\\
&= \det \begin{pmatrix} x + a_{k-1} & -1 & 0 & \cdots & 0 \\ a_{k-2} & x & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_1 & 0 & 0 & x & -1 \\ a_0 & 0 & 0 & 0 & x \end{pmatrix}\\
&= (-1)^{k+1} a_0 \det \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ x & -1 & 0 & \cdots & 0 \\ 0 & x & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & x & -1 \end{pmatrix} + (-1)^{2k} x \det \begin{pmatrix} x + a_{k-1} & -1 & \cdots & 0 \\ a_{k-2} & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & 0 & 0 & x \end{pmatrix}\\
&= a_0 + x(x^{k-1} + a_{k-1}x^{k-2} + \cdots + a_1)\\
&= f(x).
\end{aligned}
$$

Thus, we have the result by induction. □

The next theorem gives us our first serious result towards our structure theorems.

Theorem 4.1.20. *Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in F[x]$. Let $A = C(f(x))$. Set $V = F^n$ and $T = T_A$. Let $\mathcal{E}_n = \{e_1, \ldots, e_n\}$ be the standard basis of F^n. Then the subspace $W \subseteq V$ given by*

$$W = \{g(T)(e_n) : g(x) \in F[x]\}$$

is all of V. Moreover, $m_T(x) = m_{T,e_n}(x) = f(x)$.

Proof. Observe we have $T(e_n) = e_{n-1}, T^2(e_n) = e_{n-2}$, and in general $T^k(e_n) = e_{n-k}$ for $k \leq n-1$. We have that W contains $\operatorname{span}_F(T^{n-1}(e_n), \ldots, T(e_n), e_n) = \operatorname{span}_F(e_1, \ldots, e_n) = V$. Since $W \subseteq V$, this implies that $W = V$.

Note, we have n elements in $\{T^{n-1}(e_n), \ldots, T(e_n), e_n\}$ and they span, so they are linearly independent. So any polynomial $p(x)$ such that $p(T)(e_n) = 0$ must have degree at least n. We have

$$
\begin{aligned}
T^n(e_n) &= T(e_1)\\
&= -a_{n-1}e_1 - a_{n-2}e_2 - \cdots - a_0 e_n\\
&= -a_{n-1}T^{n-1}(e_n) - \cdots - a_1 T(e_n) - a_0 e_n.
\end{aligned}
$$

Thus, $T^n(e_n) + \cdots + a_1 T(e_n) + a_0 e_n = 0$. This gives $f(T)(e_n) = 0$. Since f is degree n and is monic, we must have that $f(x) = m_{T,e_n}(x)$. We know $m_{T,e_n}(x) | m_T(x)$ and $\deg m_T(x) \leq n$, so $m_T(x) = m_{T,e_n}(x) = f(x)$. □

We now switch our attention to subspaces of V that are well-behaved with respect to T, namely, ones that are preserved by T.

Definition 4.1.21. A subspace $W \subseteq V$ that satisfies $T(W) \subseteq W$ is referred to as a *T-invariant subspace* or a *T-stable subspace.*

As is the custom, we rephrase the definition in terms of matrices.

Theorem 4.1.22. *Let V be an n-dimensional F-vector space and $W \subseteq V$ a k-dimensional subspace. Let $\mathcal{B}_W = \{v_1, \ldots, v_k\}$ be a basis of W and extend to a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ of V. Let $T \in \mathrm{Hom}_F(V, V)$. Then W is T-invariant if and only if $[T]_{\mathcal{B}}$ is block upper triangular*

$$[T]_{\mathcal{B}} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where $A \in \mathrm{Mat}_k(F)$ and is given by $A = [T|_W]_{\mathcal{B}_W}$.

Note that if W is a T-invariant subspace, then $T|_W \in \mathrm{Hom}_F(W, W)$. Moreover, we have a natural map $\overline{T} : V/W \to V/W$ given by $T(v + W) = T(v) + W$. This is well-defined precisely because W is T-invariant, which one should check as an exercise. We also have the following simple relations between the annihilating polynomials of $\overline{T}$ and T as well as the minimal polynomials. These will be useful in induction proofs that follow.

Lemma 4.1.23. *Let $W \subseteq V$ be T-invariant and let $\overline{T}$ be the induced linear map on V/W. Let $v \in V$. Then $m_{\overline{T},[v]}(x) | m_{T,v}(x)$ where $[v] = v + W$.*

Proof. Observe we have

$$\begin{aligned} m_{T,v}(\overline{T})([v]) &= m_{T,v}(T)(v) + W \\ &= 0 + W. \end{aligned}$$

Thus, $m_{\overline{T},[v]}(x) | m_{T,v}(x)$. □

The following result can either be proved by the same methods as used in the previous lemma, or one can use the previous result and the definition of the minimal polynomial.

Corollary 4.1.24. *Let $W \subseteq V$ be T-invariant and let $\overline{T}$ be the induced linear map on V/W. Then $m_{\overline{T}}(x) | m_T(x)$.*

Definition 4.1.25. Let $T \in \mathrm{Hom}_F(V, V)$ and let $\mathcal{A} = \{v_1, \ldots, v_k\}$ be a set of vectors in V. The *T-span of $\mathcal{A}$* is the subspace

$$W = \left\{ \sum_{i=1}^{k} p_i(T)(v_i) : p_i(x) \in F[x] \right\}.$$

We say $\mathcal{A}$ *T-generates W*.

Exercise 4.1.26. Check that W given in the previous definition is a T-invariant a subspace of V. Moreover, show it is the smallest (with respect to inclusion) T-invariant subspace of V that contains $\mathcal{B}$.

The following lemma, while elementary, will be used repeatedly in this chapter.

Lemma 4.1.27. *Let $T \in \mathrm{Hom}_F(V,V)$. Let $w \in V$ and let W be the subspace of V that is T-generated by w. Then*

$$\dim_F W = \deg m_{T,w}(x).$$

Proof. Let $\deg m_{T,w}(x) = k$. Then $\{w, T(w), \ldots, T^{k-1}(w)\}$ spans W and if any subset spanned we would have $\deg m_{T,w}(x) < k$. This gives the result. □

Up to this point we have dealt with two separate polynomials associated to a linear map: the characteristic and minimal polynomials. It is natural to ask if there is a relation between the two. In fact, there is a very nice relation given by the following theorem.

Theorem 4.1.28. *Let $T \in \mathrm{Hom}_F(V,V)$. Then*

(a) $m_T(x) | c_T(x)$;

(b) Every irreducible factor of $c_T(x)$ is a factor of $m_T(x)$.

Proof. We proceed by induction on $\dim_F V = n$. If $n = 1$ the result is true trivially, so our base case is done. Let $\deg m_T(x) = k \leq n$. Let $v \in V$ such that $m_T(x) = m_{T,v}(x)$. Let W_1 be the T-span of v, so by Lemma 4.1.27 we have $\dim_F W_1 = k$. Set $v_k = v$ and $v_{k-i} = T^i(v)$ for $i = 0, \ldots, k-1$. Then we claim $\mathcal{B}_1 = \{v_1, \ldots, v_k\}$ is a basis for W_1. Note that this set has the correct number of elements, so we only need to show it is linearly independent or spans. Suppose there exists $a_1, \ldots, a_k \in F$ so that $a_1 v_1 + \cdots + a_k v_k = 0$. Using the definition of the v_i we have $a_1 T^{k-1}(v) + \cdots + a_k v = 0$. This implies that $m_{T,v}(x) \mid (a_1 x^{k-1} + \cdots + a_{k-1})$, which is a contradiction unless $a_1 = \cdots a_{k-1} = 0$ since $\deg m_{T,v}(x) = 0$. This shows the set is a basis. Moreover, we have $[T|_{W_1}]_{\mathcal{B}_1} = C(m_T(x))$, the companion matrix.

If $k = n$, then $W_1 = V$ So $[T]_{\mathcal{B}_1} = C(m_T(x))$ has characteristic polynomial $m_T(x)$, i.e., $c_T(x) = m_T(x)$ and we are done.

Now suppose $k < n$. Let V_2 be the orthogonal complement of W_1 in V, and $\mathcal{B}_2$ a basis of V_2. Set $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. We have $[T]_{\mathcal{B}} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ with A

the companion matrix of $m_T(x)$. This allows us to observe that

$$\begin{aligned} c_T(x) &= \det(xI_n - [T]_{\mathcal{B}}) \\ &= \det \begin{pmatrix} xI_k - A & -B \\ 0 & xI_{n-k} - D \end{pmatrix} \\ &= \det(xI_k - A)\det(xI_{n-k} - D) \\ &= m_T(x)\det(xI_{n-k} - D). \end{aligned}$$

Thus, we see $m_T(x)|c_T(x)$. This gives the first claim of the theorem.

It remains to show $c_T(x)$ and $m_T(x)$ have the same irreducible factors. Note that since $m_T(x) \mid c_T(x)$ we certainly have every irreducible factor of $m_T(x)$ is an irreducible factor of $c_T(x)$. If $m_T(x)$ has degree n, then $m_T(x) = c_T(x)$ and the result is trivial. Assume $\deg m_T(x) < n$. Consider $\overline{T} : V/W_1 \to V/W_1$. Set $\overline{\mathcal{B}} = \pi_{W_1}(\mathcal{B})$. We have $[\overline{T}]_{\overline{\mathcal{B}}} = D$. Since $\dim_F V/W_1 < \dim_F V$, we can use induction to conclude $c_{\overline{T}}(x)$ and $m_{\overline{T}}(x)$ have the same irreducible factors. Let $p(x)$ be an irreducible factor of $c_T(x)$. As above write $c_T(x) = m_T(x)\det(xI_{n-k} - D)$, which we have is equal to $m_T(x)c_{\overline{T}}(x)$. Since $p(x)$ is irreducible, it divides $m_T(x)$ or $c_{\overline{T}}(x)$. If $p(x)|m_T(x)$, we are done. If not, $p(x)|c_{\overline{T}}(x)$, and so $p(x)$ is an irreducible factor of $c_{\overline{T}}(x)$. However, $m_{\overline{T}}(x)$ and $c_{\overline{T}}(x)$ have the same irreducible factors so $p(x)$ is an irreducible factor of $m_{\overline{T}}(x)$. We now use that $m_{\overline{T}}(x)|m_T(x)$ to conclude $p(x)|m_T(x)$. □

Corollary 4.1.29. *Let $\dim_F V = n$, $T \in \operatorname{Hom}_F(V,V)$. Then V is T-generated by a single element if and only if $m_T(x) = c_T(x)$.*

Proof. Let $w \in V$ and let W be the subspace of V that is T-generated by w. We know that $\dim_F W = \deg m_{T,w}(x)$. Thus, if there is an element that T-generates V we have $\deg m_{T,w}(x) = n$, and since $m_{T,w}(x) \mid m_T(x)$ for every w, we have $\deg m_T(x) = n$ and so it must be equal to $c_T(x)$. Conversely, if $m_T(x) = c_T(x)$ then we use the fact that there is an element $w \in V$ so that $m_{T,w}(x) = m_T(x)$. Thus, $\deg m_{T,w}(x) = n$ and so $\dim_F W = n$, i.e., $W = V$. □

These results trivially give the Cayley-Hamilton theorem.

Theorem 4.1.30. *[Cayley-Hamilton Theorem]*

(a) Let $T \in \operatorname{Hom}_F(V,V)$ with $\dim_F V < \infty$. Then $c_T(T) = 0$.

(b) Let $A \in \operatorname{Mat}_n(F)$ and $c_A(x)$ the characteristic polynomial. Then $c_A(A) = 0$.

As mentioned above, the term $c_A(A)$ is a matrix and the content of the theorem above is that this is the zero matrix.

4.2 T-invariant complements

We now turn our focus to invariant subspaces and the question of when a T-invariant subspace has a T-invariant complement. Recall that given any subspace $W \subset V$, we always have a subspace $W' \subset V$ so that $V = W \oplus W'$. In particular, if W is T-invariant we have a complement W'. However, it is not necessarily the case that W' will also be T-invariant. As we saw in the last section, this essentially comes down to whether there is a basis $\mathcal{B}$ so that the matrix for T is block diagonal.

Definition 4.2.1. Let $W_1, \dots, W_k$ be subspaces of V such that $V = W_1 \oplus \cdots \oplus W_k$. Let $T \in \mathrm{Hom}_F(V, V)$. We say $V = W_1 \oplus \cdots \oplus W_k$ is a *T-invariant direct sum* if each W_i is T-invariant. If $V = W \oplus W'$ is a T-invariant direct sum, we say W' is the *T-invariant complement* of W.

We now give two fundamental examples of this. Not only are they useful for understanding the definition, they will be useful in understanding the arguments to follow.

Example 4.2.2. Let $T \in \mathrm{Hom}_F(V, V)$ and assume $m_T(x) = c_T(x)$. For convenience write $f(x) = m_T(x)$. Suppose we can factor $f(x) = g(x)h(x)$ with $\gcd(g, h) = 1$. Let $v_0 \in V$ be so that $m_{T,v_0}(x) = m_T(x)$, i.e., V is T-generated by v_0.

Set $W_1 = h(T)(V)$ and $W_2 = g(T)(V)$. We claim that $V = W_1 \oplus W_2$ is a T-invariant direct sum. Note it is clear that W_1 and W_2 are both T-invariant, so we only need to show that $V = W_1 \oplus W_2$ to see it is a T-invariant direct sum.

First, we show that $W_1 \cap W_2 = \{0\}$. Let $w_1 \in W_1$. Then $w_1 = h(T)(v_1)$ for some $v_1 \in V$. We have $g(T)(w_1) = g(T)h(T)(v_1) = f(T)(v_1) = m_T(T)(v_1) = 0$. Thus, $m_{T,w_1} \mid g(x)$. Similarly, if $w_2 \in W_2$, then $m_{T,w_2} \mid h(x)$. If $w \in W_1 \cap W_2$, then $m_{T,w}(x) \mid g(x)$ and $m_{T,w}(x) \mid h(x)$. Thus $m_{T,w}(x) = 1$. This implies we must have $w = 0$, as desired.

It remains to show that $V = W_1 + W_2$. Since $\gcd(g, h) = 1$, there exists $s, t \in F[x]$ such that $1 = g(x)s(x) + h(x)t(x)$. Thus,

$$\begin{aligned} v_0 &= (g(T)s(T) + h(T)t(T))v_0 \\ &= g(T)s(T)v_0 + h(T)t(T)v_0 \\ &= w_1 + w_2 \end{aligned}$$

where

$$\begin{aligned} w_1 &= h(T)t(T)v_0 \\ &= h(T)(t(T)v_0) \in h(T)(V) = W_1 \end{aligned}$$

and

$$\begin{aligned} w_2 &= g(T)s(T)v_0 \\ &= g(T)(s(T)v_0) \in g(T)(V) = W_2. \end{aligned}$$

Thus, $v_0 \in W_1 + W_2$. Let $v \in V$. There exists $b(x) \in F[x]$ such that $v = b(T)(v_0)$. This gives

$$\begin{aligned} v &= b(T)v_0 \\ &= b(t)(w_1 + w_2) \\ &= b(T)(w_1) + b(T)(w_2) \in W_1 + W_2. \end{aligned}$$

Thus $V = W_1 \oplus W_2$ as T-invariant spaces.

In summary, if $m_T(x) = c_T(x)$ and we can write $m_T(x) = g(x)h(x)$ with $\gcd(g, h) = 1$, then there exists T-invariant subspaces W_1 and W_2 so that $V = W_1 \oplus W_2$. Let $\mathcal{B}_i$ be a basis for W_i. Then we have

$$[T]_{\mathcal{B}_1 \cup \mathcal{B}_2} = \begin{pmatrix} [T]_{\mathcal{B}_1} & 0 \\ 0 & [T]_{\mathcal{B}_2} \end{pmatrix}.$$

Example 4.2.3. As in the previous example, assume $m_T(x) = c_T(x)$ and again write $f(x) = m_T(x)$. However, in this case we assume $f(x) = g(x)h(x)$ with $\gcd(g, h) > 1$. For example, $f(x)$ could be a power of an irreducible polynomial. Let $v_0 \in V$ such that $m_T(x) = m_{T,v_0}(x)$, i.e., V is T-generated by v_0. Set $W_1 = h(T)(V)$. Clearly we have W_1 is T-invariant. We will now show W_1 does not have a T-invariant complement. Suppose $V = W_1 \oplus W_2$ with W_2 a T-invariant subspace. Write $T_1 = T|_{W_1}$ and $T_2 = T|_{W_2}$.

We claim that $m_{T_1}(x) = g(x)$. Let $w_1 \in W_1$ and write $w_1 = h(T)(v_1)$ for some $v_1 \in V$. We have

$$\begin{aligned} g(T)(w_1) &= g(T)h(T)(v_1) \\ &= f(T)(v_1) \\ &= m_T(T)(v_1) \\ &= 0. \end{aligned}$$

Thus, we must have $m_{T_1}(x) \mid g(x)$. Set $w_1' = h(T)(v_0)$ and $k(x) = m_{T,w_1'}(x)$. Then we have

$$\begin{aligned} 0 &= k(T)(w_1') \\ &= k(T)h(T)(v_0). \end{aligned}$$

Thus, we have $m_{T,v_0}(x) = g(x)h(x) \mid k(x)h(x)$. This gives $g(x) \mid k(x) = m_{T,w_1'}(x)$. However, $m_{T,w_1'}(x) \mid m_{T_1}(x)$ and so $g(x) \mid m_{T_1}(x)$. This gives $g(x) = m_{T_1}(x)$ as claimed.

Our second claim is that $m_{T_2}(x)|h(x)$. Let $w_2 \in W_2$. Then $h(T)(w_2) \in W_1$ because $h(T)(V) = W_1$. We also have $h(T)(w_2) \in W_2$ because W_2 is T-invariant by assumption. However, $W_1 \cap W_2 = \{0\}$ so it must be the case that $h(T)(w_2) = 0$. Hence $m_{T_2}(x) \mid h(x)$ as claimed.

As we are assuming $V = W_1 + W_2$ we can write $v_0 = w_1 + w_2$ for some $w_i \in W_i$. Set $b(x) = \operatorname{lcm}(g(x), h(x))$. We have $b(T)(v_0) = b(T)(w_1) +$

$b(T)(w_2) = 0$ since $m_{T_1}(x) = g(x) \mid b(x)$ and $m_{T_2}(x) \mid h(x) \mid b(x)$. Thus, $b(x)$ is divisible by $f(x) = m_{T,v_0}(x)$. However, since $\gcd(g,h) \neq 1$ we have $b(x)$ is a proper factor of $f(x)$ that vanishes on v_0, so on V. This contradicts $m_{T,v_0}(x) = f(x) = c_T(x)$. Thus, W_1 does not have a T-invariant complement.

Now that we have these examples at our disposal, we return to the general situation.

Theorem 4.2.4. *Let $T \in \mathrm{Hom}_F(V,V)$. Let $m_T(x) = p_1(x) \cdots p_k(x)$ be a factorization into relatively prime polynomials. Set $W_i = \ker(p_i(T))$ for $i = 1, \dots, k$. Then W_i is T-invariant for each i and $V = W_1 \oplus \cdots \oplus W_k$ is a T-invariant direct sum decomposition of V.*

Proof. Let $w_i \in W_i$. We have

$$\begin{aligned} p_i(T)(T(w_i)) &= T(p_i(T)(w_i)) \\ &= T(0) \\ &= 0. \end{aligned}$$

Thus, $T(w_i) \in \ker(p_i(T)) = W_i$. This gives each W_i is T-invariant so it only remains to show V is a direct sum of the W_i.

For each i set $q_i(x) = m_T(x)/p_i(x)$. The collection $\{q_1, \dots, q_k\}$ is relatively prime (not in pairs, just overall) so there are polynomials $r_1, \dots, r_k \in F[x]$ such that

$$1 = q_1 r_1 + \cdots + q_k r_k.$$

Let $v \in V$. We have

$$\begin{aligned} v &= 1 \cdot v \\ &= r_1(T) q_1(T) v + \cdots + r_k(T) q_k(T) v \\ &= w_1 + \cdots + w_k \end{aligned}$$

where we set $w_i = r_i(T) q_i(T)(v)$. We claim that $w_i \in W_i$. Observe

$$\begin{aligned} p_i(T)(w_i) &= p_i(T) q_i(T) r_i(T)(v) \\ &= r_i(T) m_T(T)(v) \\ &= 0. \end{aligned}$$

Thus, $w_i \in W_i$ as claimed. This shows we have $V = W_1 + \cdots + W_k$.

Suppose there exists $w_i \in W_i$ so that $0 = w_1 + \cdots + w_k$. We need to show this implies $w_i = 0$ for each i. We have

$$\begin{aligned} 0 &= q_1(T)(0) \\ &= q_1(T)(w_1 + \cdots + w_k) \\ &= q_1(T)(w_1) + \cdots + q_1(T)(w_k) \\ &= q_1(T)(w_1) \end{aligned}$$

where we have used $p_i \mid q_1$ for all $i \neq 1$ so $q_1(T)(w_i) = 0$ for $i \neq 1$. Thus, we have $q_1(T)(w_1) = 0$ and by definition $p_1(T)(w_1) = 0$. Since p_1 and q_1 are relatively prime there exists $r, s \in F[x]$ so that

$$1 = r(x)p_1(x) + s(x)q_1(x).$$

Thus,

$$\begin{aligned} w_1 &= (r(T)p_1(T) + s(T)q_1(T))w_1 \\ & 0. \end{aligned}$$

The same argument gives $w_i = 0$ for all i. Thus, $V = W_1 \oplus \cdots \oplus W_k$ with W_i T-invariant. □

One thing to note in the previous result is that given T as in the statement of the theorem, the map $\pi_i = r_i(T)q_i(T)$ is a projection map from V onto W_i. This fact is important in the following result. We can now determine exactly how all T-invariant subspaces of V arise from the minimal polynomial of T.

Corollary 4.2.5. *Let $T \in \mathrm{Hom}_F(V, V)$, write $m_T(x) = p_1(x)^{e_1} \cdots p_k(x)^{e_k}$ be the irreducible factorization of $m_T(x)$. Let $W_i = \ker(p_i(T)^{e_i})$. For $i = 1, \ldots, k$ let U_i be a T-invariant subspace of W_i. (Note we allow U_i to be zero.) Then $U = U_1 \oplus \cdots \oplus U_k$ is a T-invariant subspace of V. Moreover, if W is any T-invariant subspace of V, we have*

$$W = (W \cap W_1) \oplus \cdots \oplus (W \cap W_k),$$

i.e., all T-invariant subspaces arise by looking at T-invariant subspaces of the W_i.

Proof. The previous result shows $V = W_1 \oplus \cdots \oplus W_k$ with the W_i being T-invariant, so it is clear such a U is a T-invariant subspace.

Let W be a T-invariant subspace of V. Let $w \in W$ and write $w = w_1 + \cdots + w_k$ with $w_i \in W_i$. We know that $w_i = \pi_i(w) = h_i(T)(w)$ for some $h_i(x) \in F[x]$ (see the remark preceding this result.) Now since W is T-invariant, each $w_i \in W$ as well. Thus, each $w_i \in W \cap W_i$. This is a unique expression since $V = W_1 \oplus \cdots \oplus W_k$. This gives

$$W = (W \cap W_1) \oplus \cdots \oplus (W \cap W_k),$$

as desired. □

This result reinforces the importance of the minimal polynomial in understanding the structure of T. We obtain T-invariant subspaces by factoring the minimal polynomial and using the irreducible factors.

The following lemma is essential to prove the theorem that will allow us to give our first significant result on the structure of a linear map: the rational canonical form. The proof of this lemma is a bit of work though and can be skipped without losing any of the main ideas.

Lemma 4.2.6. *Let $T \in \operatorname{Hom}_F(V,V)$. Let $w_1 \in V$ such that $m_{T,w_1}(x) = m_T(x)$ and let W_1 be the T-invariant subspace of V that is T-generated by w_1. Suppose $W_1 \subset V$ is a proper subspace and there is a vector $v_2 \in V$ so that V is T-generated by $\{w_1, v_2\}$. Then there is a vector $w_2 \in V$ such that if W_2 is the subspace T-generated by w_2, then $V = W_1 \oplus W_2$.*

Proof. Let v_2 be as in the statement of the theorem. Let V_2 be the subspace of V that is T-generated by v_2. We certainly have $V = W_1 + V_2$, but in general we will not have $W_1 \cap V_2 = \{0\}$. We will obtain the direct sum by changing v_2 into a different element so that there is no overlap.

Let $\dim_F V = n$ and $\dim_F W_1 = k$. Then $\mathcal{B}_1 = \{T^{k-1}(w_1), \dots, T(w_1), w_1\}$ is a basis for W_1. Set $u_i = T^{k-i}(w_1)$ to ease the notation. We have V is spanned by $\mathcal{B}_1 \cup \{v_2', T(v_2'), \dots\}$ where $v_2' = v_2 + w$ for any $w \in W_1$. Note that for any such v_2' that $\{w_1, v_2'\}$ also T-generates V. The point now is to choose w carefully so that if we set $w_2 = v_2'$ we will have the result. Set $\mathcal{B}_2 = \{v_2', T(v_2'), \dots, T^{n-k-1}(v_2')\}$. The first claim is that $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for V. We know that $\mathcal{B}_1$ is linearly independent, so we can add elements to it to form a basis. We add $v_2', T(v_2')$, etc. until we obtain a set that is no longer linearly independent. Certainly we cannot go past $T^{n-k-1}(v_2')$ because then we will have more than n vectors. To see we can go all the way to $T^{n-k-1}(v_2')$, observe that if $T^j(v_2')$ is a linear combination of $\mathcal{B}_1$ and $\{v_2', \dots, T^{j-1}(v_2')\}$, then the latter set consisting of $k+j$ vectors spans V so $j \geq n-k$. (Note one must use W_1 is T-invariant to conclude this. Make sure you understand this point.) However, we know $j \leq n-k$ and so $j = n-k$ as desired. Thus, $\mathcal{B}' = \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for V. Set $u_{k+1}' = T^{n-k-1}(v_2'), \dots, u_n' = v_2'$.

We now consider $T^{n-k}(u_n')$. We can uniquely write

$$T^{n-k}(u_n') = \sum_{i=1}^{k} b_i u_i + \sum_{i=k+1}^{n} b_i u_i'$$

for some $b_i \in F$. Set

$$p(x) = x^{n-k} - b_{k+1}x^{n-k-1} - \cdots - b_{n-1}x - b_n.$$

Set $u = p(T)(v_2')$ and observe

$$\begin{aligned} u &= p(T)(v_2') \\ &= p(T)(u_n') \\ &= \sum_{i=1}^{k} b_i u_i \in W_1. \end{aligned}$$

We now break into two cases:

<u>Case 1:</u> Suppose $u = 0$.

In this case we have $\sum_{i=1}^{k} b_i u_i = 0$, and so $b_i = 0$ for $i = 1, \ldots, k$ since the u_i are linearly independent. Set $V_2' = \operatorname{span}_F \mathcal{B}_2$. We have

$$\begin{aligned} T^{n-k}(v_2') &= T^{n-k}(u_n') \\ &= \sum_{i=k+1}^{n} b_i u_i' \in \operatorname{span}_F \mathcal{B}_2. \end{aligned}$$

Thus, we have $T^j(v_2') \in \operatorname{span}_F \mathcal{B}_2$ for all j, so V_2 is a T-invariant subspace of V. By construction we have $W_1 \cap V_2' = \{0\}$, so we have that $V = W_1 \oplus V_2'$ is a T-invariant direct sum decomposition of V and we are done in this case.

Case 2: Suppose $u \neq 0$.

The goal here is to reduce this case to the previous case. We must adjust v_2' for this to work. We now set V_2' to be the space T-generated by v_2'. We claim that $b_{2k-n+1}, \ldots, b_k$ are all zero so $u = \sum_{i=1}^{2k-n} b_i u_i$. Suppose there exists b_m with $2k-n+1 \leq m \leq k$ so that $b_m \neq 0$ and let m be the largest such index. Since T acts by shifting the u_i, we have

$$\begin{aligned} T^{m-1}(u) &= b_m u_1 \\ T^{m-2}(u) &= b_m u_2 + b_{m-1} u_1 \\ &\vdots \end{aligned}$$

Thus, we have

$$\{T^{m-1}p(T)(v_2'), T^{m-2}p(T)(v_2'), \ldots, p(T)v_2', T^{n-k-1}(v_2'), T^{n-k-2}(v_2'), \ldots, v_2'\}$$

is a linearly independent subset of V_2'. Thus, $\dim_F V_2' \geq m+n-k \geq k+1$. This gives that the degree of $m_{T,v_2'}(x)$ is at least $k+1$. However, since $m_{T,v_2'}(x)$ must divide $m_T(x)$, and $m_T(x) = m_{T,w_1}(x)$ which has degree k, this is a contradiction. Thus, $b_{2k-n+1} = \cdots = b_k = 0$ as claimed.

Set

$$w = -\sum_{i=1}^{2k-n} b_i u_{i+n-k}.$$

Note that the u_i's range from u_{n-k+1} to u_k, so $w \in W_1$. Set $w_2 = v_2' + w$. Define $\mathcal{B}_1 = \{u_1, \ldots, u_k\}$ as above, but now set $\mathcal{B}_2 = \{u_{k+1}, \ldots, u_n\} =$

$\{T^{n-k-1}(w_2), \ldots, w_2\}$. Set $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. We have

$$\begin{aligned} T^{n-k}(u_n) &= T^{n-k}(v_2' + w) \\ &= T^{n-k}(v_2') + T^{n-k}(w) \\ &= \sum_{i=1}^{2k-n} b_i u_i + T^{n-k}\left(-\sum_{i=1}^{2k-n} b_i u_{i+n-k}\right) \\ &= \sum_{i=1}^{2k-n} b_i u_i + \sum_{i=1}^{2k-n} (-b_i u_i) \\ &= 0. \end{aligned}$$

Thus, we are back in the previous case so we have the result. □

This lemma now allows us to prove the following theorem that will be essential to developing the rational and Jordan canonical forms.

Theorem 4.2.7. *Let $T \in \mathrm{Hom}_F(V, V)$ and let $w_1 \in V$ such that $m_{T,w_1}(x) = m_T(x)$. Let W_1 be the subspace T-generated by w_1. Then W_1 has a T-invariant complement W_2.*

Proof. If $W_1 = V$, we can set $W_2 = 0$ and we are done. Suppose that W_1 is properly contained in V. Consider the collection of T-invariant subspaces of V that intersect W_1 trivially. We have this is a nonempty set because $\{0\}$ is a subspace that intersects W_1 trivially. Partially ordering this by inclusion, we apply Zorn's lemma to choose a maximal subspace W_2 that is T-invariant and intersects W_1 trivially. We now show that $V = W_1 \oplus W_2$.

Suppose that $W_1 \oplus W_2$ is properly contained in V. Let $v \in V$ so that $v \notin W_1 \oplus W_2$. Let V_2 be the subspace of V that is T-generated by v. Set $U_2 = W_2 + V_2$. If $W_1 \cap U_2 = \{0\}$ we have a contradiction to W_2 being maximal. Thus, we must have $W_1 \cap U_2 \neq \{0\}$. Set $V' = W_1 + U_2$. We have V' is a T-invariant subspace of V. Set $T' = T|_{V'}$. Since W_2 is a T-invariant space, it is a T'-invariant subspace of V'. Consider

$$\overline{T}' : V'/W_2 \to V'/W_2.$$

Set $X = V'/W_2$ and $S = \overline{T}'$. Let $\pi_{W_2} : V' \to X$ be the natural projection map and set $\overline{w}_1 = \pi_{W_2}(w_1)$ and $\overline{v}_2 = \pi_{W_2}(v_2)$. Set $Y_1 = \pi_{W_2}(W_1) \subset X$ and $Z_2 = \pi_{W_2}(U_2) \subset X$. Clearly we have Y_1 and Z_2 are S-invariant subspaces of X. The space Y_1 is S-spanned by $\overline{w}_1$ and Z_2 is S-spanned by $\overline{v}_2$, so X is S-spanned by $\{\overline{w}_1, \overline{v}_2\}$. Finally, since $W_1 \cap W_2 = \{0\}$ we have $\pi_{W_2}|_{W_1}$ is injective.

We have that $m_{T'}(x) | m_T(x)$. We also have $m_S(x) \mid m_{T'}(x)$. We assumed $m_{T,w_1}(x) = m_T(x)$ and since $\pi_{W_2} : W_1 \to Y_1$ is injective, we must have $m_{S,\overline{w}_1}(x) = m_{T,w_1}(x)$. Since $w_1 \in V'$, $m_{T,w_1}(x) \mid m_{T'}(x)$. Finally, $m_{S,\overline{w}_1}(x) \mid m_S(x)$. Combining all of these gives

$$m_{S,\overline{w}_1}(x) \mid m_S(x) \mid m_{T'}(x) \mid m_T(x) = m_{T,w_1}(x) = m_{S,\overline{w}_1}(x),$$

which gives equality throughout.

We are now in a position to apply the previous lemma to $S, X, \overline{w}_1$, and $\overline{w}_2$. This gives a vector $\overline{w}_2$ so that $X = Y_1 \oplus Y_2$ where Y_2 is the subspace of X that is S-generated by $\overline{w}_2$. Let w_2' be any element of V' with $\pi_{W_2}(w_2') = \overline{w}_2$ and set V_2' to be the subspace of V' that is T'-generated by w_2' (equivalently, the subspace of V that is T-generated by w_2'.) This gives $\pi_{W_2}(V_2') = Y_2$.

We are finally in a position to finish the proof. Observe we have

$$V'/W_2 = X = Y_1 + Z_2 = Y_1 \oplus Y_2.$$

Set $U_2' = W_2 + V_2'$. Then

$$\begin{aligned} V &= W_1 + V_2' + W_2 \\ &= W_1 + (W_2 + V_2') \\ &= W_1 + U_2'. \end{aligned}$$

We have $W_1 \cap U_2' = \{0\}$. To see this, observe that if $x \in W_1 \cap U_2'$, then $\pi_{W_2}(x) \in \pi_{W_2}(W_1) \cap \pi_{W_2}(U_2') = Y_1 \cap Y_2 = \{0\}$. However, if $x \in W_1 \cap V_2'$, then $x \in W_1$ and $\pi_{W_2}|_{W_1}$ is injective so $x = 0$. Thus, we have $V' = W_1 \oplus U_2'$ and U_2' properly contains W_2. This contradicts the maximality of W_2. □

4.3 Rational canonical form

In this section we will give the rational canonical form for a linear transformation. The idea is that we show a "nice" basis exists so that the matrix with respect to the linear transformation is particularly simple. One important feature of the rational canonical form is that this result works over any field. This is in contrast to the Jordan canonical form, which will be presented in the next section.

Definition 4.3.1. Let $T \in \mathrm{Hom}_F(V, V)$. An ordered set $\mathcal{C} = \{w_1, \ldots, w_k\}$ is a *rational canonical T-generating set of V* if it satisfies

(a) $V = W_1 \oplus \cdots \oplus W_k$ where W_i is the subspace T-generated by w_i;

(b) for all $i = 1, \ldots, k-1$ we have $m_{T,w_{i+1}}(x) \mid m_{T,w_i}(x)$.

One should note that some textbooks will reverse the order of divisibility of the annihilating polynomials.

The first goal of this section is to show such a set exists. Showing such a set exists is straightforward given Theorem 4.2.7; the work now lies in showing uniqueness.

Theorem 4.3.2. *Let $T \in \mathrm{Hom}_F(V,V)$. Then V has a rational canonical T-generating set $\mathcal{C} = \{w_1, \dots, w_k\}$. Moreover, if $\mathcal{C}' = \{w_1', \dots, w_l'\}$ is any other rational canonical T-generating set, then $k = l$ and $m_{T,w_i}(x) = m_{T,w_i'}(x)$ for $i = 1, \dots, k$.*

Proof. Let $\dim_F V = n$. We induct on n to prove existence. Let $w_1 \in V$ such that $m_T(x) = m_{T,w_1}(x)$ and let W_1 be the subspace T-generated by w_1. If $W_1 = V$ we are done, so assume not. Let W' be the T-invariant complement of W_1, which we know exists by Theorem 4.2.7. Consider $T' = T \mid_{W'}$. Clearly we have $m_{T'}(x) \mid m_T(x)$. Moreover, $\dim_F W' < n$. By induction W' has a rational canonical T-generating set $\{w_2, \dots, w_k\}$. The rational canonical T-generating set of V is just $\{w_1, \dots, w_k\}$. This gives existence. It remains to prove uniqueness.

Let $\mathcal{C} = \{w_1, \dots, w_k\}$ and $\mathcal{C}' = \{w_1', \dots, w_l'\}$ be rational canonical T-generating sets with corresponding decompositions

$$V = W_1 \oplus \cdots \oplus W_k$$

and

$$V = W_1' \oplus \cdots \oplus W_l'.$$

Let $p_i(x) = m_{T,w_i}(x)$, $p_i'(x) = m_{T,w_i'}(x)$, $d_i = \deg p_i(x)$, and $d_i' = \deg p_i'(x)$. We proceed by induction on k. If $k = 1$, then $V = W_1 = W_1' \oplus \cdots \oplus W_l'$. We have $p_1(x) = m_{T,w_1}(x) = m_T(x) = m_{T,w_1'}(x) = p_1'(x)$. This gives $d_1 = n$. However, we also have $\dim_F W_1 = n = \deg m_{T,w_1}(x) = \deg m_T(x) = \deg p_1'(x)$. Thus, $\dim_F W_1' = n$ and so $l = 1$ and we are done in this case.

Now suppose for some $m \geq 1$ we have $p_i'(x) = p_i(x)$ for all $1 \leq i \leq m$. If $V = W_1 \oplus \cdots \oplus W_m$, then $n = d_1 + \cdots + d_m = d_1' + \cdots + d_m'$. Thus $k = m = m' = l$ and we are done. The same argument works if $V = W_1' \oplus \cdots \oplus W_m'$.

Suppose now that $V \neq W_1 \oplus \cdots \oplus W_m$, i.e., $k > m$. Consider $p_{m+1}(T)(V)$. This is T-invariant. By assumption we have

$$V = W_1 \oplus \cdots \oplus W_m \oplus W_{m+1} \oplus \cdots .$$

This gives

$$p_{m+1}(T)(V) = p_{m+1}(W_1) \oplus \cdots \oplus p_{m+1}(T)(W_m) \oplus p_{m+1}(T)(W_{m+1}) \cdots .$$

Since $p_{m+1}(T)(x) = m_{T,w_{m+1}}(x)$, we have $p_{m+1}(T)(w_{m+1}) = 0$. We now use that W_{m+1} is generated by w_{m+1} to conclude $p_{m+1}(T)(W_{m+1}) = 0$ as well. We also have that $p_{m+j}(x) \mid p_{m+1}(x)$ for all $j \geq 1$. This gives $p_{m+1}(T)(W_{m+j}) = 0$ for all $j \geq 1$. Thus,

$$p_{m+1}(T)(V) = p_{m+1}(T)(W_1) \oplus \cdots \oplus p_{m+1}(T)(W_m).$$

Since $p_{m+1}(x) \mid p_i(x)$ for $1 \leq i \leq m$, we get $\dim_F p_{m+1}(T)(W_i) = d_i - d_{m+1}$. (See the homework problems.) Thus,

$$\dim_F p_{m+1}(T)(V) = d = (d_1 - d_{m+1}) + \cdots + (d_m - d_{m+1}).$$

We do the same thing to $V = W_1' \oplus \cdots \oplus W_l'$. We now apply the same argument to $V = W_1' \oplus \cdots \oplus W_l' \oplus \cdots$ to obtain

$$p_{m+1}(T)(V) = \bigoplus_{j \geq 1} p_{m+1}(T)(W_j').$$

This has a subspace of dimension d given by $\bigoplus_{j=1}^{m} p_{m+1}(T)(W_j')$ by our induction hypothesis. Thus, this must be the entire space since it has dimension equal to the dimension of $p_{m+1}(T)(V)$. Thus, $p_{m+1}(T)(W_j') = 0$ for $j \geq m+1$. The annihilator of W_{m+1}' is $p_{m+1}'(x)$, so we must have $p_{m+1}(x) \mid p_{m+1}'(x)$. We now run the same argument with $p_{m+1}'(x)$ instead of $p_{m+1}(x)$ to obtain $p_{m+1}'(x) \mid p_{m+1}(x)$. Thus, $p_{m+1}(x) = p_{m+1}'(x)$ and we are done by induction. □

We now rephrase this in terms of matrices.

Definition 4.3.3. Let $M \in \mathrm{Mat}_n(F)$. We say M is in *rational canonical form* if M is a block diagonal matrix

$$M = \begin{pmatrix} C(p_1(x)) & & & \\ & C(p_2(x)) & & \\ & & \ddots & \\ & & & C(p_k(x)) \end{pmatrix}$$

where $C(p_i(x))$ denotes the companion matrix of $p_i(x)$ where $p_1(x), \dots, p_k(x)$ is a sequence of polynomials satisfying $p_{i+1}(x) \mid p_i(x)$ for $i = 1, \dots, k-1$.

Corollary 4.3.4. *(a) Let $T \in \mathrm{Hom}_F(V, V)$. Then V has a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}} = M$ is in rational canonical form. Moreover, M is unique.*

(b) Let $A \in \mathrm{Mat}_n(F)$. Then A is similar to a unique matrix in rational canonical form.

Proof. Let $\mathcal{C} = \{w_1, \dots, w_k\}$ be a rational canonical T-generating set for V with $p_i(x) = m_{T,w_i}(x)$. Set $d_i = \deg p_i(x)$. Then

$$\mathcal{B} = \{T^{d_1-1}(w_1), \dots, T(w_1), w_1, \dots, T^{d_k-1}(w_k), \dots, w_k\}$$

is the desired basis. To prove the second statement just apply the first part to $T = T_A$. □

Definition 4.3.5. Let T have rational canonical form with diagonal blocks $C(p_1(x)), \dots, C(p_k(x))$ with $p_i(x)$ divisible by $p_{i+1}(x)$. We call the polynomials $p_i(x)$ the *invariant factors of T*.

One should note that in some places, such as [4],this are referred to as *the elementary divisors.* Please see Section 4.6 for clarification on this.

Remark 4.3.6. (a) The rational canonical form is a special case of the fundamental structure theorem for modules over a principal ideal domain. This is where the terminology comes from.

(b) Some sources will use invariant factors so that $p_1(x) \mid p_2(x) \mid \cdots \mid p_k(x)$. This is clearly equivalent to our presentation upon a change of basis.

Corollary 4.3.7. *Let $A \in \mathrm{Mat}_n(F)$.*

(a) The matrix A is determined up to similarity by its sequence of invariant factors $p_1(x), \dots, p_k(x)$.

(b) The sequence of invariant factors is determined recursively as follows.

i. Set $p_1(x) = m_T(x)$.

ii. Let $w_1 \in V$ so that $m_T(x) = m_{T,w_1}(x)$ and let W_1 be the subspace T-generated by w_1.

iii. Let $\overline{T} : V/W_1 \to V/W_1$ and set $p_2(x) = m_{\overline{T}}(x)$.

iv. Now repeat this process where the starting space for the next step is V/W_1.

Proof. The proof is left as an exercise. □

Corollary 4.3.8. *Let $T \in \mathrm{Hom}_F(V,V)$ have invariant factors $p_1, \dots, p_k$. Then we have*

(a) $p_1(x) = m_T(x)$;

(b) $c_T(x) = p_1(x) \cdots p_k(x)$.

Proof. We have already seen the first statement. The second statement follows immediately from the definition of $c_T(x)$ and the fact that $\det(C(p(x))) = p(x)$. □

The important thing about rational canonical form, as opposed to Jordan canonical form, is that it is defined over any field. It does not depend on the field which one considers the vector space defined over, as the following result shows.

Corollary 4.3.9. *Let $F \subseteq K$ fields. Let $A, B \in \mathrm{Mat}_n(F) \subseteq \mathrm{Mat}_n(K)$.*

(a) The rational canonical form of A is the same whether computed over F or K. The minimal polynomial, characteristic polynomial, and invariant factors are the same whether considered over F or K.

(b) The matrices A and B are similar over K if and only if they are similar over F. In particular, there exists some $P \in \mathrm{GL}_n(K)$ such that $B = PAP^{-1}$ if and only if there exists $Q \in \mathrm{GL}_n(F)$ such that $B = QAQ^{-1}$.

Proof. Write M_F for the rational canonical form of A computed over F. Note this also satisfies the condition for being a rational canonical form over K as well so uniqueness of the rational canonical form gives that M_F is the rational canonical form for A over K as well. Thus, the invariant factors must agree. However, one now just uses Corollary 4.3.8 to obtain the statement about the minimal and characteristic polynomials.

Suppose that A and B are similar over F, i.e., there exists $Q \in \mathrm{GL}_n(F)$ such that $B = QAQ^{-1}$. Since $Q \in \mathrm{GL}_n(K)$ as well, this gives A and B are similar over K as well. If A and B are similar over K then A and B have the same rational canonical form over K. The first part of the corollary now tells us they have the same rational canonical form over K and F, so they are similar over F as well. □

It is important to note for the above result to be applied both matrices must actually be defined over F! This does not say if one has two matrices defined over K that they are similar over any subfield of K since they may not even be defined over that subfield.

It is particularly easy to compute the rational canonical form for a matrix in $\mathrm{Mat}_2(F)$ or $\mathrm{Mat}_3(F)$ since the invariant factors are completely determined by the characteristic and minimal polynomials in this case. We illustrate this in the following example. However, we then work the same example with a general method that works for any size matrix. We do not prove this method works as it relies on working with modules over principal ideal domains.

Example 4.3.10. Set

$$A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix} \in \mathrm{Mat}_3(\mathbb{Q}).$$

We have

$$\begin{aligned} c_A(x) &= \det \begin{pmatrix} x-2 & 2 & -14 \\ 0 & x-3 & 7 \\ 0 & 0 & x-2 \end{pmatrix} \\ &= (x-2)^2(x-3). \end{aligned}$$

Since we know $m_A(x)$ must have the same irreducible factors as $c_A(x)$, the only possibilities for $m_A(x)$ are $(x-2)(x-3)$ and $(x-2)^2(x-3)$. One easily checks that $(A - 2 \cdot 1_3)(A - 3 \cdot 1_3) = 0$ so $m_A(x) = (x-2)(x-3)$.

Thus we have $p_1(x) = (x-2)(x-3) = x^2-5x+6$. We now use that $c_A(x)$ is the product of the invariant factors to conclude that $p_2(x) = (x-2)$. Thus, we have $C(p_1(x)) = \begin{pmatrix} 5 & 1 \\ -6 & 0 \end{pmatrix}$ and $C(p_2(x)) = 2$. Thus, the rational canonical form for A is $\begin{pmatrix} 5 & 1 & 0 \\ -6 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Example 4.3.11. Note the minimal and characteristic polynomials do not in general provide enough information to determine the invariant factors for matrices in $\mathrm{Mat}_n(F)$ for $n \geq 4$. For example, if $A \in \mathrm{Mat}_4(\mathbb{Q})$ with $c_A(x) = (x-1)^4$, $m_A(x) = (x-1)^2$, then the invariant factors could be $p_1(x) = (x-1)^2, p_2(x) = (x-1)^2$ or $p_1(x) = (x-1)^2, p_2(x) = (x-1)$, $p_3(x) = x-1$. Without more information about A we cannot determine which is the correct list of invariant factors.

Example 4.3.12. We now compute the rational canonical form of $A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix} \in \mathrm{Mat}_3(\mathbb{Q})$ in a way that generalizes to matrices in $\mathrm{Mat}_n(F)$. Consider the matrix

$$x \cdot 1_3 - A = \begin{pmatrix} x-2 & 2 & -14 \\ 0 & x-3 & 7 \\ 0 & 0 & x-2 \end{pmatrix}.$$

We now apply elementary row and column operations on this matrix to diagonalize it. The fact that this matrix is always diagonalizable is a fact from abstract algebra having to do with the fact that $F[x]$ is a principal ideal domain. We use standard notation for elementary row and column operations. For example, $R_1 + R_2 \to R_1$ means we replace row 1 by row

1 + row 2.

$$\begin{aligned}
x \cdot 1_n - A &\longrightarrow \begin{pmatrix} x-2 & x-1 & -7 \\ 0 & x-3 & 7 \\ 0 & 0 & x-2 \end{pmatrix} \quad (\text{via } R_1 + R_2 \to R_1) \\
&\longrightarrow \begin{pmatrix} 1 & x-1 & -7 \\ x-3 & x-3 & 7 \\ 0 & 0 & x-2 \end{pmatrix} \quad (\text{via } -C_1 + C_2 \to C_1) \\
&\longrightarrow \begin{pmatrix} 1 & x-1 & -7 \\ 0 & -x^2+5x-6 & 7x-14 \\ 0 & 0 & x-2 \end{pmatrix} \quad (\text{via } -(x-3)R_1 + R_2 \to R_2) \\
&\longrightarrow \begin{pmatrix} 1 & 0 & -7 \\ 0 & -x^2+5x-6 & 7x-14 \\ 0 & 0 & x-2 \end{pmatrix} \quad (\text{via } -(x-1)C_1 + C_2 \to C_2) \\
&\longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -x^2+5x-6 & 7x-14 \\ 0 & 0 & x-2 \end{pmatrix} \quad (\text{via } 7C_1 + C_3 \to C_3) \\
&\longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -x^2+5x-6 & 0 \\ 0 & 0 & x-2 \end{pmatrix} \quad (\text{via } R_2 - 7R_3 \to R_2).
\end{aligned}$$

Once the matrix has been diagonalized, the polynomials of degree 1 or greater are the invariant factors. In this case we have $p_1(x) = x^2 - 5x + 6$ and $p_2(x) = x - 2$, as was found in the previous example. If one keeps track of the elementary row and column operations one can compute P that converts A to its rational canonical form as well.

Example 4.3.13. In this example we compute a representative of each similarity class of matrices $A \in \mathrm{Mat}_3(\mathbb{Q})$ that satisfy $A^4 = 1_3$. Note that if $A^4 = 1_3$, then $m_A(x) \mid x^4 - 1$. We can factor $x^4 - 1$ into irreducibles over $\mathbb{Q}$ to obtain $x^4 - 1 = (x-1)(x+1)(x^2+1)$. Thus, $m_A(x)$ is a polynomial of degree at most 3 that divides $(x-1)(x+1)(x^2+1)$. Conversely, if $B \in \mathrm{Mat}_3(\mathbb{Q})$ satisfies that $m_B(x) \mid x^4 - 1$, then $B^4 = 1_3$. We now list all possible minimal polynomials:

(a) $x - 1$;
(b) $x + 1$;
(c) $x^2 + 1$;
(d) $(x-1)(x+1)$;
(e) $(x-1)(x^2+1)$;
(f) $(x+1)(x^2+1)$.

Note we cannot have $m_A(x) = x^4 - 1$ because $\deg m_A(x) \leq 3$. We can now list the possible invariant factors, keeping in mind the product must have degree 3 and they must divide each other. The possible invariant factors are

(a) $x-1, x-1, x-1$;

(b) $x+1, x+1, x+1$;

(c) $(x-1)(x+1), x-1$;

(d) $(x-1)(x+1), x+1$;

(e) $(x-1)(x^2+1)$;

(f) $(x+1)(x^2+1)$.

Note the first factor cannot be x^2+1 because then we cannot obtain $p_2(x)$ so that $p_2(x) \mid x^2+1$ and $p_1(x)p_2(x)$ has degree 3. Thus, the elements of $\mathrm{GL}_3(\mathbb{Q})$ of order dividing 4, up to similarity, are given by

(a) 1_3;

(b) -1_3;

(c) $\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}$;

(d) $\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & -1 \end{pmatrix}$;

(e) $\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$;

(f) $\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

Note in the above matrices we omit the 0's in some spots to help emphasize the blocks giving the rational canonical form.

In the previous example the possibilities were limited by how $m_A(x)$ factored over our field. For the next example we consider the same set-up, but working over a different field.

Example 4.3.14. Consider $A \in \mathrm{Mat}_3(\mathbb{Q}(i))$ satisfying $A^4 = 1_3$. In this example we classify all such matrices up to similarity. As above, we have for such an A that $m_A(x) \mid x^4-1$, but unlike in the previous example we have $x^4-1 = (x-1)(x+1)(x-i)(x+i)$ when we factor it into irreducibles over $\mathbb{Q}(i)$. This vastly increases the possibilities for the minimal polynomials and hence invariant factors. The possible minimal polynomials are given by

(a) $x-1$;

(b) $x+1$;

(c) $x-i$;

(d) $x + i$;

(e) $x^2 - 1$;

(f) $x^2 + 1$;

(g) $(x - 1)(x - i)$;

(h) $(x - 1)(x + i)$;

(i) $(x + 1)(x - i)$;

(j) $(x + 1)(x + i)$;

(k) $(x - 1)(x^2 + 1)$;

(l) $(x + 1)(x^2 + 1)$;

(m) $(x - i)(x^2 - 1)$;

(n) $(x + i)(x^2 - 1)$.

From this, we see the possible invariant factors are given by

(a) $x - 1, x - 1, x - 1$;

(b) $x + 1, x + 1, x + 1$;

(c) $x - i, x - i, x - i$;

(d) $x + i, x + i, x + i$;

(e) $x^2 - 1, x - 1$;

(f) $x^2 - 1, x + 1$;

(g) $x^2 + 1, x - i$;

(h) $x^2 + 1, x + i$;

(i) $(x - 1)(x - i), x - 1$;

(j) $(x - 1)(x - i), x - i$;

(k) $(x - 1)(x + i), x - 1$;

(l) $(x - 1)(x + i), x + i$;

(m) $(x + 1)(x - i), x + 1$;

(n) $(x + 1)(x - i), x - i$;

(o) $(x + 1)(x + i), x + 1$;

(p) $(x + 1)(x + i), x + i$;

(q) $x^3 - x^2 + x - 1$;

(r) $x^3 + x^2 + x + 1$;

(s) $x^3 - ix^2 - x + i$;

(t) $x^3 + ix^2 - x - i$.

This gives the following possible rational canonical forms:

(a) 1_3;

(b) -1_3;

(c) $i \cdot 1_3$;

(d) $-i \cdot 1_3$;

(e) $\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}$;

(f) $\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & -1 \end{pmatrix}$;

(g) $\begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & -i \end{pmatrix}$;

(h) $\begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & i \end{pmatrix}$;

(i) $\begin{pmatrix} 1+i & 1 & \\ -i & 0 & \\ & & 1 \end{pmatrix}$;

(j) $\begin{pmatrix} 1+i & 1 & \\ -i & 0 & \\ & & i \end{pmatrix}$;

(k) $\begin{pmatrix} 1-i & 1 & \\ i & 0 & \\ & & 1 \end{pmatrix}$;

(l) $\begin{pmatrix} 1-i & 1 & \\ i & 0 & \\ & & -i \end{pmatrix}$;

(m) $\begin{pmatrix} i-1 & 1 & \\ i & 0 & \\ & & -1 \end{pmatrix}$;

(n) $\begin{pmatrix} i-1 & 1 & \\ i & 0 & \\ & & i \end{pmatrix}$;

(o) $\begin{pmatrix} -i-1 & 1 & \\ -i & 0 & \\ & & -1 \end{pmatrix}$;

(p) $\begin{pmatrix} -i-1 & 1 & \\ -i & 0 & \\ & & -i \end{pmatrix}$;

(q) $\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$;

(r) $\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$;

(s) $\begin{pmatrix} i & 1 & 0 \\ 1 & 0 & 1 \\ -i & 0 & 0 \end{pmatrix}$;

(t) $\begin{pmatrix} -i & 1 & 0 \\ 1 & 0 & 1 \\ i & 0 & 0 \end{pmatrix}$.

4.4 Jordan canonical form

In this section we use the results on the rational canonical form to deduce the Jordan canonical form. One nice aspect of the Jordan canonical form is that it is given in terms of the eigenvalues of the matrix. One serious drawback is it is only defined over fields containing the splitting field of the minimal polynomial, i.e., the field must contain all the roots of the minimal polynomial. Most treatments of the Jordan canonical form work over an algebraically closed field, and one can certainly do this to be safe. However, for a particular matrix it is not necessary to move all the way to an algebraically closed field so we do not do so.

Definition 4.4.1. Let $T \in \mathrm{Hom}_F(V, V)$ and $\lambda \in F$. If $\ker(T - \lambda\,\mathrm{id}) \neq 0$, then we say λ is an *eigenvalue of* T. Any nonzero vector in this kernel is called an *eigenvector of* T or a λ-*eigenvector of* T if we need to emphasize the eigenvalue. The space $E_\lambda^1 = \ker(T - \lambda\,\mathrm{id})$ is called the *eigenspace associated to* λ. More generally, for $k \geq 1$ the k^{th} *eigenspace of* T is given by

$$E_\lambda^k = \{v \in V : (T - \lambda\,\mathrm{id})^k v = 0\} = \ker((T - \lambda\,\mathrm{id})^k).$$

We refer to a nonzero element in E_λ^k as a *generalized* λ-*eigenvector* of T. Write $E_\lambda^\infty = \cup_{k=1}^\infty E_\lambda^k$.

One should note that $E_\lambda^1 \subset E_\lambda^2 \subset \cdots \subset E_\lambda^k \subset \cdots$.

Example 4.4.2. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V and consider $T \in \mathrm{Hom}_F(V, V)$ with matrix given by

$$A = [T]_{\mathcal{B}} = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \\ 0 & \cdots & 0 & \lambda & 1 & 0 \\ 0 & \cdots & 0 & 0 & \lambda & 1 \\ 0 & \cdots & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

We have that each v_k is a generalized eigenvector. Note that

$$A - \lambda \cdot 1_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\begin{aligned} (T - \lambda \cdot \mathrm{id})(v_1) &= 0 \\ (T - \lambda \cdot \mathrm{id})(v_2) &= v_1 \\ &\vdots \\ (T - \lambda \cdot \mathrm{id})(v_n) &= v_{n-1}. \end{aligned}$$

This clearly gives $v_2, \dots, v_n \notin E_\lambda^1$. Moreover, observe that we have $\{v_1, \dots, v_{n-1}\} \subset \mathrm{Im}(T - \lambda\,\mathrm{id})$, so $\dim_F \mathrm{Im}(T - \lambda\,\mathrm{id}) \geq n - 1$. However, since $\dim_F V = n$ and $\dim_F \ker(T - \lambda\,\mathrm{id}) \geq 1$, this gives $\dim_F \ker(T - \lambda\,\mathrm{id}) = 1$ and $\dim_F \mathrm{Im}(T - \lambda\,\mathrm{id}) = n - 1$. This allows us to conclude that $E_\lambda^1 = \mathrm{span}_F\{v_1\}$. Next we consider E_λ^2. It is immediate that $\{v_1, v_2\} \subset E_\lambda^2$. Since $(A - \lambda \cdot 1_n)[v_k]_{\mathcal{B}} = [v_{k-1}]_{\mathcal{B}}$, we have $v_k \notin E_\lambda^2$ for $k > 2$. Thus, as above we have $E_\lambda^2 = \mathrm{span}_F\{v_1, v_2\}$. More generally, the same argument gives $E_\lambda^k = \mathrm{span}_F\{v_1, \dots, v_k\}$ for $k = 1, \dots, n$ and $E_\lambda^\infty = E_\lambda^n = V$.

Exercise 4.4.3. Describe the generalized eigenspaces of the map given by

$$A = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$$

where λ_1, λ_2, and λ_3 are distinct elements of F.

Exercise 4.4.4. Let $T \in \mathrm{Hom}_F(V, V)$. Then $c_T(x)$ has a root in F if and only if T has an eigenvalue in F.

Note the previous exercise shows that if $(x - \lambda) \mid c_T(x)$, then there is necessarily a nonzero vector in V of eigenvalue λ. Since $c_T(x)$ and $m_T(x)$ have the same irreducible factors, this is equivalent to saying λ is an eigenvalue of T if and only if $(x - \lambda) \mid m_T(x)$. In fact, we can do better in terms of describing the dimensions of the eigenspaces in terms of the minimal polynomial and characteristic polynomial.

Lemma 4.4.5. *Let $T \in \mathrm{Hom}_F(V, V)$ and suppose that $m_T(x) = (x - \lambda)^e p(x)$ with $p(\lambda) \neq 0$. Then $E_\lambda^\infty = E_\lambda^e$.*

Proof. Let $v \in V$ and let m be the least positive integer so that $(T - \lambda\,\mathrm{id})^m(v) = 0$. Suppose that $m > e$. Then we have $m_{T,v}(x) \mid (x-\lambda)^m$, but $m_{T,v}(x) \nmid (x-\lambda)^{m-1}$ and so $m_{T,v}(x) = (x-\lambda)^m$. However, we know $m_{T,v}(x) \mid m_T(x)$, which is a contradiction if $m > e$. Thus, $E_\lambda^\infty = E_\lambda^e$. □

Lemma 4.4.6. *Let $T \in \mathrm{Hom}_F(V,V)$ and suppose $m_T(x) = c_T(x) = (x-\lambda)^n$ for some $\lambda \in F$. Then V is T-generated by a single element w_1 and V has a basis $\{v_1, \dots, v_n\}$ where $v_n = w_1$, and $v_i = (T - \lambda\,\mathrm{id})(v_{i+1})$ for $i = 1, \dots, n-1$.*

Proof. Let $w_1 \in V$ be such that $m_{T,w_1}(x) = m_T(x) = c_T(x)$. Let W_1 be the subspace T-generated by w_1. Then $\dim_F W_1 = \deg m_{T,w_1}(x) = \deg c_T(x) = \dim_F V$ and so $W_1 = V$. Set $v_n = w_1$ and define $v_i = (T - \lambda\,\mathrm{id})^{n-i}(v_n)$. Observe we have

$$\begin{aligned} v_i &= (T - \lambda\,\mathrm{id})^{n-i}(v_n) \\ &= (T - \lambda\,\mathrm{id})(T - \lambda I)^{n-i-1}(v_n) \\ &= (T - \lambda\,\mathrm{id})(v_{i+1}). \end{aligned}$$

Thus, we only need to show that $\mathcal{B} = \{v_1, \dots, v_n\}$ is a basis for V. This has $\dim_F V$ elements, so it is enough to check $\mathcal{B}$ is linearly independent. Suppose there are scalars $c_1, \dots, c_n \in F$ such that

$$c_1 v_1 + \cdots + c_n v_n = 0.$$

This gives

$$c_1 (T - \lambda\,\mathrm{id})^{n-1}(v_n) + \cdots + c_{n-1}(T - \lambda\,\mathrm{id})v_n + c_n v_n = 0.$$

If we set $p(x) = c_1(x-\lambda)^{n-1} + \cdots + c_{n-1}(x-\lambda) + c_n$, then $p(T)(v_n) = 0$, i.e., $p(T)(w_1) = 0$. This gives $m_{T,w_1}(x) \mid p(x)$. However, $m_{T,w_1}(x) = c_T(x)$ has degree n and $\deg p(x) = n-1$, so it must be that $p(x) = 0$, i.e., $c_j = 0$ for all j. □

Corollary 4.4.7. *Let T and $\mathcal{B}$ be as in the previous lemma. Then*

$$[T]_{\mathcal{B}} = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \\ 0 & \cdots & 0 & \lambda & 1 & 0 \\ 0 & \cdots & 0 & 0 & \lambda & 1 \\ 0 & \cdots & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Proof. We have $(T - \lambda\,\mathrm{id})v_1 = 0$, so $Tv_1 = \lambda v_1$. For $i > 1$ we have $(T - \lambda\,\mathrm{id})v_{i+1} = v_i$, so $T(v_{i+1}) = v_i + \lambda v_{i+1}$. This gives the correct form for the matrix. □

Definition 4.4.8. A basis $\mathcal{B}$ as in Lemma 4.4.6 is called a *Jordan basis for* V. Moreover, generally if $V = V_1 \oplus \cdots \oplus V_k$ is a T-invariant decomposition and each V_i has a Jordan basis $\mathcal{B}_i$, then we call $\mathcal{B} = \cup \mathcal{B}_i$ a *Jordan basis for* V.

Definition 4.4.9. (a) A matrix $A \in \mathrm{Mat}_k(F)$ of the form

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \\ 0 & \cdots & 0 & \lambda & 1 & 0 \\ 0 & \cdots & 0 & 0 & \lambda & 1 \\ 0 & \cdots & 0 & 0 & 0 & \lambda \end{pmatrix}$$

is called a $k \times k$ *Jordan block* associated to the eigenvalue λ.

(b) A matrix J is said to be in *Jordan canonical form* if it is a block diagonal matrix where each J_i is a Jordan Block.

Theorem 4.4.10. *(a) Let $T \in \mathrm{Hom}_F(V,V)$. Suppose that*

$$c_T(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_k)^{e_k}$$

over F. Then V has a basis $\mathcal{B}$ such that $J = [T]_{\mathcal{B}}$ is in Jordan canonical form. Moreover, J is unique up to the order of the blocks.

(b) Let $A \in \mathrm{Mat}_n(F)$. Suppose

$$c_A(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_k)^{e_k}$$

over F. Then A is similar to a matrix J in Jordan canonical form. Moreover, J is unique up to the order of the blocks

Proof. Set $p_i(x) = (x - \lambda_i)^{e_i}$. Set $V^i = E_{\lambda_i}^{e_i} = \ker(p_i(T))$. Note this V^i is the λ_i-eigenspace of V by Lemma 4.4.5. We can apply Theorem 4.2.4 to conclude that

$$V = E_{\lambda_1}^{e_1} \oplus \cdots \oplus E_{\lambda_k}^{e_k}.$$

Set $T_i = T|_{V^i}$. Then one can show that $c_{T_i}(x) = (x - \lambda_i)^{e_i}$. Choose a rational canonical T_i-generating set $\mathcal{C} = \{w_1^i, \dots, w_{m_i}^i\}$ with corresponding direct sum decomposition

$$V^i = W_1^i \oplus \cdots \oplus W_{m_i}^i$$

where W_j^i is the T_i-generated subspace by w_j^i. Note that each W_j^i satisfies the hypotheses of Lemma 4.4.6 so we have a Jordan basis $\mathcal{B}_j^i$ for W_j^i with respect to T_i. The desired basis is

$$\mathcal{B} = \bigcup_{j=1}^{k} \bigcup_{i=1}^{j} B_{m_i}^{j}.$$

The uniqueness follows from uniqueness of the rational canonical form and our construction.

The second claim follows easily from the first. □

One should note that if F is algebraically closed the characteristic polynomial always splits completely into linear factors. In particular, if F contains all the roots of $c_T(x)$ then T can be put in Jordan canonical form over F. If $c_T(x)$ does not split over F we cannot put T in Jordan canonical form over F. This makes the rational canonical form more useful in such situations.

We now give an algorithm for computing the Jordan canonical form. Our first worked example is particularly simple as the matrix is already in Jordan canonical form. However, it is nice to see how the algorithm works on a very simple example before giving a less trivial one.

Example 4.4.11. Consider the matrix

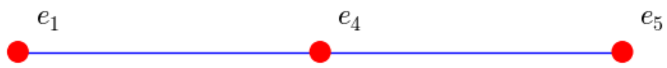
$$A = \begin{pmatrix} 6 & 1 & 0 & & & & & \\ 0 & 6 & 1 & & & & & \\ 0 & 0 & 6 & & & & & \\ & & & 6 & & & & \\ & & & & 6 & & & \\ & & & & & 7 & 1 & \\ & & & & & 0 & 7 & \\ & & & & & & & 7 \end{pmatrix}.$$

One easily calculate that $c_A(x) = (x-6)^5(x-7)^3$. This immediately gives the only nontrivial generalized eigenspaces are associated to 6 and 7. Moreover, from the powers on the linear terms we know $E_6^\infty = E_6^j$ for some $j = 1, \dots, 5$ and $E_7^\infty = E_7^i$ for some $i = 1, 2, 3$.

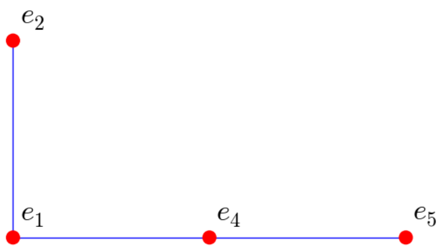
Consider $\lambda = 6$. We compute the dimension of $\ker(A - 6 \cdot 1_8)$. Note that

$$A - 6 \cdot 1_8 = \begin{pmatrix} 0 & 1 & 0 & & & & & \\ 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 1 & 1 & \\ & & & & & 0 & 1 & \\ & & & & & & & 1 \end{pmatrix}.$$

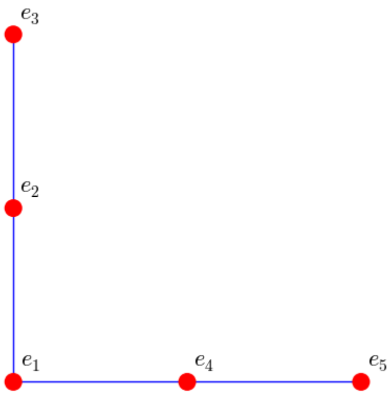
It is now immediate that $\dim_F E_6^1 = \dim_F(A - 6 \cdot 1_8) = 3$ and has basis $\{e_1, e_4, e_5\}$. The next step is to compute $E_6^2 = \ker(A - 6 \cdot 1_8)^2$. This has dimension 4 with basis $\{e_1, e_2, e_4, e_5\}$. Finally, $E_6^3 = \ker(A - 6 \cdot 1_8)^3$ has dimension 5 and is spanned by $\{e_1, e_2, e_3, e_4, e_5\}$. We represent this graphically as follows. We begin with a horizontal line with nodes the basis elements of E_6^1.



We then add a second row of nodes by adding basis elements that are in $E_6^2 - E_6^1$. In this case it means we only add e_2. Note we can put it over any element of the first row since we are only looking for the size of the blocks here, so for convenience we put it over e_1.

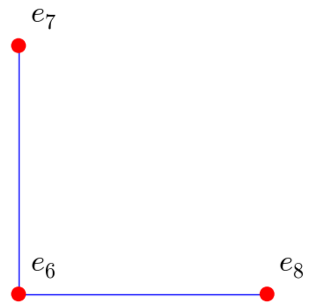


Finally, we build on that by adding a third row with basis elements that are in $E_6^3 - E_6^2$. Since there is only one element, it must go over the e_2 as there can be no gaps in the vertical lines through the nodes we add.



This gives us all the information we need for the Jordan blocks associated to the eigenvalue 6. The number of nodes on the first horizontal line tells us the number of Jordan blocks with 6 on the diagonal, and the height over each of these nodes tells the size of the block. So we have three blocks with 6's on the diagonal: one of size 3, and two of size 1.

We now move on to the eigenvalue 7. We compute $E_7^1 = \ker(A - 7 \cdot 1_8)$ has dimension 2 with basis $\{e_6, e_8\}$. We have E_7^2 has dimension 7 with basis $\{e_6, e_7, e_8\}$. We now position these on a graph as above, giving only the final result here:



Thus, we have two blocks with 7's on the diagonal, one of size 2 and one of size 1. Putting this into matrix form gives us what we already knew, namely, that A is already in Jordan canonical form.

Our next example is a bit more involved.

Example 4.4.12. Consider the matrix

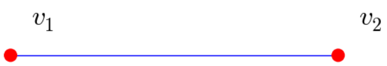
$$A = \begin{pmatrix} 3 & 3 & 0 & 0 & 0 & -1 & 0 & 2 \\ -3 & 4 & 1 & -1 & -1 & 0 & 1 & -1 \\ 0 & 6 & 3 & 0 & 0 & -2 & 0 & -4 \\ -2 & 4 & 0 & 1 & -1 & 0 & 2 & -5 \\ -3 & 2 & 1 & -1 & 2 & 0 & 1 & -2 \\ -1 & 1 & 0 & -1 & -1 & 3 & 1 & -1 \\ -5 & 10 & 1 & -3 & -2 & -1 & 6 & -10 \\ -3 & 2 & 1 & -1 & -1 & 0 & 1 & 1 \end{pmatrix}.$$

One calculates

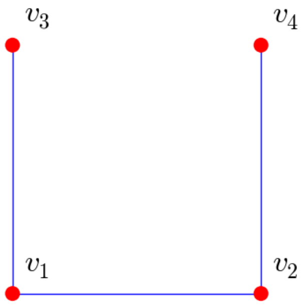
$$\begin{aligned} c_A(x) &= (x-2)(x-3)^5(x^2-6x+21) \\ &= (x-2)(x-3)^5(x-(3+2\sqrt{-3}))(x-(3-2\sqrt{-3})). \end{aligned}$$

Thus, A does not have Jordan canonical form over $\mathbb{Q}$, but does over $\mathbb{Q}(\sqrt{-3})$. We now compute the Jordan canonical form over $\mathbb{Q}(\sqrt{-3})$. Note that the eigenvalues $2, 3 \pm 2\sqrt{-3}$ are very easily to deal with. Since they only occur to multiplicity one, there can only be one Jordan block for each of these eigenvalues, each of size 1. Thus, the only work comes in determining the block structure for the eigenvalue 3.

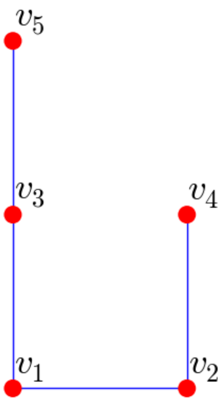
We compute the Jordan block structure for the eigenvalue 3 just as in the previous example. The difference here is it requires more work to compute the size of the eigenspaces, and we don't keep track of the bases of the eigenspaces since we are only looking for the canonical form and not the basis that realizes the form. One could keep track of the bases using elementary linear algebra at each step if one so desired. One computes, using elementary linear algebra, that $\dim_{\mathbb{Q}(\sqrt{-3})} E_3^1 = 2$. Let $\{v_1, v_2\}$ be a basis for this space. One then calculates $\dim_{\mathbb{Q}(\sqrt{-3})} E_3^2 = 4$, so we add vectors v_3 and v_4 to obtain a basis $\{v_1, v_2, v_3, v_4\}$ for E_3^2. We know E_3^∞ has dimension 5, so it must be that $E_3^\infty = E_3^3$ and so we have a basis $\{v_1, v_2, v_3, v_4, v_5\}$ for E_3^3. Graphically, we have for the first step:



Since we add two vectors going from E_3^1 to E_3^2, we add v_3 over v_1 and v_4 over v_2 obtaining:



Finally, we add the final vector over v_3 to obtain the final graph:



Thus, we have two Jordan blocks associated to the eigenvalue 3, one of size 3 and one of size 2. Thus, the Jordan canonical form of A is given by

$$\begin{pmatrix} 3 & 1 & 0 & & & & & \\ 0 & 3 & 1 & & & & & \\ 0 & 0 & 3 & & & & & \\ & & & 3 & 1 & & & \\ & & & 0 & 3 & & & \\ & & & & & 2 & & \\ & & & & & & 3+2\sqrt{-3} & \\ & & & & & & & 3-2\sqrt{-3} \end{pmatrix}.$$

4.5 Semi-simple and diagonalizable operators

In the past few sections we have seen examples of how to pick a nice basis so that a linear map has a nice form with respect to this basis. In this section we use these results to recover some results from elementary linear algebra as well as add a few results that can be obtained when one does not have Jordan canonical form or when one has multiple linear transformations.

Definition 4.5.1. Let $T \in \mathrm{Hom}_F(V,V)$. We say T is *diagonalizable* if there is a basis $\mathcal{B}$ of V so that $[T]_{\mathcal{B}}$ is a diagonal matrix.

We have the following result from elementary linear algebra on diagonalizability.

Corollary 4.5.2. *Let $T \in \mathrm{Hom}_F(V,V)$. If $c_T(x)$ does not split into a product of linear factors, then T is not diagonalizable. If $c_T(x)$ does split into a product of linear factors then the following are equivalent:*

(a) T is diagonalizable;

(b) $m_T(x)$ splits into a product of distinct linear factors;

(c) For every eigenvalue λ, $E_\lambda^\infty = E_\lambda^1$;

(d) For every eigenvalue λ of T if $c_T(x) = (x-\lambda)^{e_\lambda}p(x)$ with $p(\lambda) \neq 0$, then $e_\lambda = \dim_F E_\lambda^1$;

(e) If we set $d_\lambda = \dim_F E_\lambda^1$, then $\sum d_\lambda = \dim_F V$;

(f) If $\lambda_1, \ldots, \lambda_m$ are the distinct eigenvalues of T, then $V = E_{\lambda_1}^1 \oplus \cdots \oplus E_{\lambda_m}^1$.

Proof. Suppose T is diagonalizable. There is a basis $\mathcal{B}$ such that

$$[T]_{\mathcal{B}} = D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}$$

for some $\lambda_1, \ldots, \lambda_n \in F$ where the λ_i are not assumed to be distinct. Then $c_T(x) = \det(x1_m - D) = (x-\lambda_1)\cdots(x-\lambda_m)$. Thus, $c_T(x)$ splits into linear factors. Equivalently, if $c_T(x)$ does not split into linear factors then T is not diagonalizable.

Now suppose $c_T(x) = (x-\lambda_1)^{e_1}\cdots(x-\lambda_m)^{e_m}$ for some $\lambda_i \in F$ and $e_i \in \mathbb{Z}_{\geq 1}$ where we now assume the λ_i are distinct. Note that since $c_T(x)$ splits into linear factors, the Jordan canonical form for T exists over F.

Suppose T is diagonalizable. We have from the proof of the existence of the Jordan canonical form of T that $V = V^1 \oplus \cdots \oplus V^m$ where $V^i = \ker(T-\lambda_i)^{e_i} = E_{\lambda_i}^{e_i}$. Each V^i splits into a direct sum of subspaces coming from the rational canonical form of $T_i = T|_{V^i}$. In particular, we can write

$$V^i = W_1^i \oplus \cdots \oplus W_{m_i}^i$$

where each W_j^i is generated by an element w_j^i. The dimension of each W_j^i corresponds to the size of the corresponding block in the rational canonical form. Thus, if $\dim_F W_j^i > 1$, then the rational canonical form for T restricted to W_j^i is larger than a 1 by 1 matrix, i.e., T is not diagonalizable. This shows we have each W_j^i is 1-dimensional. Now we have W_j^i is generated by a single element w_j^i, $m_{T_i,w_{j+1}^i}(x) \mid m_{T_i,w_j^i}(x)$ for each i, and $m_{T_i}(x) = m_{T_i,w_1^i}(x)$. Thus, $m_{T_i}(x) = (x-\lambda_i)$. We have $m_T(x)$ is the least common multiple of the $m_{T_i}(x)$, and so $m_T(x)$ splits into a product of distinct linear factors, i.e., we have 1) implies 2).

Now suppose that $m_T(x)$ splits into a product of distinct linear terms. We saw above that if $m_T(x) = (x-\lambda_1)\cdots(x-\lambda_m)$ with the λ_i distinct, then Lemma 4.4.5 gives $E_\lambda^\infty = E_\lambda^1$ and so 2) implies 3).

Observe that if $c_T(x) = (x-\lambda)^e p(x)$, then we have $\dim_F E_\lambda^\infty = e$. (This is a homework exercise.) Assume that for every eigenvalue λ we have $E_\lambda^\infty = E_\lambda^1$. This gives the result and so 3) implies 4).

Suppose that we are given for each λ that $\dim_F E_\lambda^1 = e_\lambda$. Then since $\deg c_T(x) = n$, we must have $d_{\lambda_1} + \cdots + d_{\lambda_m} = e_{\lambda_1} + \cdots + e_{\lambda_m} = n$, i.e., 4) implies 5).

We now show 5) implies 6). Suppose $d_{\lambda_1} + \cdots + d_{\lambda_m} = n$. We know $E_{\lambda_1}^1 \oplus \cdots \oplus E_{\lambda_m}^1 \subset V$. However, since they have the same dimension they must be equal.

Finally, it only remains to show 6) implies 1). Suppose we have $V = E_{\lambda_1}^1 \oplus \cdots \oplus E_{\lambda_m}^1$. This gives that each Jordan block can have size at most 1, so in particular the Jordan canonical form is a diagonal matrix. □

One does not need the existence of Jordan canonical form to prove the above result as there are elementary proofs. However, this proof has the added benefit of reinforcing the important points of the proof of the existence of Jordan canonical form. One should note that $c_T(x)$ factoring into linear factors does not imply T is diagonalizable (this is the entire point of the Jordan canonical form), but if it splits into distinct linear factors then one does have T is diagonalizable since $m_T(x) \mid c_T(x)$.

We saw in the previous section that a linear map could fail to be diagonalizable for a couple of reasons. The first is fairly easy to deal with. For instance, consider the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{R})$. The characteristic polynomial of this is $c_A(x) = x^2 + 1$, so A is not diagonalizable over $\mathbb{R}$ because its eigenvalues, namely $\pm i$, are not in $\mathbb{R}$. However, if we consider this as a matrix in $\mathrm{Mat}_2(\mathbb{C})$, then since the eigenvalues are distinct we see the Jordan canonical form is $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, and so it is diagonalizable over $\mathbb{C}$. This motivates the following definition.

Definition 4.5.3. Let $T \in \mathrm{Hom}_F(V, V)$. We say T is *potentially diagonalizable* if T is diagonalizable over a field extension K/F. (This is also referred to as *absolutely semi-simple.*)

We have a criterion for a linear map to be diagonalizable in terms of its minimal polynomial, namely, the minimal polynomial should split into distinct linear factors over the field. We also have a criterion in terms of the minimal polynomial for a linear map to be potentially diagonalizable. First, we recall a definition from abstract algebra.

Definition 4.5.4. A polynomial $f \in F[x]$ is said to be *separable* if it has no repeated roots.

Exercise 4.5.5. Show f is separable if and only if $\gcd(f, f') = 1$ where f' denotes the formal derivative of f.

The power of the previous exercise is it allows one to check for separability without having to find the roots of f. Finding the greatest common divisor of two polynomials depends only on the Euclidean algorithm in $F[x]$, which is very fast and easy to use; finding roots of polynomials is very hard.

Proposition 4.5.6. *A linear map $T \in \mathrm{Hom}_F(V, V)$ is potentially diagonalizable if and only if the minimal polynomial $m_T(x) \in F[x]$ is separable.*

Proof. Suppose T is potentially diagonalizable. Let K/F be the field so that T is diagonalizable over K. Then we know that the minimal polynomial of T is the same when considered over F or K. Since T is diagonalizable over K, the minimal polynomial can have no repeated roots over K, so certainly no repeated roots over F.

Now suppose $m_T(x)$ is separable. Let K be the splitting field of $m_T(x)$ over F, i.e., the field obtained upon adjoining all the roots of $m_T(x)$ to F. Since $m_T(x)$ is separable and K contains all the roots of $m_T(x)$, it splits into distinct linear factors over K. Thus, T is diagonalizable over K. □

Of course, there are linear maps that are not diagonalizable or potentially diagonalizable. For example, we know the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Q})$ is not even potentially diagonalizable because it is already in Jordan canonical form, so it cannot be diagonalized over a larger field because the Jordan canonical form is unique when it exists. We consider two more types of linear maps here.

Definition 4.5.7. Let $T \in \mathrm{Hom}_F(V, V)$. We say T is *simple* if the only T-invariant subspaces of V are $\{0\}$ and V.

Example 4.5.8. Consider $V = \mathbb{R}^2$ and let T be the linear map given by rotation by any fixed α radians with $\alpha \neq 0$. This is clearly a linear map, but the only fixed subspaces are V and $\{0\}$, so it is a simple map.

As in the above cases, we can characterize simple linear maps in terms of the minimal polynomial.

Theorem 4.5.9. *Let $T \in \mathrm{Hom}_F(V, V)$. The following are equivalent:*

(a) T is simple,

(b) $m_T(x)$ is irreducible in $F[x]$ and has degree $\dim_F V$,

(c) $c_T(x)$ is irreducible in $F[x]$,

(d) $c_T(x) = m_T(x)$ is irreducible in $F[x]$.

Proof. If $m_T(x)$ is reducible or has degree less than $n = \dim_F V$, we use the rational canonical form of T to obtain a contradiction. Namely, write $c_T(x) = h(x)m_T(x) = h(x)g_1(x)g_2(x)$. Recall that if $p_1(x), \ldots, p_k(x)$ are the invariant factors, then $c_T(x) = p_1(x) \cdots p_k(x)$ and $p_1(x) = m_T(x)$. Thus, we must have $\deg p_2(x) \geq 1$, and so $\ker p_2(T)$ gives a non-trivial proper T-invariant subspace. So it must be the case that $\deg m_T(x) = \deg c_T(x)$. If $m_T(x)$ is reducible, then Theorem 4.2.4 gives non-trivial proper T-invariant subspaces of V. Thus, we see that if $m_T(x)$ is reducible or has degree less than the degree of $c_T(x)$, then T is not simple. Thus, we have (1) implies (2).

If $m_T(x)$ is irreducible and has degree $\dim_F V$, then since $m_T(x) \mid c_T(x)$ and $\deg c_T(x) = n$, we have $m_T(x) = c_T(x)$ and so $c_T(x)$ is irreducible. This shows (2) implies (3) and (4). We also immediately have (3) implies (4) because $m_T(x) \mid c_T(x)$.

Finally, to see (4) implies (1) just apply Corollary 4.2.5. □

The last class of linear maps we will deal with are semi-simple linear maps.

Definition 4.5.10. Let $T \in \mathrm{Hom}_F(V,V)$. We say T is *semi-simple* if every T-invariant subspace has a T-invariant complement.

Just as above, we can classify these in terms of the minimal polynomial of T.

Theorem 4.5.11. *Let $T \in \mathrm{Hom}_F(V,V)$. Then T is semi-simple if and only if $m_T(x)$ is square-free.*

Proof. Assume $m_T(x)$ is square-free and let W be a T-invariant subspace of V. Let $m_T(x) = p_1(x)\cdots p_k(x)$ be the factorization of $m_T(x)$ into irreducible factors. We have that $V = V_1 \oplus \cdots \oplus V_k$ where $V_i = \ker p_i(T)$. Set $W_i = W \cap V_i$, so $W = \oplus_{j=1}^{k} W_j$. Finding a T-invariant complement of W in V is equivalent to finding a T-invariant complement of W_i in V_i. The easiest way to do this is to use a little bit of module theory, but it can be done with vectors as well. Observe that since $p_i(x)$ is an irreducible polynomial, we have $K = F[x]/(p_i(x))$ is a field. We claim V_i is a K-vector space. We define scalar multiplication by

$$[f(x)] \cdot v = f(T)(v).$$

The fact that $p_i(T)$ kills V_i gives that this scalar multiplication is well-defined. It is now elementary to check that V_i is a K-vector space. Moreover, we have that W_i is a subspace of the K-vector space V_i, and so has a complement W_i' as a K-vector space. Since W_i' is an K-subspace, it is necessarily an F-subspace that is also T-invariant. Thus, $W' = \sum_{i=1}^{k} W_i'$ is the T-invariant complement of W.

Now assume that $m_T(x)$ is divisible by $p_1(x)^2$ for $p_1(x)$ irreducible. We now just apply the argument given in Example 4.2.3 to construct a T-invariant subspace with not T-invariant complement. Thus, if $m_T(x)$ is not square-free then T is not semi-simple. □

As an easy corollary we have the following.

Corollary 4.5.12. *Let $T \in \operatorname{Hom}_F(V, V)$ with $c_T(x)$ square-free. Then T is semi-simple and $m_T(x) = c_T(x)$.*

Proof. We have $m_T(x) \mid c_T(x)$, so clearly $m_T(x)$ must be square-free so T is semi-simple. Moreover, $m_T(x)$ and $c_T(x)$ have the same irreducible factors, so if $c_T(x)$ is square-free necessarily $m_T(x) = c_T(x)$. □

Note that a polynomial that splits with distinct roots is clearly separable. Thus, diagonalizable implies potentially diagonalizable, which also follows immediately from the definition. One also has a separable polynomial has no repeated factors, so potentially diagonalizable implies semi-simple. Clearly a simple linear map is semi-simple as well. The reverse implications are not true. For example, we have seen above already a map that is potentially diagonalizable but not diagonalizable. We also have the following exercise.

Exercise 4.5.13. Give an example of a semi-simple linear map that is not simple.

It isn't as easy to give a linear map that is semi-simple but not potentially diagonalizable. This has to do with the subtle difference of a polynomial being separable as opposed to having distinct factors. For example, if the field one is working over has characteristic 0, it is not possible to give a polynomial that has distinct factors but is not separable. (This is true over any perfect field; it does not really require characteristic 0.) Over a perfect field semisimplicity and potential diagonalizability are the same thing; over an algebraically closed field semisimplicity is the same as diagonalizability. To see these are not the same in general, we consider a non-perfect field.

Example 4.5.14. Let $F = \mathbb{F}_2(t)$, i.e., the field consisting of ratios of polynomials with coefficients in $\mathbb{F}_2$. This is not a perfect field. For instance, t is not a square in F. Consider the matrix $A = \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}$. Then $c_A(x) = x^2 - t$, so $c_A(x)$ is irreducible and so it is square-free. Thus, A is a semi-simple linear map on F^2. (It is actually a simple linear map.) Now if we consider the splitting field of $c_A(x)$, this is given by $K = F(\sqrt{t})$. Note that over K we have $c_A(x) = (x - \sqrt{t})^2$ since K has characteristic 2. Thus, A is not potentially diagonalizable. Thus, we have a linear map that is semi-simple but not potentially diagonalizable.

We have seen in the last two sections that in many instances one can choose a basis so that the matrix representing a linear transformation is in a very nice form. However, it may be the case that one is interested in more than one linear transformation at a time. In that case, the results just given aren't as useful because while a basis might put one linear transformation into a nice form, it does not necessarily put the other transformation in a nice form. We do have the following result in this direction.

Theorem 4.5.15. *Let $S, T \in \mathrm{Hom}_F(V, V)$ with each diagonalizable. The following are equivalent:*

- *(a) There is a basis $\mathcal{B}$ of V such that $[S]_{\mathcal{B}}$ and $[T]_{\mathcal{B}}$ are diagonal, i.e., $\mathcal{B}$ consists of common eigenvalues of S and T;*
- *(b) S and T commute.*

Proof. First suppose there is a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ so that $[S]_{\mathcal{B}}$ and $[T]_{\mathcal{B}}$ are both diagonal. Then there exists $\lambda_i, \mu_i \in F$ such that $S(v_i) = \lambda_i v_i$ and $T(v_i) = \mu_i v_i$. To show S and T commute, it is enough to check they commute on a basis. We have

$$\begin{aligned} ST(v_i) &= S(\mu_i v_i) \\ &= \mu_i S(v_i) \\ &= \mu_i \lambda_i v_i \\ &= \lambda_i \mu_i v_i \\ &= \lambda_i T(v_i) \\ &= TS(v_i). \end{aligned}$$

Thus, S and T commute.

Now suppose that S and T commute. Since T is diagonalizable, we can write $V = V_1 \oplus \cdots \oplus V_k$ with V_i the μ_i-eigenspace of T. For $v_i \in V_i$ we have

$$\begin{aligned} TS(v_i) &= ST(v_i) \\ &= \mu_i S(v_i). \end{aligned}$$

Thus, $S(v_i) \in V_i$ and so the V_i are S-invariant. We claim each V_i has a basis consisting of eigenvectors of S, i.e., $S_i = S|_{V_i}$ is diagonalizable. We know from the previous section that a linear transformation is diagonalizable if and only if the minimal polynomial splits into distinct linear factors. Thus, $m_S(x)$ splits into distinct linear factors by assumption. However, $m_{S_i}(x) \mid m_S(x)$, so S_i is diagonalizable as well. Let $\mathcal{B}_i$ be a basis of V_i consisting of eigenvectors of S. The elements of $\mathcal{B}_i$ are in V_i, so are eigenvectors of T automatically. Set $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ and we are done. □

Definition 4.5.16. Let $T \in \mathrm{Hom}_F(V, V)$. We say T is *nilpotent* if there is a positive integer r so that T^r is 0.

Let $A \in \mathrm{Mat}_n(F)$ be a matrix and assume that $c_A(x)$ splits over F so that A has a Jordan canonical form. Note that if J is the Jordan canonical form of A, we can write $J = S + N$ where S is a semi-simple matrix and N is a nilpotent matrix. In particular, the matrix S is the diagonal matrix and N is the super-diagonal matrix. Moreover, S and N are unique and they satisfy $SN = NS$. This can be show in much greater generality. First, we need the following result on

Theorem 4.5.17. *Let F be a subfield of $\mathbb{C}$ and let $T \in \mathrm{Hom}_F(V,V)$. There is a unique semi-simple map $S \in \mathrm{Hom}_F(V,V)$ and a unique nilpotent map $N \in \mathrm{Hom}_F(V,V)$ so that*

(a) $T = S + N$;

(b) $SN = NS$.

Furthermore, S and N are given by polynomials in T.

Proof. Write $m_T(x) = p_1(x)^{e_1} \cdots p_k(x)^{e_k}$ be the factorization of $m_T(x)$ into irreducible components. Let $r = \max\{e_1, \ldots, e_k\}$. Set $f(x) = p_1(x) \cdots p_k(x)$. We have $f(x)$ is a product of distinct irreducibles and $f(x)^r$ is divisible by $m_T(x)$.

We construct a sequence of polynomials $g_0(x), g_1(x), \ldots$ so that $f\left(x - \sum_{j=0}^{m} g_j(x) f(x)^j\right)$ is divisible by $f(x)^{m+1}$ for $m = 0, 1, 2, \ldots$. To begin, set $g_0(x) = 0$. Then clearly we have $f(x)$ divides $f(x)$, so the result is true in this case. Suppose we have constructed $g_0(x), \ldots, g_{m-1}(x)$ with the required properties. Set

$$h(x) = x - \sum_{j=0}^{m-1} g_j(x) f(x)^j.$$

Then we have $f(x)^{m-1} \mid f(h(x))$. We apply Taylor's formula to $f(h(x) - g_m(x)f(x)^m)$ to obtain

$$f(h(x) - g_m(x)f(x)^m) = f(h(x)) - g_m(x)f(x)^m f'(h(x)) + f(x)^{m+1} b(x)$$

where $b(x) \in F[x]$ is some polynomial. Our assumption gives that $f(h(x)) = q(x)f(x)^m$ for some $q(x) \in F[x]$. Thus, if we can choose $g_m(x)$ so that $q(x) - g_m(x)f'(h(x))$ is divisible by $f(x)$. This can be done because $\gcd(f(x), f'(x)) = 1$.

Now take $m = r - 1$. Since $f(T)^r = 0$ we have

$$f\left(T - \sum_{j=0}^{r-1} g_j(T) f(T)^j\right) = 0.$$

Set

$$N = \sum_{j=1}^{r-1} g_j(T) f(T)^j = \sum_{j=0}^{r-1} g_j(T) f(T)^j$$

where the second equality is because we chose $g_0(x) = 0$. We use the fact that $\sum_{j=1}^{m} g_j(x) f(x)^j$ is divisible by f to see that $N^r = 0$ and so N is nilpotent. Set $S = T - N$. Clearly we have $T = S + N$. To see that S is semi-simple, just note that $f(S) = f(T-N) = 0$ by the construction of N and f has distinct irreducible factors by definition. It is also clear S and N are given by polynomials in T since that is how they were constructed.

It only remains to prove that S and N are unique. To prove uniqueness it is enough to work over an algebraically closed field. Suppose there exists N' and S' that satisfy S' being semi-simple, N' nilpotent, $T = S' + N'$ and $S'N' = N'S'$. We now show that $S' = S$ and $N' = N$. We immediately see since S' and N' commute with each other and $T = S' + N'$, S' and N' commute with T. Since S and N are given by polynomials in T, we have S' and N' commute with S and N as well. Observe we have $S + N = S' + N'$, i.e., $S - S' = N' - N$ and all four of these operators commute with each other. Since we are working over an algebraically closed field, semi-simple is the same as diagonalizable so S and S' are diagonalizable. Moreover, since they commute they are simultaneously diagonalizable. This gives $S - S'$ is diagonalizable, which in turns gives $N' - N$ is diagonalizable. Moreover, since N and N' are both nilpotent and commute with each other, $N' - N$ is nilpotent. In particular, we have

$$(N' - N)^m = \sum_{j=0}^{m} \binom{m}{j} (N')^{m-j}(-N)^j.$$

If $\dim_F V = n$, if we take $m = 2n$ that is large enough so that $(N' - N)^m = 0$. (In fact n is large enough, but that isn't important for this result.) So we have $S - S'$ is diagonalizable and nilpotent. The minimal polynomial for $S - S'$ must be x^r for some $r \leq m$ because the operator is nilpotent, but since $S - S'$ is semi-simple its minimal polynomial must split into distinct irreducible factors. Thus, $r = 1$ and so $S - S' = 0$, i.e., $S = S'$. Since $S = S'$, we immediately get $N = N'$ and we have the result. □

We close this section with a few further results on the Jordan and rational canonical structure of products of matrices.

Lemma 4.5.18. *Let $S, T \in \mathrm{Hom}_F(V, V)$ with V a finite dimensional F-vector space. Let $p(x) = a_r x^r + \cdots + a_1 x + a_0 \in F[x]$ with $a_0 \neq 0$. Then $\dim_F(\ker p(ST)) = \dim_F(\ker p(TS))$.*

Proof. Let $\{v_1, \ldots, v_k\}$ be a basis for $\ker(p(ST))$. We claim $\{T(v_1), \ldots, T(v_k)\}$ is linearly independent. Suppose

$$c_1 T(v_1) + \cdots + c_k T(v_k) = 0.$$

This gives $T(c_1 v_1 + \cdots + c_k v_k) = 0$, so $ST(c_1 v_1 + \cdots + c_k v_k) = 0$. Let $v = c_1 v_1 + \cdots + c_k v_k$. Then $ST(v) = 0$. Moreover, we also have $v \in \ker(p(ST))$ because $\{v_1, \ldots, v_k\}$ is a basis for $\ker p(ST)$. Thus

$$\begin{aligned} 0 &= p(ST)v \\ &= a_r (ST)^r(v) + \cdots + a_1 ST(v) + a_0 v \\ &= a_0 v \end{aligned}$$

where we have used $v \in \ker(ST)$. However, since $a_0 \neq 0$, this gives $v = 0$. Thus $0 = c_1v_1 + \cdots + c_kv_k$. But $\{v_1, \ldots, v_k\}$ is linearly independent so we must have $c_i = 0$ for all i as desired. Thus $\{T(v_1), \ldots, T(v_k)\}$ is linearly independent.

We now claim $T(v_i) \in \ker(p(TS))$ for each i. We have

$$\begin{aligned}(TS)^jT &= \underbrace{(TS)\cdots(TS)}_{j \text{ times}}T \\ &= T\underbrace{(ST)\cdots(ST)}_{j \text{ times}} = T(ST)^j.\end{aligned}$$

Thus,

$$\begin{aligned}p(TS)T(v_i) &= (a_r(TS)^r + \cdots + a_0)T(v_i) \\ &= T((a_r(ST)^r + \cdots + a_0)(v_i)) \\ &= T(p(ST)(v_i)) \\ &= T(0) \\ &= 0.\end{aligned}$$

Thus, $\{T(v_1), \ldots, T(v_k)\}$ is a linearly independent set in $\ker p(TS)$, so $\dim_F \ker(p(TS)) \geq k = \dim_F \ker(p(ST))$. One now uses the same argument with S replacing T To get the other direction of the inequality. □

Theorem 4.5.19. *Let $T \in \mathrm{Hom}_F(V,V)$ and $S \in \mathrm{Hom}_F(V,V)$ with F algebraically closed. Then ST and TS have the same nonzero eigenvalues, and for a common eigenvalue λ, they have the same Jordan block structure at λ.*

Proof. Let $\lambda \neq 0$ be a nonzero eigenvalue of ST and $v \in V$ a nonzero eigenvector. Set $w = T(v)$. Then $ST(v) = \lambda v = S(w)$. We also have

$$\begin{aligned}TS(w) &= T(\lambda v) \\ &= \lambda T(v) \\ &= \lambda w.\end{aligned}$$

We have $T(v)$ is nonzero since if it were 0 we would have $S(0) = \lambda v$ implies $\lambda = 0$ or $v = 0$, a contradiction. Thus $w \neq 0$ and so λ is an eigenvalue of TS as well. The same argument in the other direction shows every nonzero eigenvalue of TS is a nonzero eigenvalue of ST. It remains to deal with the Jordan block structure.

Consider the polynomials $p_{j,\lambda}(x) = (x-\lambda)^j$ for $j \geq 1$. The Jordan block structure of ST at λ is given by the dimensions of $\ker(p_{j,\lambda}(ST))$. However, Lemma 4.5.18 gives $\dim\ker(p_{j,\lambda}(ST)) = \dim\ker(p_{j,\lambda}(TS))$ for each j as long as the constant term of $p_{j,\lambda}(x) \neq 0$. However, as long as $\lambda \neq 0$ this is satisfied so the Jordan block structures must be equal. □

Over a general field we have the following result.

Corollary 4.5.20. *Let $S, T \in \operatorname{Hom}_F(V, V)$. Then $c_{TS}(x) = c_{ST}(x)$.*

Proof. We claim it is enough to prove the result for F algebraically closed. This follows because the characteristic polynomial is the same regardless of which field one consider S and T to be defined over.

Let $\dim_F V = n$ and let $\lambda_1, \dots, \lambda_k$ be the distinct nonzero eigenvalues of ST. Write

$$c_{ST}(x) = x^{e_0}(x - \lambda_1)^{e_1} \cdots (x - \lambda_k)^{e_k}$$

with $e_0 = n - (e_1 + \cdots + e_k)$. The previous theorem gives that ST and TS have the same Jordan block structure at $\lambda_1, \dots, \lambda_k$. Thus, $c_{TS}(x) = x^{f_0}(x - \lambda_1)^{e_1} \cdots (x - \lambda_k)^{e_k}$ where $f_0 = n - (e_1 + \cdots + e_k) = e_0$. Thus $c_{TS}(x) = c_{ST}(x)$. □

The previous results do not give that the matrices associated to ST and TS are similar. Consider the following example.

Example 4.5.21. Let $A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $AB = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ has characteristic polynomial $c_{AB}(x) = x^2$ and $BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ has characteristic polynomial $c_{BA}(x) = x^2$. Note the rational canonical form of BA is BA, but the rational canonical form of AB is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus, they cannot be similar.

Theorem 4.5.22. *Let $T \in \operatorname{Hom}_F(V, V)$. Then the following are equivalent:*

(a) V is T-generated by a single element, i.e., the rational canonical form of T is a single block;

(b) every linear transformation $S \in \operatorname{Hom}_F(V, V)$ that commutes with T is a polynomial in T.

Proof. Suppose that V is T-generated by a single element v_0. Let $S \in \operatorname{Hom}_F(V, V)$ commute with T. There exists $p_0(x) \in F[x]$ so that $S(v_0) = p_0(T)(v_0)$ since v_0 is a T-generator of V. Let $v \in V$ and write $v = g(T)(v_0)$ for some $g(x) \in F[x]$. We have

$$\begin{aligned} S(v) &= S(g(T)(v_0)) \\ &= g(T)S(v_0) \\ &= g(T)p_0(T)(v_0) \\ &= p_0(T)g(T)(v_0) \\ &= p_0(T)(v). \end{aligned}$$

Thus, $S = p_0(T)$ since they agree on every element of V.

Now suppose that V is not T-generated by a single element. Let $\{v_1, \dots, v_k\}$ be a rational canonical generating set for T. This gives $V = V_1 \oplus \cdots \oplus V_k$ with V_i T-generated by v_i. In particular the V_i are T-invariant. Define $S : V \to V$ by

$$S(v) = \begin{cases} 0 & \text{if } v \in V_1 \\ v & \text{if } v \in V_i \text{ for } i > 0. \end{cases}$$

Since the direct sum is T-invariant we get that S and T commute. Suppose $S = p(T)$ for some $p(x) \in F[x]$. We have $0 = S(v_1) = p(T)(v_1)$. Thus, $p_1(x) \mid p(x)$ where $p_j(x) = m_{T,v_j}(x)$. However, $p_j(x) \mid p_1(x)$ for all $j \geq 1$ and so $p_j(x) \mid p(x)$ for all j. Thus, $S(v_j) = p(T)(v_j) = 0$ because $p_j(x) \mid p(x)$. This is a contradiction if $k > 1$. □

4.6 Canonical forms via modules

This section is not part of the main course and is simply included to provide some clarification to those that have had some exposure to modules and would like to fit the contents of this chapter into what they already know. We will assume basic familiarity with modules as given in Section 2.5.

The following example is the key example for relating modules to what we've done in this chapter.

Example 4.6.1. Let F be a field and V a finite dimensional F-vector space. Let $T \in \mathrm{Hom}_F(V, V)$. We have that V is an $F[x]$-module via $f(x) \cdot v = f(T)(v)$. Note that here we have V is a cyclic $F[x]$-module if and only if V is T-generated by a single element. Now consider what it means for a subspace W of V to be T-invariant. This means that $T(W) \subset W$ and W is a subspace of V. However, one can easily see this is equivalent to requiring that W be an $F[x]$-submodule of V. Thus, the entire theory of T-invariant subspaces amounts to studying submodules of V when considered as an $F[x]$-module with x acting via T.

We can now state the first form of the Fundamental Theorem of Finitely Generated Modules over a principal ideal domain (PID).

Theorem 4.6.2. *Let R be a PID and M a finitely generated R-module. Then*

$$M \cong R^r \oplus R/a_1R \oplus R/a_2R \oplus \cdots \oplus R/a_mR \tag{4.1}$$

for some integer $r \geq 0$ and some nonzero elements $a_1, \dots, a_m \in R$ which are not units in R and satisfy the divisibility relations $a_m \mid a_{m-1} \mid \cdots \mid a_2 \mid a_1$.

The elements $a_1, \dots, a_m$ in the above theorem are referred to as the *invariant factors* of M and r is referred to as the *rank* of M. Equation 4.1 is the *invariant factor decomposition* of M.

We can prove this theorem fairly easily given Theorem 2.5.21. We include a proof here of Theorem 4.6.2 because it is relevant for our proof of Smith Normal Form below, which justifies the algorithm used earlier for finding the rational canonical form of a matrix.

Proof. Let $z_1, \dots, z_n$ be a set of generators for M of minimal cardinality. Define a surjective R-linear map $\pi : R^n \to M$ given by $\pi(e_i) = z_i$ where $e_1, \dots, e_n$ is the standard basis of R^n. Since this map is surjective, we immediately obtain an isomorphism of R-modules $R^n / \ker(\pi) \cong M$. We now apply Theorem 2.5.21 to R^n and $\ker(\pi)$ to obtain a new basis $y_1, \dots, y_n$ of R^n so that $a_1 y_1, \dots, a_m y_m$ is a basis of $\ker(\pi)$ for some $a_i \in R$ with $a_m \mid a_{m-1} \mid \cdots \mid a_1$. Thus, we have

$$\begin{aligned} M &\cong R^n / \ker(\pi) \\ &\cong (Ry_1 \oplus \cdots \oplus Ry_n)/(Ra_1 y_1 \oplus \cdots \oplus Ra_m y_m). \end{aligned}$$

Now consider the surjective R-linear map

$$\begin{aligned} &Ry_1 \oplus \cdots \oplus Ry_n \longrightarrow R/a_1 R \oplus \cdots \oplus R/a_m R \oplus R^{n-m} \\ &(r_1 y_1, \dots, r_n y_n) \mapsto (r_1 \pmod{a_1 R}, \dots, r_m \pmod{a_m R}, r_{m+1}, \dots, r_n). \end{aligned}$$

The kernel of this map is easily seen to be $Ra_1 y_1 \oplus \cdots \oplus Ra_m y_m$, which gives the desired isomorphism. □

We now show Theorem 4.6.2 implies every matrix has a rational canonical form. Let $T \in \mathrm{Hom}_F(V, V)$ and view V as an $F[x]$-module as in the example about. It is fairly straightforward to show that V is a torsion $F[x]$-module, and so $r = 0$ in the above theorem. Thus, the theorem gives $p_1(x), \dots, p_m(x)$ in $F[x]$ with $p_m(x) \mid p_{m-1}(x) \mid \cdots \mid p_1(x)$ so that

$$V \cong F[x]/(p_1(x)) \oplus \cdots \oplus F[x]/(p_m(x)).$$

Now to see that this gives rational canonical form we just need to choose an appropriate basis for each space $F[x]/(p_j(x))$. Write $p_j(x) = x^{n_j} + a_{n_j-1} x^{n_j-1} + \cdots + a_1 x + a_0$. Consider the basis $\mathcal{B} = \{[x]^{n_j-1}, [x]^{n_j-1}, \cdots, [x], [1]\}$. We have that T acts via multiplication by x, thus $T(x^k) = x^{k+1}$. Thus, in $F[x]/(p_j(x))$ we have T acts as

$$\begin{aligned} [1] &\mapsto [x] \\ [x] &\mapsto [x]^2 \\ &\vdots \\ [x]^{n_j-1} &\mapsto [x]^{n_j} = -a_{n_j-1}[x]^{n_j-1} - \cdots - a_1[x] - a_0. \end{aligned}$$

This shows that the matrix of T with respect to $\mathcal{B}$ on $F[x]/(p_j(x))$ is precisely the companion matrix. Thus, we have recovered rational canonical form.

One should keep in mind that even though it was very easy to deduce rational canonical form from the structure theorem, proving the structure theorem itself requires a considerable amount of effort. The reason this method is generally preferable to the approach given earlier in this chapter is that the structure theorem for finitely generated modules has many other applications, such as classifying all finitely generated abelian groups.

We can give a different version of the structure theorem that is useful for obtaining Jordan canonical form. We assumed above that R is a PID. This implies that R is necessarily a unique factorization domain as well, i.e., a UFD. Given any $a \in R$ there are primes $p_1, \ldots, p_s$, a unit u, and positive integers $e_1, \ldots, e_s$ so that

$$a = up_1^{e_1} \cdots p_s^{e_s}.$$

Recall here that "prime" means that if $p \mid ab$, then $p \mid a$ or $p \mid b$. The factorization of a is unique up to units. We can apply this to the structure theorem by decomposing each a_i into its prime factorization, i.e., we can write

$$R/aR \cong R/p_1^{e_1}R \oplus \cdots \oplus R/p_s^{e_s}R.$$

We now restate the theorem in this form.

Theorem 4.6.3. *Let R be a PID and M a finitely generated R-module. Then we have*

$$M \cong R^r \oplus R/p_1^{e_1}R \oplus \cdots \oplus R/p_t^{e_t}R$$

where $r \geq 0$ is an integer and the $p_i^{e_i}$ are positive powers of primes in R. (Note we do not assume the p_i are distinct here!)

The $p_i^{e_i}$ are referred to as the *elementary divisors* of M.

One can now easily recover Jordan canonical form from rational canonical form. Assume Jordan canonical form exists over F. For each invariant factor $a_i(x)$ we completely factor it as

$$a_i(x) = \prod_{j=1}^{n_i} (x - \lambda_j^{(i)})^{e_j^{(i)}}.$$

The elementary divisors are then given by the $(x - \lambda_j^{(i)})^{e_j^{(i)}}$ as i runs over the various invariant factors.

One last thing to tie up is the method used to calculate the rational canonical form. This follows immediately from the *Smith Normal Form* of a matrix, which we now develop.

Theorem 4.6.4. *Let $A \in \mathrm{Mat}_n(F)$ Using the elementary row and column operations of*

- *(a) interchanging two row or columns*
- *(b) adding an $F[x]$-multiple of one row or column to another*
- *(c) multiplying any row or column by a nonzero element of F*

the matrix $x1_n - A \in \mathrm{Mat}_n(F[x])$ can be put into diagonal form, called the Smith Normal Form for A,

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & p_m(x) & & & \\ & & & & p_{m-1}(x) & & \\ & & & & & \ddots & \\ & & & & & & p_1(x) \end{pmatrix}$$

with $p_1(x), \ldots, p_m(x)$ the invariant factors of A.

Proof. The key ingredient to this proof is that $F[x]$ is a Euclidean domain, which allows us to find greatest common divisors of elements. Let V be an n-dimensional vector space with basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ and let $T \in \mathrm{Hom}_F(V, V)$ be defined by

$$T(v_j) = \sum_{i=1}^{n} a_{ij} v_i \quad \text{for } j = 1, \ldots, n$$

where $A = (a_{ij})$. We consider the free $F[x]$-module of rank n, $M = F[x]^n$. Let $z_1, \ldots, z_n$ denote a basis of M over $F[x]$. We clearly have a natural surjective F-linear map $\varphi : M \to V$ just given by sending z_j to v_j. From the proof of Theorem 4.6.2 we see that the proof comes down to finding the correct generators for M with relations for $\ker(\varphi)$.

By definition of the module structure we have x acts on V via the linear transformation T so

$$x(v_j) = \sum_{i=1}^{n} a_{ij} v_i \quad \text{for } j = 1, \ldots, n.$$

Set

$$w_j = -a_{1j} z_1 - \cdots - a_{j-1j} z_{j-1} + (x - a_{jj}) z_j - a_{j+1j} z_{j+1} - \cdots - a_{nj} z_n$$

for $j = 1, \ldots, n$. We clearly have that $w_j \in \ker(\varphi)$. Solving the equation defining w_j for xz_j we see that

$$xz_j = w_j + f_j$$

where $f_j \in Fz_1 + \cdots + Fz_n$. This immediately gives that we have

$$F[x]z_1 + \cdots + F[x]z_n = (F[x]w_1 + \cdots + F[x]w_n) + (Fz_1 + \cdots + Fz_n).$$

We now claim that $\ker(\varphi)$ is generated by $w_1, \ldots, w_n$. Let $f_1(x)z_1 + \cdots + f_n(x)z_n \in \ker(\varphi)$. Note we can write any element of M in this form because $z_1, \ldots, z_n$ is a set of generators. We can write

$$f_1(x)z_1 + \cdots + f_n(x)z_n = (g_1(x)w_1 + \cdots g_n(x)w_n) + (c_1z_1 + \cdots + c_nz_n)$$

for $c_i \in F$ by our above decomposition. Since $w_i \in \ker(\varphi)$ for each i, this gives

$$c_1v_1 + \cdots + c_nv_n = 0.$$

However, $v_1, \ldots, v_n$ forms a basis for V over F, so we must have $c_1 = \cdots = c_n = 0$, and thus any element of the kernel of φ is in $F[x]w_1 + \cdots + F[x]w_n$, as claimed.

We now just need to observe that the matrix for $\{w_1, \ldots, w_n\}$ in terms of $\{z_1, \ldots, z_n\}$ is given by

$$x1_n - {}^tA = \begin{pmatrix} x - a_{11} & -a_{21} & \cdots & -a_{n1} \\ -a_{12} & x - a_{22} & \cdots & -a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & x - a_{nn} \end{pmatrix}.$$

Now one just proceeds to diagonalize this as was done in Section 4.3. □

4.7 Problems

For all of these problems V is a finite dimensional F-vector space.

1. Let $T \in \mathrm{Hom}_F(V, V)$. Prove that the intersection of any collection of T-invariant subspaces of V is T-invariant.

2. Let $T \in \mathrm{Hom}_F(V, V)$ and $w \in V$. Let $m_{T,w}(x) \in F[x]$ be the annihilating polynomial of w.
(a) Show that if $m_{T,w}(x) = p(x)q(x)$, then $p(x) = m_{T,q(T)(w)}(x)$.
(b) Let W be the subspace of V that is T-generated by w. If $\deg m_{T,w}(x) = d$ and $\deg q(x) = e$, show that $\dim_W q(T)(W) = d - e$.

3. Let $A \in \mathrm{Mat}_4(\mathbb{Q})$ be defined by

$$\begin{pmatrix} 1 & 2 & -4 & 4 \\ 2 & -1 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 3 \end{pmatrix}.$$

First find the rational canonical form of A by hand. Check your answer using SAGE.

4. Prove that two 3×3 matrices are similar if and only if they have the same characteristic and minimal polynomials. Give an explicit counterexample to this assertion for 4×4 matrices.

5. We say $A \in \mathrm{Mat}_2(F)$ has multiplicative order n if $A^n = I$ and $A^m \neq I$ for any $0 < m < n$. Show that $x^5 - 1 = (x-1)(x^2 - 4x + 1)(x^2 + 5x + 1)$ in $\mathbb{F}_{19}[x]$. Use this to determine all similarity all elements of $\mathrm{Mat}_2(\mathbb{F}_{19})$ of multiplicative order 5.

6. In a group G we say two elements a and b are *conjugate* if there exists $g \in G$ so that $a = gbg^{-1}$. The *conjugacy class* of an element is the collection of all elements conjugate to it. Given a conjugacy class $\mathcal{C}$, any element in $\mathcal{C}$ is referred to as a *representative* for $\mathcal{C}$. Determine representatives for all the conjugacy classes for $\mathrm{GL}_3(\mathbb{F}_2)$.

7. Prove that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of a matrix $A \in \mathrm{Mat}_n(F)$, then $\lambda_1^k, \ldots, \lambda_n^k$ are the eigenvalues of A^k for any $k \geq 0$.

8. Let $c_T(x) = (x - \lambda)^e p(x)$ with $p(\lambda) \neq 0$. Show that $\dim_F E_\lambda^\infty = e$.

9. Prove that the matrices $\begin{pmatrix} 2 & 0 & 0 & 0 \\ -4 & -1 & -4 & 0 \\ 2 & 1 & 3 & 0 \\ -2 & 4 & 9 & 1 \end{pmatrix}$ and $\begin{pmatrix} 5 & 0 & -4 & -7 \\ 3 & -8 & 15 & -13 \\ 2 & -4 & 7 & -7 \\ 1 & 2 & -5 & 1 \end{pmatrix}$ are similar.

10. Prove that the matrices $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 5 & 2 & -8 & -8 \\ -6 & -3 & 8 & 8 \\ -3 & -1 & 3 & 4 \\ 3 & 1 & -4 & -5 \end{pmatrix}$ both have characteristic polynomial $(x-3)(x+1)^3$. Determine the Jordan canonical form for each matrix and determine if they are similar.

11. Determine all possible Jordan canonical forms for a linear transformation with characteristic polynomial $(x-2)^3(x-3)^2$.

12. Prove that any matrix $A \in \mathrm{Mat}_n(\mathbb{C})$ satisfying $A^3 = A$ can be diagonalized. Is the same statement true over any field F? If so, prove it. If not, give a counterexample.

13. Determine the Jordan canonical form for a matrix $A \in \mathrm{Mat}_n(\mathbb{Q})$ with entries all equal to 1.

Chapter 5

Bilinear and sesquilinear forms

In this chapter we study bilinear and sesquilinear forms. In the special cases of symmetric, skew-symmetric, Hermitian, and skew-Hermitian forms we provide the standard classification theorems.

5.1 Basic definitions and facts

In this chapter we will be interested in maps $V \times V \to F$ that are linear in each variable separately. In particular, we will be interested in bilinear forms.

Definition 5.1.1. Let V be an F-vector space. A function $\varphi : V \times V \to F$ is said to be a *bilinear form* if φ is an F-linear map in each variable separately, i.e., for all $v_1, v_2, v \in V$ and $c \in F$ we have

(a) $\varphi(cv_1 + v_2, v) = c\varphi(v_1, v) + \varphi(v_2, v)$;

(b) $\varphi(v, cv_1 + v_2) = c\varphi(v, v_1) + \varphi(v, v_2)$.

We denote the collection of bilinear forms by $\mathrm{Hom}_F(V, V; F)$.

Exercise 5.1.2. Show that $\mathrm{Hom}_F(V, V; F)$ is an F-vector space.

We now give some elementary examples of bilinear forms. Checking each of these satisfy the criterion to be bilinear form is left as an exercise.

Example 5.1.3. The first example, and the one that should be kept in mind throughout this and the next chapter, is that familiar example of a dot product from multivariable calculus. Let $V = \mathbb{R}^n$ for some $n \in \mathbb{Z}_{\geq 1}$. Define $\varphi : V \times V \to \mathbb{R}$ by setting

$$\begin{aligned}\varphi(v, w) &= v \cdot w\\ &= {}^t\!vw\\ &= \sum_{i=1}^{n} a_i b_i\end{aligned}$$

where $v = {}^t(a_1, \ldots, a_n)$ and $w = {}^t(b_1, \ldots, b_n)$.

Example 5.1.4. Let $A \in \mathrm{Mat}_n(F)$. We have a bilinear form φ_A on $V = F^n$ defined by

$$\varphi_A(v, w) = {}^t\!vAw.$$

Much as one saw that upon choosing a basis any linear map between finite dimensional vector spaces could be realized as a matrix, we will soon see that any bilinear form on a finite dimensional vector space can be represented as a matrix as in this example upon choosing a basis for V.

Example 5.1.5. Let $V = C^0([0, 1], \mathbb{R})$ be the $\mathbb{R}$-vector space of continuous functions from $[0, 1]$ to $\mathbb{R}$. Define $\varphi : V \times V \to \mathbb{R}$ by setting

$$\varphi(f, g) = \int_0^1 f(x)g(x)dx.$$

This gives a bilinear form. More generally, given any positive integer n, one can consider the vector space of paths $V = C^0([0, 1], \mathbb{R}^n)$. Given $f, g \in V$, we have $f(x), g(x) \in \mathbb{R}^n$ for each $x \in [0, 1]$. Thus, for each x we have $f(x) \cdot g(x)$ is well-defined from Example 5.1.3 above. We can define

$$\varphi(f, g) = \int_0^1 f(x) \cdot g(x)dx.$$

This defines a bilinear form on V.

In the next section we will be interested in classifying certain "nice" bilinear forms. In particular, we will often restrict to the case our bilinear form is non-degenerate.

Definition 5.1.6. Let $\varphi \in \mathrm{Hom}_F(V, V; F)$ be a bilinear form. We say that φ is *right non-degenerate* if given $w_0 \in V$ so that $\varphi(v, w_0) = 0$ for every $v \in V$ one has $w_0 = 0$. We say it is *left non-degenerate* if given any $v_0 \in V$ so that $\varphi(v_0, w) = 0$ for every $w \in V$ one has $v_0 = 0$. We say φ is *non-degenerate* if it is left and right non-degenerate.

We will see below that if V is a finite dimensional vector space that φ is left non-degenerate if and only if it is right non-degenerate. However, if V is infinite dimensional these may not be the same as the following example shows.

Example 5.1.7. Consider the $\mathbb{R}$-vector space

$$\ell^2 = \left\{ (a_n)_{n\geq 1} : \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}.$$

Define a bilinear form φ on ℓ^2 by setting

$$\varphi(a,b) = \sum_{n=1}^{\infty} a_{n+1}b_n$$

for $a = (a_n)$ and $b = (b_n)$. Set $x = (1,0,0,\dots)$. Then we have $\varphi(x,b) = 0$ for all b, so φ is not left non-degenerate, i.e., it is left degenerate. However, if $\varphi(a,b_0) = 0$ for all $a = (a_n)$, it is easy to check this implies b_0 is the sequence consisting of all 0's. Thus, φ is right non-degenerate.

Non-degenerate forms arise in a very natural way in a context we have already studied, namely, in terms of isomorphisms between a finite dimensional vector space and its dual. In particular, recall that when studying dual spaces we saw that for a finite dimensional F-vector space V that one has $V \cong V^\vee$, but that this isomorphism depends upon picking a basis and so is non-canonical. It turns out that non-degenerate bilinear forms are in bijection with the collection of such isomorphisms for finite dimensional vector spaces. For infinite dimensional spaces one only obtains V injects into $V^\vee$.

Let $\varphi \in \mathrm{Hom}_F(V,V;F)$. If we fix $v_0 \in V$ then we have a linear map $\varphi(\cdot, v_0) \in V^\vee$ given by $w \mapsto \varphi(w,v_0)$. In particular, we define $R_\varphi : V \to V^\vee$ by setting

$$R_\varphi(v) = \varphi(\cdot, v).$$

Note that for any $v \in V$, the map $R_\varphi(v) \in V^\vee$ is given by $R_\varphi(v)(w) = \varphi(w,v)$. We claim that R_φ is a linear map. We want to show that for any $a \in F$ and $v_1, v_2 \in V$ that $R_\varphi(av_1+v_2) = aR_\varphi(v_1)+R_\varphi(v_2)$. As this is an equality of maps, we must show these maps agree on elements. We have

$$\begin{aligned} R_\varphi(av_1+v_2)(w) &= \varphi(w, av_1+v_2) \\ &= a\varphi(w,v_1) + \varphi(w,v_2) \\ &= aR_\varphi(v_1)(w) + R_\varphi(v_2)(w), \end{aligned}$$

as desired. Thus, $R_\varphi \in \mathrm{Hom}_F(V,V^\vee)$.

Of course, one could just as easily have fixed $w_0 \in V$ and considered the linear map $\varphi(w_0,\cdot) \in V^\vee$ given by $v \mapsto \varphi(w_0,v)$. In this case we define $L_\varphi : V \to V^\vee$ by setting

$$L_\varphi(w) = \varphi(w,\cdot).$$

Just as above one obtains $L_\varphi \in \mathrm{Hom}_F(V,V^\vee)$.

Lemma 5.1.8. *A bilinear form φ is non-degenerate if and only if L_φ and R_φ are injections.*

Proof. First, suppose that L_φ and R_φ are injective. Let $w \in V$. If $\varphi(v, w) = 0$ for all $v \in V$, this implies that $R_\varphi(w)(v) = 0$ for every $v \in V$. However, this says that $R_\varphi(w)$ is the zero map. Since we are assuming R_φ is injective, this gives $w = 0$. Thus, φ is right non-degenerate. Similarly, L_φ injective implies that φ is left non-degenerate.

Now suppose that L_φ or R_φ is not injective. As the argument is essentially the same, we suppose there exists $w \in V$ with $w \neq 0$ so that $R_\varphi(w) = 0$, i.e., R_φ is not an injection. This translates to the statement that $0 = R_\varphi(w)(v) = \varphi(v, w)$ for all $v \in V$. This gives that φ is not right non-degenerate, i.e., φ is degenerate. □

One can note in the previous result that if V is finite dimensional then R_φ being an injection is the same as R_φ being an isomorphism since V and $V^\vee$ have the same dimension. So in the finite dimensional case one can replace the condition L_φ and R_φ are injective with them being isomorphisms.

One can define the left and right kernel of φ by setting

$$\ker_R \varphi = \ker R_\varphi = \{w \in V : \varphi(v, w) = 0, \text{ for every } v \in V\}$$

and

$$\ker_L \varphi = \ker L_\varphi = \{v \in V : \varphi(v, w) = 0, \text{ for every } w \in V\}.$$

This allows us to rephrase the condition that φ is non-degenerate to be that the left and right kernels of φ are trivial.

In the case that V is a finite dimensional vector space one only needs to consider left or right non-degenerate as the other comes for free. This is due to the following result.

Theorem 5.1.9. *Let V be a finite dimensional vector space over a field F. Let $\varphi \in \mathrm{Hom}_F(V, V; F)$. Then L_φ and R_φ are dual to each other, namely, given $L_\varphi : V \to V^\vee$, if we dualize this to obtain $L_\varphi^\vee : (V^\vee)^\vee \to V^\vee$, then upon identifying $(V^\vee)^\vee$ with V via the canonical isomorphism we have $L_\varphi^\vee = R_\varphi$. Similarly one has $R_\varphi^\vee = L_\varphi$.*

Proof. Recall that given F-vector spaces V, W and $T \in \mathrm{Hom}_F(V, W)$, the dual map $T^\vee$ is defined by

$$T^\vee(\psi)(v) = \psi(T(v))$$

for $v \in V$ and $\psi \in V^\vee$. We also recall the canonical isomorphism between V and $(V^\vee)^\vee$ is given by sending v to the evaluation map eval_v. For

$v, w \in V$ we have

$$\begin{aligned} L_\varphi^\vee(\mathrm{eval}_v)(w) &= \mathrm{eval}_v(L_\varphi(w)) \\ &= \mathrm{eval}_v(\varphi(w, \cdot)) \\ &= \varphi(w, v) \\ &= R_\varphi(v)(w). \end{aligned}$$

Thus, we have $L_\varphi^\vee = R_\varphi$ upon identifying V and $(V^\vee)^\vee$. □

This result shows that if V is finite dimensional then R_φ is injective if and only if L_φ is injective. Equivalently, φ is non-degenerate if and only if R_φ is an isomorphism.

Lemma 5.1.10. *Let V be a finite dimensional vector space. There is a bijection between isomorphisms $T : V \to V^\vee$ and non-degenerate bilinear forms $\varphi \in \mathrm{Hom}_F(V, V; F)$.*

Proof. Let φ be a non-degenerate bilinear form. Then we associate the required isomorphism by sending φ to R_φ.

Now suppose we have an isomorphism $T : V \to V^\vee$. Define $\varphi \in \mathrm{Hom}_F(V, V; F)$ by

$$\varphi(v, w) = T(w)(v).$$

It is elementary to check this is a bilinear form and $R_\varphi = T$, so is non-degenerate. Moreover, since $R_\varphi = T$ it is immediate that these maps are inverse to each other and so provide the required bijection. □

It is well known from multivariable calculus that Example 5.1.3 is used to define the length of a vector in $\mathbb{R}^n$. However, if we make the same definition on $\mathbb{C}$ we do not recover the length of a complex number. In order to find the length of $z \in \mathbb{C}$, we want to consider $z\overline{z}$. This leads to the definition of a different type of form, a sesquilinear form, that keeps track of conjugation as well. Before defining these forms, we need to define conjugation for more general fields than $\mathbb{C}$.

Definition 5.1.11. Let F be a field and $\mathrm{conj} : F \to F$ a map that satisfies:

(a) $\mathrm{conj}(\mathrm{conj}(x)) = x$ for every $x \in F$;

(b) $\mathrm{conj}(z + y) = \mathrm{conj}(x) + \mathrm{conj}(y)$ for every $x, y \in F$;

(c) $\mathrm{conj}(xy) = \mathrm{conj}(x)\,\mathrm{conj}(y)$ for every $x, y \in F$.

We call such a map a *conjugation map* on F. We say conj is nontrivial if conj is not the identity map on F.

We give a couple of familiar examples of conjugation maps.

Example 5.1.12. Let $F = \mathbb{C}$. Then conj is the familiar conjugation map sending $x + iy$ to $x - iy$ for $x, y \in \mathbb{R}$.

Example 5.1.13. Let $D \in \mathbb{Z}$ and consider $F = \mathbb{Q}(\sqrt{D})$ where we recall

$$\mathbb{Q}(\sqrt{D}) = \{a + b\sqrt{D} : a, b \in \mathbb{Q}\}.$$

It is easy to check the map sending $a + b\sqrt{D}$ to $a - b\sqrt{D}$ is a nontrivial conjugation map on F if D is not a perfect square.

To emphasize that one should think of conjugation as a generalization of complex conjugation as well as to save writing we will denote conjugation maps by $x \mapsto \overline{x}$.

Lemma 5.1.14. *Let F be a field with nontrivial conjugation and assume* $\operatorname{char}(F) \neq 2$. *Then:*

(a) Let $F_0 = \{z \in F : \overline{z} = z\}$. Then F_0 is a subfield of F.

(b) There is a nonzero element $j \in F$ so that $\overline{j} = -j$.

(c) Every element of F can be written uniquely as $z = x + jy$ for some $x, y \in F_0$.

Proof. The fact that F_0 is a subfield of F is left as an exercise. Let $a \in F$ so that $\overline{a} \neq a$. Since the conjugation is nontrivial there is always such an a. Set $j = \frac{a - \overline{a}}{2}$. Then one has $\overline{j} = -j$. Given $z \in F$, we have $\frac{z + \overline{z}}{2}$ and $\frac{z - \overline{z}}{2j}$ are both elements of F_0 and $z = \left(\frac{z+\overline{z}}{2}\right) + j\left(\frac{z-\overline{z}}{2j}\right)$. □

Example 5.1.15. Returning to the above examples, in the case $F = \mathbb{C}$ we have $F_0 = \mathbb{R}$ and $j = \sqrt{-1}$. In the case $F = \mathbb{Q}(\sqrt{D})$ we have $F_0 = \mathbb{Q}$ and $j = \sqrt{D}$.

Given a field F that admits a conjugation map and a vector space V over F, we can also consider a conjugation map on V.

Definition 5.1.16. Let V be an F-vector space where F is a field with conjugation. A *conjugation map* on V is a map $\operatorname{conj} : V \to V$ that satisfies

(a) $\operatorname{conj}(\operatorname{conj}(v)) = v$ for every $v \in V$;

(b) $\operatorname{conj}(v + w) = \operatorname{conj}(v) + \operatorname{conj}(w)$ for every $v, w \in V$;

(c) $\operatorname{conj}(av) = \overline{a}\operatorname{conj}(v)$ for every $a \in F$, $v \in V$.

As in the case of conjugation on a field, we will often write $v \mapsto \overline{v}$ to denote a conjugation map.

Given a finite dimensional F-vector space, one always has a conjugation map on V if F has a conjugation map. Namely, upon choosing a basis one has $V \cong F^n$ for $n = \dim_F V$. Given $v \in V$, we have an element

$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in F^n$. Since F has a nontrivial conjugation, we have $\begin{pmatrix} \overline{a_1} \\ \vdots \\ \overline{a_n} \end{pmatrix} \in F^n$. We set $\overline{v}$ to be the element of V corresponding to this element. This clearly gives a non-trivial conjugation on V.

The third property in the definition of conjugation for a vector space gives immediately that linear maps are not the correct maps to work with if one wants to take nontrivial conjugation into account. To remedy this, we define conjugate linear maps.

Definition 5.1.17. Let F be a field with conjugation and let V and W be F-vector spaces. A map $T : V \to W$ is said to be *conjugate linear* if it satisfies

(a) $T(v_1 + v_2) = T(v_1) + T(v_2)$ for every $v_1, v_2 \in V$;

(b) $T(av) = \overline{a}T(v)$ for every $a \in F$, $v \in V$.

We say T is a *conjugate isomorphism* if it is conjugate linear and bijective.

Of course, one might not wish to introduce a completely new definition to deal with conjugation. We can get around this by defining a new vector space associated to V that takes into account the conjugation. Let $\overline{V}$ be equal to V as a set, and let $\overline{V}$ have the same addition as V. However, we define scalar multiplication on $\overline{V}$ by setting

$$\begin{aligned} F \times V &\to V \\ (a, v) &\mapsto a \cdot v := \overline{a}v. \end{aligned}$$

Exercise 5.1.18. Check that $\overline{V}$ is an F-vector space.

We now consider maps in $\mathrm{Hom}_F(\overline{V}, W)$. Let $T \in \mathrm{Hom}_F(\overline{V}, W)$. Then we have

(a) $T(v_1 + v_2) = T(v_1) + T(v_2)$ for every $v_1, v_2 \in V$;

(b) $T(a \cdot v) = aT(v)$ for every $a \in F$, $v \in V$, i.e., $T(\overline{a}v) = aT(v)$ for every $a \in F$, $v \in V$.

The second equation translates to $T(av) = \overline{a}T(v)$, i.e., T is a conjugate linear map. Thus, the set of conjugate linear maps forms an F-vector space, in particular, it is equal to $\mathrm{Hom}_F(\overline{V}, W)$. In the case that $W = F$ we write $\overline{V}^\vee = \mathrm{Hom}_F(\overline{V}, F)$.

There is one subtlety to keep in mind here. We have a natural F-vector space structure on $\mathrm{Hom}_F(\overline{V}, W)$ just as we would for any set of linear maps between two F-vector spaces, namely, given $a \in F$ and $T \in \mathrm{Hom}_F(\overline{V}, W)$, we set $a \cdot T$ to be the map defined by $a \cdot T(v) = aT(v)$. The scalar multiplication on $\mathrm{Hom}_F(\overline{V}, W)$ is really derived from the scalar multiplication in

W. The scalar multiplication here is not given by the conjugate multiplication; if we wanted that vector space we would have to write $\overline{\mathrm{Hom}_F(\overline{V}, W)}$, or, equivalently, we would have $\mathrm{Hom}_F(\overline{V}, \overline{W})$.

Definition 5.1.19. A *sesquilinear form* $\varphi : V \times V \to F$ is a map that is linear in the first variable and conjugate linear in the second variable.

While bilinear forms are linear in each variable, a sesquilinear form is linear in the first variable and conjugate linear in the second variable. Note that "sesqui" means one and a half, and this is what it refers to.

Exercise 5.1.20. Show that a sesquilinear form $\varphi : V \times V \to F$ is the same as a bilinear form from $V \times \overline{V}$ to F. As such, we can denote the collection of sesquilinear forms by $\mathrm{Hom}_F(V, \overline{V}; F)$.

The above examples of bilinear forms are easily adjusted to give sesquilinear forms.

Example 5.1.21. Let $V = \mathbb{C}^n$. Define $\varphi : V \times V \to \mathbb{C}$ by

$$\begin{aligned}\varphi(v, w) &= {}^t v \overline{w} \\ &= \sum_{i=1}^{n} v_i \overline{w}_i\end{aligned}$$

where $v = {}^t(v_1, \ldots, v_n)$ and $w = {}^t(w_1, \ldots, w_n)$. Observe that $\varphi(v, v) = ||v||^2$, and so for any $v \in \mathbb{C}^n$ we have $\varphi(v, v) \in \mathbb{R}_{\geq 0}$.

Example 5.1.22. Let $V = F^n$ where F is a field with conjugation. Let $A \in \mathrm{Mat}_n(F)$ and define $\varphi : V \times V \to F$ by

$$\varphi(v, w) = {}^t v A \overline{w}.$$

This is a sesquilinear form.

Example 5.1.23. Let $V = C^0([0, 1], \mathbb{C})$, i.e., V is the set of paths in $\mathbb{C}$. Define $\varphi : V \times V \to \mathbb{C}$ to be the function

$$\varphi(f, g) = \int_0^1 f(z) \overline{g(z)} dz.$$

One can easily check that φ is a sesquilinear form.

We define left and right non-degenerate as well as non-degenerate for sesquilinear forms just as for bilinear forms. We would like to give a classification for this in terms of maps analogous to L_φ and R_φ as for bilinear forms. There is one major difference here though in that above L_φ and R_φ could both be viewed as maps from V to $V^\vee$, where here we need to take into account the conjugate linearity of the maps. As in the case of bilinear forms for each $v \in V$ we define the map $L_\varphi(v)$ by setting $L_\varphi(v)(u) = \varphi(v,u)$ for each $u \in V$. In this case for $v, v_1, v_2, u, u_1, u_2 \in V$ and $c \in F$ we have

$$\begin{aligned} L_\varphi(v_1 + cv_2)(u) &= \varphi(v_1 + cv_2, u) \\ &= \varphi(v_1, u) + c\varphi(v_2, u) \\ &= L_\varphi(v_1)(u) + cL_\varphi(v_2)(u) \end{aligned}$$

and

$$\begin{aligned} L_\varphi(v)(u_1 + cu_2) &= \varphi(v, u_1 + cu_2) \\ &= \varphi(v, u_1) + \overline{c}\varphi(v, u_2) \\ &= L_\varphi(v)(u_1) + \overline{c}L_\varphi(v)(u_2). \end{aligned}$$

Thus, we see that $L_\varphi : V \to \overline{V}^\vee$. As above, for each $v \in V$ define $R_\varphi(v)$ to be the map defined by $R_\varphi(v)(u) = \varphi(u,v)$ for each $u \in V$. Now for $v_1, v_2, u \in V$ and $c \in F$ we have

$$\begin{aligned} R_\varphi(v_1 + cv_2)(u) &= \varphi(u, v_1 + cv_2) \\ &= \varphi(u, v_1) + \overline{c}\varphi(u, v_2) \\ &= R_\varphi(v_1)(u) + \overline{c}R_\varphi(v_2)(u), \end{aligned}$$

i.e., $R_\varphi(v_1 + cv_2) = R_\varphi(v_1) + \overline{c}R_\varphi(v_2)$. Moreover, for each fixed v and all $u_1, u_2 \in V$, $c \in F$, it is easy to check that

$$R_\varphi(v)(u_1 + cu_2) = R_\varphi(v)(u_1) + cR_\varphi(v)(u_2).$$

This shows that R_φ is a conjugate-linear map and for each $v \in V$ we have $R_\varphi(v)$ is a linear map, i.e., $R_\varphi : \overline{V} \to V^\vee$.

Our next goal is to relate L_φ and R_φ. This proceeds much as in the bilinear case, but the scalar actions can be a bit confusing so we provide the details here. The first thing to recall is the canonical isomorphism $V \to (V^\vee)^\vee$; in our case we will need

$$\begin{aligned} \Phi : \overline{V} &\to (\overline{V}^\vee)^\vee \\ v &\mapsto \mathrm{eval}_v\,. \end{aligned}$$

We just briefly recall the proof that Φ is a linear map due to the potential confusion for the scalar multiplications. In particular, recall for $v \in V$ and $a \in F$ we have $a \cdot v = \overline{a}v$ but the scalar multiplication on $\overline{V}^\vee$ and $(\overline{V}^\vee)^\vee$ is given by the scalar multiplication induced by the multiplication in F. So, for example if $f \in \overline{V}^\vee$, $a \in F$, and $v_1, v_2 \in V$, then we have

$$f(a \cdot v_1 + v_2) = af(v_1) + f(v_2)$$

i.e.,

$$f(\overline{a}v_1 + v_2) = af(v_1) + f(v_2).$$

Thus, given $v_1, v_2 \in V$, $a \in F$, and $f \in \overline{V}^{\vee}$ we have

$$\begin{aligned}\Phi(a \cdot v_1 + v_2)(f) &= eval_{\overline{a}v_1+v_2}(f)\\ &= f(\overline{a}v_1 + v_2)\\ &= af(v_1) + f(v_2)\\ &= a\Phi(v_1)(f) + \Phi(v_2)(f)\\ &= (a\Phi(v_1) + \Phi(v_2))(f).\end{aligned}$$

The proof that this is an injection in general and an isomorphism for finite dimensional cases is omitted as there is nothing confusing added to that. We now follow the argument given for bilinear forms to relate L_φ and R_φ. We have $L_\varphi^\vee : (\overline{V}^\vee)^\vee \to V^\vee$ and $R_\varphi : \overline{V} \to V^\vee$. We identify $\overline{V}$ and $(\overline{V}^\vee)^\vee$ as above. The argument that gives $L_\varphi^\vee = R_\varphi$ upon making this identification now goes through verbatim from the case of bilinear forms.

We can define the left and right kernels of φ just as was done for bilinear forms. Once again we obtain φ is non-degenerate if and only if L_φ and R_φ are injections. If V is finite-dimensional, this is equivalent to L_φ and R_φ being isomorphisms.

Exercise 5.1.24. Show there is a bijection between isomorphisms $T : V \to \overline{V}^\vee$ and non-degenerate sesquilinear forms $\varphi : V \times V \to V$.

We now define the matrix associated to a bilinear (resp. sesquilinear) form.

Definition 5.1.25. Let $\varphi : V \times V \to F$ be a bilinear or sesquilinear form. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of V. Set $a_{ij} = \varphi(v_i, v_j)$. The *matrix associated to* φ is given by

$$[\varphi]_{\mathcal{B}} = A = (a_{ij}) \in \mathrm{Mat}_n(F).$$

Theorem 5.1.26. *Let φ be a bilinear or sesquilinear form on a finite dimensional vector space V. Let $\mathcal{B}$ be a basis of V. Then for $w, v \in V$ we have*

$$\varphi(v, w) = {}^t[v]_{\mathcal{B}} A [w]_{\mathcal{B}}$$

if φ is bilinear and

$$\varphi(v, w) = {}^t[v]_{\mathcal{B}} A \overline{[w]}_{\mathcal{B}}$$

if φ is sesquilinear.

Proof. This follows immediately upon calculating on a basis. □

The definition of the matrix associated to a bilinear (resp. sesquilinear) form seems rather arbitrary in how we defined it. At this point the only justification is that the previous theorem shows this definition works how we expect it to. However, there is a more natural reason this is the correct definition to use. Let φ be a bilinear or sesquilinear form. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of V and $\mathcal{B}^\vee = \{v_1^\vee, \dots, v_n^\vee\}$ the dual basis of $V^\vee$. (Note that $\mathcal{B}$ is also a basis for $\overline{V}$.) Observe we have

$$R_\varphi(v_j) = \varphi(\cdot, v_j) \in V^\vee.$$

If φ is a bilinear map this is a linear map from V to $V^\vee$, so we can ask for $[R_\varphi]_{\mathcal{B}}^{\mathcal{B}^\vee}$. Similarly, if φ is sesquilinear this is a linear map from $\overline{V}$ to $V^\vee$ and so again we can consider $[R_\varphi]_{\mathcal{B}}^{\mathcal{B}^\vee}$. To calculate this, we need to expand $R_\varphi(v_j)$ in terms of $\mathcal{B}^\vee$ for each j. Write

$$R_\varphi(v_j) = c_1 v_1^\vee + \cdots + c_n v_n^\vee$$

for $c_i \in F$. The goal is to find c_i. Observe we have

$$\begin{aligned} a_{ij} &= \varphi(v_i, v_j) \\ &= R_\varphi(v_j)(v_i) \\ &= c_1 v_1^\vee(v_i) + \cdots + c_n v_n^\vee(v_i) \\ &= c_i. \end{aligned}$$

Thus, $a_{ij} = c_i$ and so

$$R_\varphi(v_j) = a_{1j} v_1^\vee + \cdots + a_{nj} v_n^\vee,$$

which gives

$$[R_\varphi]_{\mathcal{B}}^{\mathcal{B}^\vee} = [\varphi]_{\mathcal{B}}.$$

This shows that the matrix of a bilinear or sesquilinear form φ is really just the matrix associated to the linear map R_φ.

Corollary 5.1.27. *A bilinear or sesquilinear form φ on a finite dimensional vector space V is non-degenerate if and only if $[\varphi]_{\mathcal{B}}$ is nonsingular for any basis $\mathcal{B}$.*

Proof. This follows immediately from the definition of non-degenerate and relation between $[\varphi]_{\mathcal{B}}$ and the linear map R_φ. □

Just as when studying linear maps, we would like to know how a change of basis effects the matrix of φ.

Theorem 5.1.28. *Let V be a finite dimensional F-vector space and let $\mathcal{B}$ and $\mathcal{C}$ be bases of V. If P is the change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$ then if φ is bilinear we have*

$$[\varphi]_{\mathcal{C}} = {}^tP[\varphi]_{\mathcal{B}}P$$

and if φ is sesquilinear we have

$$[\varphi]_{\mathcal{C}} = {}^tP[\varphi]_{\mathcal{B}}\overline{P}.$$

Proof. We consider the case that φ is sesquilinear. The proof when φ is bilinear can be obtained by taking the conjugation to be trivial in this proof.

We have from the definition that for every $v, w \in V$

$$\varphi(v, w) = {}^t[v]_{\mathcal{B}}[\varphi]_{\mathcal{B}}\overline{[w]}_{\mathcal{B}}$$

and

$$\varphi(v, w) = {}^t[v]_{\mathcal{C}}[\varphi]_{\mathcal{C}}\overline{[w]}_{\mathcal{C}}.$$

Moreover, since P is the change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$ we have $[v]_{\mathcal{B}} = P[v]_{\mathcal{C}}$ and $\overline{[w]}_{\mathcal{B}} = \overline{P}\overline{[w]}_{\mathcal{C}}$. Combining all of this we obtain

$$\begin{aligned}
{}^t[v]_{\mathcal{C}}[\varphi]_{\mathcal{C}}\overline{[w]}_{\mathcal{C}} &= \varphi(v, w) \\
&= {}^t[v]_{\mathcal{B}}[\varphi]_{\mathcal{B}}\overline{[w]}_{\mathcal{B}} \\
&= {}^t(P[v]_{\mathcal{C}})[\varphi]_{\mathcal{B}}\overline{P}\overline{[w]}_{\mathcal{C}} \\
&= {}^t[v]_{\mathcal{C}}\,{}^tP\varphi_{\mathcal{B}}\overline{P}\overline{[w]}_{\mathcal{C}}.
\end{aligned}$$

Since this is true for all $v, w \in V$, we must have

$$[\varphi]_{\mathcal{C}} = {}^tP[\varphi]_{\mathcal{B}}\overline{P}$$

as claimed. □

It is very important to note the difference between this and changing bases for linear maps. We do not conjugate the matrix here, i.e., in Chapter 3 we had the equation $[T]_{\mathcal{C}} = P^{-1}[T]_{\mathcal{B}}P$. This means that you cannot in general use the results from Chapter 4 to find a nice basis for the matrix $[\varphi]_{\mathcal{B}}$. Later we will discuss this further and determine some cases when one can use rational and Jordan canonical forms to obtain nice bases for forms as well.

Definition 5.1.29. Let $A, B \in \mathrm{Mat}_n(F)$. We say A and B are *congruent* (resp. *conjugate congruent*) if there exists $P \in \mathrm{GL}_n(F)$ so that $B = {}^tPAP$ (resp. $B = {}^tPA\overline{P}$.)

Exercise 5.1.30. Show that congruence on matrices defines an equivalence relation.

The above results show that matrices A and B in $\mathrm{Mat}_n(F)$ represent the same bilinear (resp. sesquilinear) form if and only if A and B are congruent.

5.2 Symmetric, skew-symmetric, and Hermitian forms

In this section we specialize the forms we are looking at. This allows us to prove some very nice structure theorems. For the rest of this section, all of the bilinear and sesquilinear forms will be one of the forms given in the following definition. When we wish to emphasize a form φ on V, we write (V, φ) for the vector space.

Definition 5.2.1. Let V be a vector space over a field F.

(a) A bilinear form φ on V is said to be *symmetric* if $\varphi(v, w) = \varphi(w, v)$ for all $v, w \in V$.

(b) A bilinear form φ on V is said to be *skew-symmetric* if $\varphi(v, w) = -\varphi(w, v)$ for all $v, w \in V$ and $\varphi(v, v) = 0$ for all $v \in V$.

(c) A sesquilinear form φ on V is said to be *Hermitian* if $\varphi(v, w) = \overline{\varphi(w, v)}$ for all $v, w \in V$.

(d) A sesquilinear form φ on V is said to be *skew-Hermitian* if $\mathrm{char}(F) \neq 2$ and $\varphi(v, w) = -\overline{\varphi(w, v)}$ for all $v, w \in V$.

Exercise 5.2.2. Show that if $\mathrm{char}(F) \neq 2$ then φ is skew-symmetric if and only if $\varphi(v, w) = -\varphi(w, v)$ for all $v, w \in V$.

Exercise 5.2.3. Show that if φ is any of the forms given in this definition $\ker \varphi$ is well-defined since $\ker_R \varphi = \ker_L \varphi$.

One should note that it is not often one encounters skew-Hermitian forms. The reason for this is as follows. Let F be a field with $\mathrm{char}(F) \neq 2$ with non-trivial conjugation. Then we saw before that there is an element $j \in F$ so that $\overline{j} = -j$ and every element $z \in F$ can be written uniquely in the form $z = x + jy$ with $x, y \in F_0$. Let φ be a skew-Hermitian form. Set $\psi(v, w) = j\varphi(v, w)$. We claim that ψ is Hermitian. To see this, observe

$$\begin{aligned} \overline{\psi(v, w)} &= \overline{j\varphi(v, w)} \\ &= \overline{j}\,\overline{\varphi(v, w)} \\ &= (-j)(-\varphi(w, v)) \\ &= \psi(v, w). \end{aligned}$$

Similarly, given a Hermitian form one can multiply by j to obtain a skew-Hermitian form. Thus, we see that we can move back and forth between Hermitian and skew-Hermitian forms simply by multiplying by j. This means there is essentially nothing new in studying skew-Hermitian forms.

Lemma 5.2.4. *Let (V, φ) be a finite dimensional F-vector space. Let $\mathcal{B}$ be a basis for V and set $A = [\varphi]_{\mathcal{B}}$. Then we have:*

(a) φ is symmetric if and only if ${}^tA = A$.

(b) φ is skew-symmetric if and only if ${}^tA = -A$.

(c) φ is Hermitian if and only if ${}^tA = \overline{A}$.

The proof of this lemma is left as an exercise. It amounts to converting the definition to matrix language.

Definition 5.2.5. Let $A \in \mathrm{Mat}_n(F)$.

(a) We say A is *symmetric* if ${}^tA = A$.

(b) We say A is *skew-symmetric* if ${}^tA = -A$.

(c) We say A is *Hermitian* if ${}^tA = \overline{A}$.

One should keep in mind that the notion we are generalizing is that of the dot product on $\mathbb{R}^n$. As such, the correct notion of equivalence between spaces (V, φ) and (W, ψ) should preserve "distance", i.e., it should be a generalization of the notion of isometry from Euclidean geometry.

Definition 5.2.6. Let (V, φ) and (W, ψ) be F-vector spaces. We say $T \in \mathrm{Hom}_F(V, W)$ is an *isometry* if T is an isomorphism and satisfies

$$\varphi(v_1, v_2) = \psi(T(v_1), T(v_2))$$

for all $v_1, v_2 \in V$.

As is the custom, we rephrase this definition in terms of matrices.

Lemma 5.2.7. *Let $T \in \mathrm{Hom}_F(V, W)$ with (V, φ) and (W, ψ) finite dimensional F-vector spaces of dimension n. Let $\mathcal{B}$ be a basis for V and $\mathcal{C}$ a basis for W. Set $P = [T]_{\mathcal{B}}^{\mathcal{C}}$. Then we have T is an isometry if and only if $P \in \mathrm{GL}_n(F)$ and*

$${}^tP[\psi]_{\mathcal{C}}P = [\varphi]_{\mathcal{B}}$$

for φ and ψ bilinear and

$${}^tP[\psi]_{\mathcal{C}}\overline{P} = [\varphi]_{\mathcal{B}}$$

if φ and ψ are sesquilinear.

Proof. Exercise. □

Definition 5.2.8. Let φ be a form on V. The *isometry group of φ* is given by

$$\mathrm{Isom}(V, \varphi) = \{T \in \mathrm{Hom}_F(V, V) : T \text{ is an isometry}\}.$$

When V is clear from context we will write $\mathrm{Isom}(\varphi)$.

Exercise 5.2.9. Let (V, φ) and (W, ψ) be F-vector spaces. Show that (V, φ) is isometric to (W, ψ) if and only if the matrix relative to φ is congruent to the matrix of ψ if φ and ψ are bilinear. If they are sesquilinear, show the same statement with conjugate congruent.

Exercise 5.2.10. Show that $\mathrm{Isom}(V, \varphi)$ is a group under composition.

One should keep in mind that studying isometry groups of a space gives information back about the geometry of the space (V, φ). We will see more detailed examples of this in the next chapter when we restrict ourselves to inner product spaces.

Depending on the type of form φ we give the isometry group special names.

(a) Let φ be a nondegenerate symmetric bilinear form. We write $\mathrm{O}(\varphi)$ for the isometry group and refer to this as the *orthogonal group* associated to φ. We call isometries in this group *orthogonal maps.*

(b) Let φ be a nondegenerate Hermitian form. We write $\mathrm{U}(\varphi)$ for the isometry group and refer to this as the *unitary group* associated to φ. We call isometries in this group *unitary maps.*

(c) Let φ be a skew-symmetric bilinear form. We write $\mathrm{Sp}(\varphi)$ for the isometry group and refer to this as the *symplectic group* associated to φ. We call isometries in this group *symplectic maps.*

Recall that when studying linear transformations $T \in \mathrm{Hom}_F(V, V)$ in Chapter 4, the first step was to break V into a direct sum of T-invariant subspaces. We want to follow the same general pattern here, but first need the appropriate notion of breaking V up into pieces for a form φ.

Definition 5.2.11. Let φ be a form on V. We say $v, w \in V$ are *orthogonal with respect to* φ if

$$\varphi(v, w) = \varphi(w, v) = 0.$$

We say subspaces $V_1, V_2 \subset V$ are *orthogonal* if

$$\varphi(v_1, v_2) = \varphi(v_2, v_1) = 0$$

for all $v_1 \in V_1$, $v_2 \in V_2$.

We once again emphasize that one should be keeping in mind the notion of a dot product on $\mathbb{R}^n$ when thinking of these definitions. One then sees that in this special case this notion of orthogonal is the same as the notion that was introduced in vector calculus.

Definition 5.2.12. Let $V_1, V_2 \subset V$ be orthogonal subspaces. We say V is the *orthogonal direct sum* of V_1 and V_2 and write $V = V_1 \perp V_2$ if $V = V_1 \oplus V_2$.

Exercise 5.2.13. (a) Show that $V = V_1 \perp V_2$ if $V = V_1 \oplus V_2$ and given any $v, v' \in V$, when we write $v = v_1 + v_2$ and $v' = v_1' + v_2'$ for $v_1, v_1' \in V_1$, $v_2, v_2' \in V_2$ we have

$$\varphi(v, v') = \varphi(v_1, v_1') + \varphi(v_2, v_2').$$

(b) Suppose $V = V_1 \perp V_2$ and V is finite dimensional. Let $\varphi_1 = \varphi|_{V_1 \times V_1}$ and $\varphi_2 = \varphi|_{V_2 \times V_2}$. If we let $\mathcal{B}_1$ be a basis for V_1, $\mathcal{B}_2$ a basis for V_2, and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, then show that

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} [\varphi_1]_{\mathcal{B}_1} & 0 \\ 0 & [\varphi_2]_{\mathcal{B}_2} \end{pmatrix}.$$

The next result shows that even if one is given a degenerate form, one can split off the kernel of this form and then be left with a nondegenerate form. This will allow us to focus our attention on nondegenerate forms when proving our classification theorems.

Lemma 5.2.14. *Let φ be a form on a finite dimensional F-vector space V. Then there is a subspace V_1 of V so that $\varphi_1 = \varphi|_{V_1}$ is nondegenerate and $V = \ker(\varphi) \perp V_1$. Moreover, (V_1, φ_1) is well-defined up to isometry.*

Proof. The exercise above gives that $\ker(\varphi)$ is a subspace of V, so it certainly has a complement V_1, i.e., we can write $V = \ker(\varphi) \oplus V_1$ for some subspace $V_1 \subset V$. Set $\varphi_1 = \varphi|_{V_1}$. The definition of $\ker(\varphi)$ immediately gives that $V = \ker(\varphi) \perp V_1$.

The next step is to show that φ_1 is nondegenerate, i.e., R_{φ_1} is injective. (Note we use finite dimensional here to conclude R_{φ_1} being injective is equivalent to being an isomorphism.) Suppose there exists $v_1 \in V_1$ so that $R_{\varphi_1}(v_1) = 0$, i.e., $0 = R_{\varphi_1}(v_1)(w_1) = \varphi_1(w_1, v_1) = \varphi(w_1, v_1)$ for every $w_1 \in V_1$. Moreover, if $w \in V - V_1$, then $w \in \ker(\varphi)$ and so $\varphi(w, v_1) = 0$. Thus, $\varphi(w, v_1) = 0$ for all $w \in V$, i.e. $v_1 \in \ker(\varphi)$. However, this is a contradiction since $V_1 \cap \ker(\varphi) = \{0\}$. Thus, φ_1 must be nondegenerate.

The last thing to show is that V_1 is well-defined up to isometry. Set $V' = V/\ker(\varphi)$. We define a form φ' on V' as follows. Let $\pi : V \to V'$ be the standard projection map. For $v', w' \in V'$, let $v, w \in V$ such that $\pi(v) = v'$, $\pi(w) = w'$. Set

$$\varphi'(v', w') = \varphi(v, w).$$

This is a well-defined form on V' as we are quotienting out by the kernel. Thus, $\pi|_{V_1}$ is an isometry between (V_1, φ_1) and (V, φ). This gives the result. □

Definition 5.2.15. Let φ be a form on a finite dimensional vector space V. The *rank* of φ is the dimension of $V/\ker(\varphi)$.

We saw in the proof of the previous lemma that given φ, there is a subspace $V_1 \subset V$ so that $V = \ker(\varphi) \perp V_1$. In general, given a subspace $W \subset V$, we can ask if there is a subspace $W^{\perp}$ so that $V = W \perp W^{\perp}$.

Definition 5.2.16. Let $W \subset V$ be a subspace. The *orthogonal complement* of W is given by

$$W^{\perp} = \{v \in V : \varphi(w,v) = 0 \, \text{for all } w \in W.\}.$$

Exercise 5.2.17. Show that $W^{\perp}$ is a subspace of V.

Lemma 5.2.18. *Let W be a finite dimensional subspace of (V,φ) and set $\psi = \varphi|_{W \times W}$. If ψ is nondegenerate then $V = W \perp W^{\perp}$. If V is finite dimensional and φ is nondegenerate as well then $\psi^{\perp} = \varphi|_{W^{\perp} \times W^{\perp}}$ is also nondegenerate.*

Proof. By definition we have W and $W^{\perp}$ are orthogonal, so to show $V = W \perp W^{\perp}$ we need only show that $V = W \oplus W^{\perp}$.

Let $v_0 \in W \cap W^{\perp}$. Observe that $\varphi(v_0, v_0) = 0$ because $v_0 \in W$ and $v_0 \in W^{\perp}$. Using that ψ is nondegenerate we obtain $v_0 = 0$ and so $W \cap W^{\perp} = \{0\}$.

Let $v_0 \in V$. Note that since ψ is assumed to be nondegenerate and W is finite dimensional, $R_{\psi} : W \to W^{\vee}$ is an isomorphism. Thus, for any $T \in W^{\vee}$ there is an element $w_T \in W$ so that $R_{\psi}(w_T) = T$. We now apply this fact to our situation. We have $R_{\varphi}(v_0)|_W \in W^{\vee}$, so there is a $w_0 \in W$ with

$$R_{\psi}(w_0) = R_{\varphi}(v_0)|_W,$$

i.e., for every $w \in W$ we have

$$\begin{aligned} \psi(w, w_0) &= R_{\psi}(w_0)(w) \\ &= R_{\varphi}(v_0)|_W(w) \\ &= \varphi(w, v_0). \end{aligned}$$

Thus, we have

$$\begin{aligned} \varphi(w, v_0) &= \psi(w, w_0) \\ &= \varphi(w, w_0) \end{aligned}$$

where we have used $\varphi_{W \times W} = \psi$. Subtracting we have that for every $w \in W$

$$\varphi(w, v_0 - w_0) = 0,$$

i.e., $v_0 - w_0 \in W^{\perp}$. This allows us to write

$$v_0 = w_0 + (v_0 - w_0) \in W + W^{\perp}.$$

Thus, $V = W \perp W^{\perp}$ as desired.

Now suppose that φ is nondegenerate. Let $v_0 \in W^{\perp}$. Since φ is nondegenerate there exists $v \in V$ with $\varphi(v, v_0) \neq 0$. Write $v = w_1 + w_2$ with $w_1 \in W$ and $w_2 \in W^{\perp}$. Then

$$\begin{aligned} 0 &\neq \varphi(v, v_0) \\ &= \varphi(w_1 + w_2, v_0) \\ &= \varphi(w_1, v_0) + \varphi(w_2, v_0) \\ &= \varphi(w_2, v_0) \end{aligned}$$

where we have used that $\varphi(w_1, v_0) = 0$ because $w_1 \in W$ and $v_0 \in W^{\perp}$. Thus, $R_{\varphi}(v_0)(w_2) \neq 0$ and since v_0 was arbitrary, this gives $R_{\varphi|_{W^{\perp}}}$ is injective. Since we are assuming V is finite dimensional, this is enough to conclude that $R_{\varphi|_{W^{\perp}}}$ is an isomorphism, i.e., $\varphi|_{W^{\perp}\times W^{\perp}}$ is nondegenerate. □

Example 5.2.19. It is not in general the case that if φ is a non-degenerate on a vector space V than $\varphi|_{W\times W}$ is non-degenerate for a subspace $W \subset V$. For instance, let $V = F^2$ with standard basis $\mathcal{E}_2 = \{e_1, e_2\}$. Set $W = \text{span}_F\{e_1\}$. Consider the bilinear form defined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is easy to see this is a non-degenerate form on V, but restricted to W this form is degenerate.

Lemma 5.2.20. *Let W be a subspace of (V, φ) with V finite dimensional. Assume $\varphi|_{W\times W}$ and $\varphi|_{W^{\perp}\times W^{\perp}}$ are both nondegenerate. Then $(W^{\perp})^{\perp} = W$.*

Proof. We can write $V = W \perp W^{\perp}$ and $V = W^{\perp} \perp (W^{\perp})^{\perp}$ by using that W and $W^{\perp}$ are both subspaces of V. This immediately gives that $\dim_F W = \dim_F (W^{\perp})^{\perp}$. It is now easy to see using the definition that $W \subset (W^{\perp})^{\perp}$. Since they have the same dimension, we must have equality. □

We will later see how to prove the above theorem in the case that W is finite dimensional without the need to assume that V is finite dimensional as well. Our next step is to classify forms on finite dimensional vector spaces where possible. The following result is essential in the case of φ symmetric or Hermitian.

Lemma 5.2.21. *Let (V, φ) be an F-vector space and assume that φ is nondegenerate. If $\text{char}(F) \neq 2$, assume φ is symmetric or Hermitian. If $\text{char}(F) = 2$, assume φ is Hermitian. Then there is a vector $v \in V$ with $\varphi(v, v) \neq 0$.*

Proof. Let $v_1 \in V$ with $v_1 \neq 0$. If $\varphi(v_1, v_1) \neq 0$ we are done, so assume $\varphi(v_1, v_1) = 0$. The fact that φ is nondegenerate gives a nonzero $v_2 \in V$

so that $\varphi(v_1, v_2) \neq 0$. Set $b = \varphi(v_1, v_2)$. If $\varphi(v_2, v_2) \neq 0$ we are done, so assume $\varphi(v_2, v_2) = 0$. Set $v_3 = tv_1 + v_2$ where $t \in F$. Then we have

$$\begin{aligned} \varphi(v_3, v_3) &= \varphi(tv_1 + v_2, tv_1 + v_2) \\ &= t\overline{t}\varphi(v_1, v_1) + t\varphi(v_1, v_2) + \overline{t}\varphi(v_2, v_1) + \varphi(v_2, v_2) \\ &= tb + \overline{t}\overline{b} \end{aligned}$$

where $\overline{t} = t$ and $\overline{b} = b$ if φ is symmetric. Thus, if V is symmetric $\varphi(v_3, v_3) = 2tb$, so choose any $t \neq 0$ and we are done. Now suppose that φ is Hermitian so $\varphi(v_3, v_3) = tb + \overline{tb}$. Set $t = 1/b$ so $tb = 1$. We claim $\overline{1} = 1$. To see this, observe $\overline{1} = \overline{1 \cdot 1} = \overline{1} \cdot \overline{1}$, so $\overline{1} = 1$. Thus, $\varphi(v_3, v_3) = 2 \neq 0$ as long as $\mathrm{char}(F) \neq 2$. Now suppose that $\mathrm{char}(F) = 2$. We want to set $t = a/b$ where $a \neq \overline{a}, -\overline{a}$. Using that the conjugation is non-trivial, we know there exists $j \in F$ so that $\overline{j} = -j$ and every element of F can be written uniquely in the form $a = \alpha + j\beta$ with $\alpha, \beta \in F_0$. If $\beta \neq 0$ then $a \neq \overline{a}$ and if $\alpha \neq 0$ then $a \neq -\overline{a}$. Thus, just pick a to be any element with $\alpha\beta \neq 0$. Set $t = a/b$ so that $\varphi(v_3, v_3) = a + \overline{a} \neq 0$. Thus, we have the result in all cases. □

Suppose we have a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ for (V, φ) so that $V_i = \mathrm{span}_F\{v_i\}$ satisfies

$$V = V_1 \perp \cdots \perp V_n.$$

Observe this gives

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

where $a_i = \varphi(v_i, v_i)$. This leads to the following definition.

Definition 5.2.22. Let φ be a symmetric bilinear form or a Hermitian form on a finite dimensional vector space V. We say φ is *diagonalizable* if there are 1-dimensional subspaces $V_1, \dots, V_n$ so that $V = V_1 \perp \cdots \perp V_n$.

Let $a \in F$ with $a \neq 0$. We can define a bilinear form on F by considering $a \in \mathrm{Mat}_1(F)$, i.e., we define $\varphi_a(x, y) = xay$. Here we get φ_a is symmetric for free. Similarly, if F has nontrivial conjugation we can define a sesquilinear form φ_a by setting $\varphi_a(x, y) = xa\overline{y}$. This is not Hermitian for free. In particular, we have

$$\begin{aligned} \varphi_a(x, y) &= xa\overline{y} \\ &= \overline{y}a\overline{\overline{x}}. \end{aligned}$$

This is equal to $\overline{\varphi_a(y, x)}$ if and only if $\overline{a} = a$. Thus, we have φ_a is Hermitian if and only if $\overline{a} = a$. In either case, we write $[a]$ for the space (F, φ_a). This set-up allows us to rephrase the definition of diagonalizable to say φ is diagonalizable if there exists $a_1, \dots, a_n$ so that φ is isometric to $[a_1] \perp \cdots \perp [a_n]$.

Theorem 5.2.23. *Let (V, φ) be a finite dimensional vector space over a field F. If $\mathrm{char}(F) \neq 2$ we assume φ is symmetric or Hermitian. If $\mathrm{char}(F) = 2$, we require φ to be Hermitian. Then φ is diagonalizable.*

Proof. We immediately reduce to the case that φ is nondegenerate by recalling that there is a subspace V_1 so that $\varphi|_{V_1 \times V_1}$ is nondegenerate and $V = \ker(\varphi) \perp V_1$. Thus, if $\varphi|_{V_1 \times V_1}$ is diagonalizable, then certainly φ is as well since the restriction to $\ker(\varphi)$ is just 0.

We now assume φ is nondegenerate and induct on the dimension of V. The case that $\dim_F V = 1$ is trivial. Assume the result is true for all vector spaces of dimension less than n and let $\dim_F V = n$. We have a $v_1 \in V$ so that $\varphi(v_1, v_1) \neq 0$ by Lemma 5.2.21. Set $a_1 = \varphi(v_1, v_1)$ and let $V_1 = \mathrm{span}_F\{v_1\}$. Since $\varphi|_{V_1}$ is nondegenerate we have $V = V_1 \perp V_1^{\perp}$. However, this gives $\dim_F V_1^{\perp} = n-1$ and Lemma 5.2.18 gives that $\varphi_1 := \varphi|_{V_1^{\perp} \times V_1^{\perp}}$ is nondegenerate. We apply the induction hypothesis to $(V_1^{\perp}, \varphi_1)$ to obtain one dimensional subspaces $V_2, \dots, V_n$ so that $V_1^{\perp} = V_2 \perp \cdots \perp V_n$. Thus, $V = V_1 \perp V_2 \perp \cdots \perp V_n$ and we are done. □

We can use this theorem to give our first classification theorem. The following theorem is usually stated strictly over algebraically closed fields, but we state it a little more generally as algebraically closed is not required.

Theorem 5.2.24. *Let (V, φ) be an n-dimensional F-vector space with φ a nondegenerate symmetric bilinear form. Suppose that F satisfies that $\mathrm{char}(F) \neq 2$ and it is closed under square-roots, namely, given any $a \in F$, one has $f(x) = x^2 - a \in F[x]$ has a root in F. (In particular, if F is algebraically closed this is certainly the case.) Then φ is isometric to $[1] \perp \cdots \perp [1]$. In particular, all nondegenerate symmetric bilinear forms over such a field are isometric.*

Proof. The previous theorem gives one dimensional subspaces $V_1, \dots, V_n$ so that $V = V_1 \perp \cdots \perp V_n$. Let $\{v_i\}$ be a basis for V_i and set $a_i = \varphi(v_i, v_i)$. We choose v_i so that $a_i \neq 0$. By our assumption on F we have an element $b_i \in F$ so that b_i is a root of $f(x) = x^2 - 1/a_i$, i.e., $b_i^2 = 1/a_i$. Set $\mathcal{B} = \{b_1 v_1, \dots, b_n v_n\}$. Then we have

$$\begin{aligned} \varphi(b_i v_i, b_i v_i) &= b_i^2 \varphi(v_i, v_i) \\ &= b_i^2 a_i \\ &= 1. \end{aligned}$$

This gives the result. □

In the case that one has φ is isometric to $[1] \perp \cdots \perp [1]$ with n copies of $[1]$, we will write $\varphi \simeq n[1]$.

Recall that the isometry group of a symmetric bilinear form φ was denoted $\mathrm{O}(\varphi)$. Consider now the case that (V, φ) is defined over a field F that is closed under square-roots. Then we have just seen there is a basis $\mathcal{B}$ so that $[\varphi]_{\mathcal{B}} = 1_n$ where 1_n is the $n \times n$ identity matrix. Using this basis we can represent the orthogonal group as

$$\begin{aligned} \mathrm{O}_n(\varphi) &= \{M \in \mathrm{GL}_n(F) : {}^tM 1_n M = 1_n\} \\ &= \{M \in \mathrm{GL}_n(F) : {}^tM = M^{-1}\}. \end{aligned}$$

We refer to matrices $M \in \mathrm{O}_n(F)$ as *orthogonal matrices.* One important point to note here is that we saw before that change of basis matrices for bilinear and sesquilinear forms correspond to isometries. Thus, in this case we see that change of basis matrices for non-degenerate symmetric bilinear forms correspond to invertible matrices M satisfying ${}^tM = M^{-1}$. This shows that changing bases for such forms is equivalent to changing the basis of a linear transformation. In other words, one can apply all the nice results of the previous chapter to the matrices of non-degenerate symmetric bilinear forms.

Note that the proof of Theorem 5.2.24 does not work for φ a Hermitian form even if F is algebraically closed because in order to scale the v_i we would need b_i so that $b_i\bar{b}_i = a_i$. However, this equation cannot be set up as a polynomial equation and so algebraically closed does not guarantee a solution to such an equation exists. To push our classifications further we need to restrict the fields of interest some. In general one needs an ordered field. However, to save the trouble of introducing ordered fields we will consider our fields F to satisfy $\mathbb{Q} \subset F \subset \mathbb{C}$. In addition, if we wish to consider symmetric forms, we will require $F \subset \mathbb{R}$. If we wish to consider Hermitian forms, we only require F has nontrivial conjugation. One important point to note here is that with our set-up, if φ is symmetric we certainly have $\varphi(v,v) \in \mathbb{R}$ for all $v \in V$. However, if φ is Hermitian we have for any $v \in V$ that $\varphi(v,v) \in \mathbb{C}$ satisfies $\varphi(v,v) = \overline{\varphi(v,v)}$, i.e., $\varphi(v,v) \in \mathbb{R}$ in this case as well. This allows us to make the following definition.

Definition 5.2.25. Let (V,φ) be an F-vector space so that if φ is symmetric, $\mathbb{Q} \subset F \subset \mathbb{R}$ and if φ is Hermitian $\mathbb{Q} \subset F \subset \mathbb{C}$. We say φ is *positive definite* if $\varphi(v,v) > 0$ for all nonzero $v \in V$ and φ is *negative definite* if $\varphi(v,v) < 0$ for all nonzero $v \in V$. We say φ is *indefinite* if there are vectors $v, w \in V$ so that $\varphi(v,v) > 0$ and $\varphi(w,w) < 0$.

Definition 5.2.26. We say a matrix $A \in \mathrm{Mat}_n(F)$ is *positive definite* if the corresponding form φ is positive definite. Likewise, we say A is *negative definite* if the associated form is negative definite.

With these definitions we give the next classification theorem.

Theorem 5.2.27 (Sylvestor's Law of Inertia)**.** *Let (V,φ) be a finite dimensional $\mathbb{R}$-vector space with φ a nondegenerate symmetric bilinear form on V or let (V,φ) be a $\mathbb{C}$-vector space with φ a nondegenerate Hermitian* *form on V. Then φ is isometric to $p[1] \perp q[-1]$ for well-defined integers p, q with $p + q = \dim_F V$.*

Proof. The proof that φ is isometric to $p[1] \perp q[-1]$ follows the same argument as used in the proof of Theorem 5.2.24. We begin with the case that (V, φ) is a $\mathbb{R}$-vector space and φ is a nondegenerate symmetric bilinear form. As in the proof of Theorem 5.2.24, we have one-dimensional subspaces $V_1, \ldots, V_n$ so that $V_i = \mathrm{span}_{\mathbb{R}}\{v_i\}$ and $V = V_1 \perp \cdots \perp V_n$. Write $a_i = \varphi(v_i, v_i)$. Let $b_i = 1/\sqrt{|a_i|} \in \mathbb{R}$. Then $b_i^2 = 1/|a_i|$ and so

$$\begin{aligned}\varphi(b_i v_i, b_i v_i) &= b_i^2 \varphi(v_i, v_i)\\ &= \frac{a_i}{|a_i|}\\ &= \mathrm{sign}(a_i)\end{aligned}$$

where $\mathrm{sign}(a_i) = 1$ if $a_i > 0$ and $\mathrm{sign}(a_i) = -1$ if $a_i < 0$. Thus, if we let p be the number of positive a_i and q the number of negative a_i we get $\varphi \simeq p[1] \perp q[-1]$.

Now suppose (V, φ) is a $\mathbb{C}$-vector space and φ is a nondegenerate Hermitian form. Given any $v \in V$ and $b \in \mathbb{C}$ we have $\varphi(bv, bv) = b\bar{b}\varphi(v, v) = |b|^2\varphi(v, v)$. Proceeding as above, this shows that for each j we can scale $\varphi(v_j, v_j)$ by any positive real number. Since $\varphi(v_j, v_j) \in \mathbb{R}$ for all j, we again have $\varphi \simeq p[1] \perp q[-1]$ where p and q are defined as above.

It remains to show that p and q are invariants of φ and do not depend on the choice of basis. Let V_+ be the largest subspace of V so that the restriction of φ to V_+ is positive definite and V_- the largest subspace of V so that the restriction of φ to V_- is negative definite. Set $p_0 = \dim_F V_+$ and $q_0 = \dim_F V_-$. Note that p_0 and q_0 are well-defined and do not depend on any choices. We now show $p = p_0$ and $q = q_0$.

Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V so that $[\varphi]_{\mathcal{B}} = p[1] \perp q[-1]$. Set $\mathcal{B}_+ = \{v_1, \ldots, v_p\}$ and $\mathcal{B}_- = \{v_{p+1}, \ldots, v_n\}$. Let $W_+ = \mathrm{span}_F \mathcal{B}_+$ and $W_- = \mathrm{span}_F \mathcal{B}_-$. Then we have φ restricted to W_+ is positive definite so $p \leq p_0$ and similarly we obtain $q \leq q_0$. Note that this gives $n = p + q \leq p_0 + q_0$. Suppose $p \neq p_0$. Then $\dim_F V_+ + \dim_F V_- > n$, so $V_+ \cap V_- \neq \{0\}$. This is easily seen to be a contradiction, so $p = p_0$ and the same argument gives $q = q_0$, completing the proof. □

One important point to note is all that was really used in the above proof was that given any positive number $|a_i|$, one had $1/\sqrt{|a_i|}$ in F. Thus, we do not need to use $F = \mathbb{R}$ or $F = \mathbb{C}$ in the theorem. For example, if (V, φ) is an F-vector space withφ is a symmetric bilinear form and $F \subset \mathbb{R}$, to apply the result to (V, φ) we only require that $\sqrt{|a|} \in F$ for every $a \in F$. If the field does not contain all its positive square roots things are more difficult as we will see below.

Definition 5.2.28. Let $(V, \varphi), p$, and q be as in the previous theorem. The *signature* of φ is (p, q).

Corollary 5.2.29. *A nondegenerate symmetric bilinear form on a finite dimensional $\mathbb{R}$-vector space or a nondegenerate Hermitian form on a finite dimensional $\mathbb{C}$-vector space is classified up to isometry by its signature. If φ is not required to be nondegenerate, it is classified by its signature and its rank.*

Proof. This follows immediately from the previous theorem with the exception of the degenerate case. However, if we do not require φ to be nondegenerate, then write $V = \ker(\varphi) \perp V_1$ and apply Sylvestor's law to V_1 and we obtain the result. □

If we allow φ to be degenerate, the previous result gives $\varphi = p[1] \perp q[-1] \perp r[0]$ for $r = n - p - q$.

We again briefly return to the isometry groups. Let φ be a nondegenerate symmetric bilinear form on a real vector space V of signature (p, q) so $\varphi \simeq p[1] \perp q[-1]$. Let $1_{p,q}$ be given by

$$1_{p,q} = \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix}.$$

Then the isometry group of φ is given by

$$\mathrm{O}_{p,q}(\mathbb{R}) := \mathrm{O}(\varphi) = \{M \in \mathrm{GL}_{p+q}(\mathbb{R}) : {}^tM 1_{p,q} M = 1_{p,q}\}.$$

In the case that $p = n = \dim_{\mathbb{R}} V$ and $q = 0$ we recover the case given above for a positive definite symmetric bilinear form.

Let φ be a nondegenerate Hermitian form on a complex vector space V of signature (p, q) so $\varphi \simeq p[1] \perp q[-1]$. The isometry group in this case is given by

$$\mathrm{U}_{p,q}(\mathbb{C}) := U(\varphi) = \{M \in \mathrm{GL}_{p+q}(\mathbb{C}) : {}^tM 1_{p,q} \overline{M} = 1_{p,q}\}.$$

One will also often see this group denoted at $\mathrm{U}(p, q)$ when $\mathbb{C}$ is clear from context. In the case $p = n = \dim_{\mathbb{C}} V$ and $q = 0$, we have

$$\mathrm{U}_n(\mathbb{C}) := \mathrm{U}_{n,0}(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) : {}^tM 1_n \overline{M} = 1_n\}.$$

We now briefly address the situation of a nondegenerate symmetric bilinear or Hermitian form on a vector space over a field $\mathbb{Q} \subset F \subset \mathbb{C}$ that is not closed under taking square roots of positive numbers. (If F is not a subset of $\mathbb{R}$, we realize the "positive" numbers in F as $F \cap \mathbb{R}_{>0}$. Note by requiring $F \subset \mathbb{C}$ this makes sense.) Let φ be such a form. Theorem 5.2.23 gives elements $a_1, \ldots, a_n$ so that $\varphi \simeq [a_1] \perp \cdots \perp [a_n]$. We know that each a_i can be scaled by any element b_i^2 for $b_i \in F$. This gives a nice diagonal representation for such forms. Let $F^\times = F - \{0\}$ and set $(F^\times)^2 = \{b^2 : b \in F^\times\}$. The above discussion may lead one to believe that there is a bijection between nondegenerate symmetric bilinear forms on an n-dimensional F-vector space V and $(F^\times/(F^\times)^2)^n$. However, the following example shows this does not work even in the case $F = \mathbb{Q}$, namely, there is not a well-defined map from the collection of nondegenerate symmetric bilinear forms to $(F^\times/(F^\times)^2)^n$. One can relate the study of such forms to the classification of quadratic forms, but one quickly sees it is a very difficult problem in this level of generality.

Example 5.2.30. Consider the symmetric bilinear form φ on $\mathbb{Q}^2$ given by

$$\varphi(x, y) = {}^t x \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} y.$$

Let $P = \begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix}$. Then we have

$$\begin{pmatrix} 5 & 0 \\ 0 & 30 \end{pmatrix} = {}^t P \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} P,$$

and so we have φ is isometric to $\begin{pmatrix} 5 & 0 \\ 0 & 30 \end{pmatrix}$, but the diagonal entries of this do not differ from 2 and 3 by squares.

We next classify skew-symmetric forms.

Theorem 5.2.31. *Let (V, φ) be a finite dimensional vector space over a field F and let φ be a skew-symmetric form. Then $n = \dim_F V$ must be even and φ is isometric to $n/2$ copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, i.e., there is a basis $\mathcal{B}$ so that*

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

Proof. We use induction on $n = \dim_F V$. Let $n = 1$ and $\mathcal{B} = \{v_1\}$ be a basis for V. Since φ is skew-symmetric we have $\varphi(v_1, v_1) = 0$, which gives $[\varphi]_{\mathcal{B}} = [0]$. This contradicts φ being nondegenerate. Since the $n = 1$ case is reasonably trivial, we also show the $n = 2$ case as it is more illustrative of the general case. Let $v_1 \in V$ with $v_1 \neq 0$. Since φ is nondegenerate there is a $w \in V$ so that $\varphi(w, v_1) \neq 0$. Since φ is skew-symmetric w cannot be a multiple of v_1 and so $\dim \operatorname{span}_F\{v_1, w\} = 2$. Set $a = \varphi(w, v_1)$ and $v_2 = w/a$. Set $\mathcal{B}_1 = \{v_1, v_2\}$ and $V_1 = \operatorname{span}_F\{v_1, v_2\}$. Then we have

$$[\varphi|_{V_1}]_{\mathcal{B}_1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now suppose the theorem is true for all vector spaces of dimension less than n. Let $v_1 \in V$ with $v_1 \neq 0$. As in the $n = 2$ case we construct a v_2 and $V_1 = \operatorname{span}_F\{v_1, v_2\}$ so that

$$[\varphi|_{V_1}]_{\mathcal{B}_1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This shows $\varphi|_{V_1}$ is nondegenerate, so $V = V_1 \perp V_1^{\perp}$. We now apply the induction hypothesis to $V_1^{\perp}$. Suppose that n is odd and φ is a nondegenerate Hermitian form on V. Then $\varphi|_{V_1^{\perp}}$ is a nondegenerate Hermitian form on a vector space of dimension $n - 2$. Since $n - 2$ is odd, the induction hypothesis gives there are no nondegenerate Hermitian forms on a vector space of dimension $n-2$ if it is odd. This contradiction shows there are no nondegenerate Hermitian forms on V if n is odd. Now suppose n is even, then since $\dim_F V_1^{\perp} = n - 2$ is even we have a basis $\mathcal{B}_2 = \{v_3, \ldots, v_n\}$ of $V_1^{\perp}$ with

$$[\varphi|_{V_1^{\perp}}]_{\mathcal{B}_2} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

where there are $(n-2)/2$ copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the diagonal. The basis $\mathcal{B} = \{v_1, v_2, v_3, \ldots, v_n\}$ now gives the result. □

Exercise 5.2.32. One often sees the isometry group for a skew-symmetric bilinear form represented by a different matrix. Namely, let (V, φ) be an F-vector space of dimension $2n$ with φ a skew-symmetric bilinear form.

(a) Show that there is a basis $\mathcal{B}_1$ so that

$$[\varphi]_{\mathcal{B}_1} = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}.$$

(b) Show that there is a basis $\mathcal{B}_2$ so that

$$[\varphi]_{\mathcal{B}_2} = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}$$

Using the previous exercise, we can write

$$\operatorname{Sp}_{2n}(F) = \left\{ M \in \operatorname{GL}_{2n}(F) : {}^tM \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} M = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \right\}.$$

The last case for us to classify is the case of skew-Hermitian forms. Note that one can obtain the conclusion below simply by using the fact that if φ is skew-symmetric, then $j\varphi$ is Hermitian and then applying the earlier results on Hermitian forms. However, we give a direct proof here to illustrate the methods one last time.

Theorem 5.2.33. *Let (V, φ) be an F-vector space with φ a nondegenerate skew-Hermitian form. Then φ is isometric to $[a_1] \perp \cdots \perp [a_n]$ for some $a_i \in F$ with $a_i \neq 0$, $\overline{a}_i = -a_i$. (Note that $\overline{a}_i = -a_i$ is equivalent to $a_i = jb_i$ for some $b_i \in F_0$.)*

Proof. Our first step is to show there is a $v \in V$ so that $\varphi(v, v) \neq 0$. Let $v_1 \in V$ be any nonzero element. If $\varphi(v_1, v_1) \neq 0$, set $v = v_1$. Otherwise, choose $v_2 \in V$ so that $\varphi(v_2, v_1) \neq 0$. Such a v_2 exists because φ is nondegenerate. Set $a = \varphi(v_1, v_2)$. If $\varphi(v_2, v_2) \neq 0$, set $v = v_2$. Otherwise, for any $c \in F$ set $v_3 = v_1 + \overline{c}v_2$. We have

$$\begin{aligned}\varphi(v_3, v_3) &= \varphi(v_1, v_1) + \varphi(v_1, \overline{c}v_2) + \varphi(\overline{c}v_2, v_1) + \varphi(\overline{c}v_2, \overline{c}v_2)\\ &= c\varphi(v_1, v_2) + \overline{c}\varphi(v_2, v_1)\\ &= ac - \overline{ac}.\end{aligned}$$

Set $c = j/a$. Then we have

$$\varphi(v_3, v_3) = j - \overline{j} = 2j \neq 0.$$

We now proceed by induction on the dimension of V. If $\dim_F V = 1$, we construct v_3 as above and we have φ is isometric to $[2j]$. Now assume the result is true for all vector spaces of dimension less than n. Let $v_1 \in V$ be a vector so that $\varphi(v_1, v_1) \neq 0$ as above. Set $V_1 = \operatorname{span}_F\{v_1\}$. We have $\varphi|_{V_1 \times V_1} \neq 0$, so $\varphi|_{V_1 \times V_1}$ is nondegenerate so we have $V = V_1 \perp V_1^{\perp}$. We now apply the induction hypothesis to $V_1^{\perp}$. This gives the result. □

Exercise 5.2.34. Let (V, φ) be an $\mathbb{C}$-vector space and φ a nondegenerate skew-Hermitian form. Then φ is isometric to $p[i] \perp q[-i]$ for some well-defined integers p and q with $p + q = \dim_{\mathbb{C}} V$.

5.3 The adjoint map

The last topic of this chapter is the adjoint map. Let (V, φ) and (W, ψ) be F-vector spaces with φ and ψ either both non-degenerate bilinear or both non-degenerate sesquilinear forms. (Throughout this section when given φ and ψ, we always assume they are non-degenerate and are both bilinear or both sesquilinear.) Let $T \in \operatorname{Hom}_F(V, W)$. The adjoint map is a linear map $T^* \in \operatorname{Hom}_F(W, V)$ that behaves nicely with respect to φ and ψ.

Definition 5.3.1. Let $T \in \mathrm{Hom}_F(V, W)$. The *adjoint map* is a map $T^* : W \to V$ satisfying

$$\varphi(v, T^*(w)) = \psi(T(v), w)$$

for all $v \in V$, $w \in W$.

It is important to note here that many books will use T^* to denote the dual map $T^\vee$. Be careful not to confuse the adjoint T^* with the dual map! We will see why books use this notation below.

The first step is to show that an adjoint map actually exists when V and W are finite dimensional.

Proposition 5.3.2. *Let (V, φ) and (W, ψ) be finite dimensional vector spaces with φ and ψ bilinear forms. Then there is an adjoint map $T^* \in \mathrm{Hom}_F(W, V)$.*

Proof. We begin by showing there is a map T^* that satisfies the equation. We then show it is unique and linear. Recall that given any $\Phi \in W^\vee$, we have $T^\vee \circ \Phi \in V^\vee$ defined by $T^\vee \circ \Phi(v) = \Phi(T(v))$ for $v \in V$. Fix $w \in W$ so we have $\varphi(\cdot, w) = R_\psi(w) \in W^\vee$. Then $T^\vee \circ R_\psi(w) \in V^\vee$ and is given by $(T^\vee \circ R_\psi(w))(v) = R_\psi(w)(T(v)) = \psi(T(v), w)$. We now use that φ is non-degenerate, i.e., $R_\varphi : V \to V^\vee$ is an isomorphism to conclude that for each $w \in W$ there exists a unique element $z_w \in V$ so that $R_\varphi(z_w) = T^\vee \circ R_\psi(w)$, i.e., for each $w \in W$ and $v \in V$ we have

$$\begin{aligned} \varphi(v, z_w) &= R_\varphi(z_w)(v) \\ &= (T^\vee \circ R_\psi(w))(v) \\ &= R_\psi(w)(T(v)) \\ &= \psi(T(v), w). \end{aligned}$$

Thus, we have a map $T^* : W \to V$ defined by sending w to z_w. Moreover, from the construction it is clear the map is unique.

One could easily show this map is linear directly, but we present a different proof that is conceptually more useful for later results. Suppose that φ and ψ are bilinear. We just saw the defining formula for T^* can be rewritten as

$$R_\varphi(T^*(w))(v) = T^\vee(R_\psi(w))(v).$$

Thus, $R_\varphi \circ T^* = T^\vee \circ R_\psi$. Since φ is assumed to be non-degenerate, R_φ is an isomorphism so R_φ^{-1} exists and we can write

$$T^* = R_\varphi^{-1} \circ T^\vee \circ R_\psi.$$

Since each term on the right hand side is linear, so is T^*. This finishes the proof of the result in the case the forms are bilinear. □

We immediately obtain the following version in coordinates.

Corollary 5.3.3. *Let $\mathcal{B}$ be a basis of (V, φ) and $\mathcal{C}$ be a basis of (W, ψ). Set $P = [\varphi]_{\mathcal{B}}$ and $Q = [\psi]_{\mathcal{C}}$. Then*

$$[T^*]^{\mathcal{B}}_{\mathcal{C}} = P^{-1}\, {}^t[T]^{\mathcal{C}}_{\mathcal{B}} Q$$

if φ and ψ are bilinear.

One now needs the same results for sesquilinear forms. If one proceeds as above the map T^* that one constructs is a map from $\overline{W}$ to $\overline{V}$; the claim is there is a map from W to V. We remedy this by introducing the conjugate of a linear map. As this is much easier to see in coordinates, we do that case first. Suppose we are given a linear map $T : \overline{V} \to \overline{W}$. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V and $\mathcal{C} = \{w_1, \ldots, w_m\}$ be a basis of W. (Note these are also bases of V and W.) To obtain the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$ we write

$$T(v_i) = a_{1i} \cdot w_1 + \cdots + a_{mi} \cdot w_m.$$

Thus, the matrix of $T : \overline{V} \to \overline{W}$ is given by $A = (a_{ij}) \in \mathrm{Mat}_{m,n}(F)$ where we view the matrices as acting on $\overline{F^n}$. However, if we wish to view this as a linear map from V to W we have

$$T(v_i) = \overline{a_{1i}} w_1 + \cdots + \overline{a_{mi}} w_m,$$

so the correct matrix in this case is $\overline{A} = (\overline{a_{ij}}) \in \mathrm{Mat}_{m,n}(F)$ where we view the matrix as acting on F^n. Thus, given a linear transformation $T : \overline{V} \to \overline{W}$ with associated matrix $A = [T]^{\mathcal{C}}_{\mathcal{B}}$, we obtain a linear transformation $\overline{T}$ given in coordinates by $[\overline{T}]^{\mathcal{C}}_{\mathcal{B}} = \overline{A}$. We can now remedy the above situation by setting $T^* = \overline{R_\varphi}^{-1} \circ \overline{T^\vee} \circ \overline{R_\psi}$. This finishes the proof of Proposition 5.3.2 and gives the following result.

Corollary 5.3.4. *Let $\mathcal{B}$ be a basis of (V, φ) and $\mathcal{C}$ be a basis of (W, ψ). Set $P = [\varphi]_{\mathcal{B}}$ and $Q = [\psi]_{\mathcal{C}}$. Then*

$$[T^*]^{\mathcal{B}}_{\mathcal{C}} = \overline{P}^{\,-1}\, \overline{{}^t[T]^{\mathcal{C}}_{\mathcal{B}}}\,\overline{Q}$$

if φ and ψ are sesquilinear.

Note the above proof fails completely when V and W are not finite dimensional vector spaces. One no longer has that R_φ and R_ψ are isomorphisms. In fact, in the infinite dimensional case it often happens that the adjoint map does not exist! One particularly important case where one knows adjoints exist in the infinite dimensional case is for bounded linear maps on Hilbert spaces. This follows from the Riesz Representation Theorem. We do not expand on this here as it takes us too far afield.

One special case of the above corollaries is when $V = W$ and $\varphi = \psi$. In this case one has $P = Q$. We can take this further with the following definition and result. These concepts will be developed more fully in the next chapter.

Definition 5.3.5. Let (V,φ) be a vector space. A basis $\mathcal{B} = \{v_i\}$ is said to be *orthogonal* if $\varphi(v_i, v_j) = 0$ for all $v_i, v_j \in \mathcal{B}$ with $i \neq j$. We say the basis is *orthonormal* if it is orthogonal and satisfies $\varphi(v_i, v_i) = 1$ for all $v_i \in \mathcal{B}$.

In general there is no reason an arbitrary vector space should have an orthonormal basis. However, if one has an orthonormal basis one obtains some very nice properties. One immediate property is the following corollary.

Corollary 5.3.6. *Let (V,φ) and (W,ψ) be finite dimensional vector spaces with orthonormal bases $\mathcal{B}$ and $\mathcal{C}$ respectively. Let $T \in \operatorname{Hom}_F(V,W)$. Then*

$$[T^*]_{\mathcal{C}}^{\mathcal{B}} = {}^t[T]_{\mathcal{B}}^{\mathcal{C}}$$

if φ and ψ are bilinear and

$$[T^*]_{\mathcal{C}}^{\mathcal{B}} = \overline{{}^t[T]_{\mathcal{B}}^{\mathcal{C}}}$$

if φ and ψ are sesquilinear.

Proof. This follows from Proposition 5.3.2 upon observing that P and Q are both the identity matrix due to the fact that $\mathcal{B}$ and $\mathcal{C}$ are orthonormal. □

Suppose we have a finite dimensional vector space (V,φ) with φ a non-degenerate symmetric bilinear form. Suppose there are orthonormal bases $\mathcal{B} = \{v_1,\ldots,v_n\}$ and $\mathcal{C} = \{w_1,\ldots,w_n\}$ of V. Let $T \in \operatorname{Hom}_F(V,W)$ be defined by $T(v_i) = w_i$. Observe we have

$$\begin{aligned}\varphi(v_i, v_j) &= \delta_{i,j}\\ &= \varphi(w_i, w_j)\\ &= \varphi(T(v_i), T(v_j))\end{aligned}$$

where $\delta_{ij} = \left\{\begin{array}{cc} 0 & i \neq j \\ 1 & i = j\end{array}\right.$. We immediately obtain that $\varphi(v,\widetilde{v}) = \varphi(T(v), T(\widetilde{v}))$ for all $v, \widetilde{v} \in V$. In particular, we have $\varphi(v,\widetilde{v}) = \varphi(v, T^*(T(\widetilde{v})))$ for all $v, w \in V$, i.e., $\varphi(v, \widetilde{v} - T^*(T(\widetilde{v}))) = 0$ for all $v,\widetilde{v} \in V$. Since φ is non-degenerate, this gives $T^* \circ T = \mathrm{id}_V$, i.e., if we set $P = [T]_{\mathcal{B}}^{\mathcal{C}}$, then ${}^tPP = 1_n$. Thus, if P is a change of basis matrix between two orthonormal bases then P is an orthogonal matrix. The same argument gives that if φ is a Hermitian form, then a change of basis matrix between two orthonormal bases is a unitary matrix.

Exercise 5.3.7. Let $\mathcal{B} = \{v_1,\ldots,v_n\}$ be an orthonormal basis of a finite dimensional vector space (V,φ) with φ a non-degenerate symmetric bilinear form. Let $T \in \operatorname{Hom}_F(V,V)$ and let $P = [T]_{\mathcal{B}}$. Show that if P is an orthogonal matrix then $\mathcal{C} = \{T(v_1),\ldots,T(v_n)\}$ is an orthonormal basis of V. Prove the analogous result when φ is Hermitian as well.

Exercise 5.3.8. Let (U, ϕ), (V, φ), and (W, ψ) be finite dimensional vector spaces. Prove the following elementary results about the adjoint map.

(a) $S, T \in \mathrm{Hom}_F(V, W)$. Show that $(S+T)^* = S^* + T^*$.

(b) $(cT)^* = \overline{c}T^*$ for every $c \in F$

(c) Let $T \in \mathrm{Hom}_F(U, V)$ and $S \in \mathrm{Hom}_F(V, W)$. Then $(S \circ T)^* = T^* \circ S^*$.

(d) Let $p(x) = a_n x^n + \cdots + a_1 x + a_0 \in F[x]$ and $T \in \mathrm{Hom}_F(V, W)$. Then $(p(T))^* = \overline{p}(T^*)$ where $\overline{p}(x) = \overline{a}_n x^n + \cdots + \overline{a}_1 x + \overline{a}_0$.

Corollary 5.3.9. *Let (V, φ) and (W, ψ) be vector spaces with φ and ψ both symmetric, skew-symmetric, Hermitian, or skew-Hermitian. Then $(T^*)^* = T$.*

Proof. We prove the Hermitian case and leave the others as exercises as the proofs are analogous. Set $S = T^*$ so that

$$\varphi(v, S(w)) = \psi(T(v), w)$$

for all $v \in V$, $w \in W$. We have

$$\begin{aligned}\varphi(S(w), v) &= \overline{\varphi(v, S(w))}\\ &= \overline{\psi(T(v), w)}\\ &= \psi(w, T(v))\end{aligned}$$

for all $v \in V$, $w \in W$. This shows we must have $T = S^*$ since the adjoint is unique and T satisfies the properties of being the adjoint of S. □

Note the above proof works for infinite dimensional spaces as well with the added assumption that the adjoint exists.

5.4 Problems

For these problems we assume V is a finite dimensional vector space over a field F.

(a) Let V_1 and V_2 be subspaces of V. Show that $V = V_1 \perp V_2$ if

(i) $V = V_1 \oplus V_2$ and

(ii) given any $v, v' \in V$, when we write $v = v_1 + v_2$ and $v' = v_1' + v_2'$ for $v_i, v_i' \in V_i$ we have

$$\varphi(v, v') = \varphi_1(v_1, v_1') + \varphi_2(v_2, v_2')$$

where $\varphi_i = \varphi|_{V_i}$.

(b) Let φ be a bilinear form on V and assume $\mathrm{char}(F) \neq 2$. Prove that B is skew-symmetric if and only if the diagonal function $V \to F$ given by $v \mapsto \varphi(v, v)$ is additive.

(c) Let D be an integer so that $\sqrt{D} \notin \mathbb{Z}$. Let $V = \mathbb{R}^2$.

(a) Show that $\varphi_D(x, y) = {}^t x \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} y$ is a bilinear form on V.

(b) Given two such integers D_1, D_2, give necessary and sufficient conditions for φ_{D_1} to be isometric to φ_{D_2}?

(c) Suppose now that $V = \mathbb{Q}^2$. Under what condition is φ_{D_1} isometric to φ_{D_2}?

(d) Let $V = \mathbb{R}^2$. Set $\varphi((x_1, y_1), (x_2, y_2)) = x_1 x_2$.

(a) Show this is a bilinear form. Give a matrix representing this form. Is this form nondegenerate?

(b) Let $W = \mathrm{span}_{\mathbb{R}}(e_1)$ where e_1 is the standard basis element. Show that $V = W \perp W^{\perp}$.

(c) Calculate $(W^{\perp})^{\perp}$.

(e) (a) Let $A \in \mathrm{Mat}_n(\mathbb{C})$. Prove that there is a unique Hermitian matrix $H \in \mathrm{Mat}_n(\mathbb{C})$ and a unique skew-Hermitian matrix $S \in \mathrm{Mat}_n(\mathbb{C})$ so that $A = H + S$. This is referred to as the Hermitian decomposition of A.

(b) Find the Hermitian decomposition of $A = \begin{pmatrix} 2i & 1+3i \\ -1+3i & -5i \end{pmatrix}$.

(f) Let $A \in \mathrm{Mat}_n(\mathbb{C})$ be a Hermitian matrix. Prove all the eigenvalues of A must be real.

(g) Let V be a 3-dimensional $\mathbb{Q}$-vector space with basis $\mathcal{B} = \{v_1, v_2, v_3\}$. Define a symmetric bilinear form on V by setting $\varphi(v_1, v_1) = 0, \varphi(v_1, v_2) = -2, \varphi(v_1, v_3) = 2, \varphi(v_2, v_2) = 2, \varphi(v_2, v_3) = -2, \varphi(v_3, v_3) = 3$.

(a) Give the matrix $[\varphi]_{\mathcal{B}}$.

(b) If possible, find a basis $\mathcal{B}'$ so that $[\varphi]_{\mathcal{B}'}$ is diagonal and give $[\varphi]_{\mathcal{B}'}$. If it is not possible to diagonalize φ, give reasons.

(c) Is the symmetric bilinear form given in (b) isometric to the symmetric bilinear form given by $\begin{pmatrix} -2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$?

(h) Let $A \in \mathrm{U}_n(\mathbb{C})$. Prove all the eigenvalues of A must lie on the unit circle in $\mathbb{C}$.

(i) Let $A \in \mathrm{O}_2(\mathbb{R})$. Prove that the first row of A has the form $(\cos\theta, \sin\theta)$ for some $\theta \in [0, 2\pi)$. Given this first row, what are all possible second rows of A?

(j) (a) Let $V = \mathbb{F}_5^3$ and let $\mathcal{E}_3$ be the standard basis for V. Let $\varphi : V \to V$ be the symmetric bilinear form given by

$$[\varphi]_{\mathcal{E}_3} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 4 \end{pmatrix}.$$

Find an orthogonal basis for V with respect to φ. Can you make this an orthonormal basis?

(b) Give a finite collection of matrices so that every nondegenerate symmetric bilinear form on V is isometric to one of the forms listed. Give a short justification on why your list is complete.

(k) Let φ be the standard inner product on $\mathbb{R}^n$ from multivariable calculus class. Define $T \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^n)$ by

$$T(x_1, \ldots, x_n) = (0, x_1, \ldots, x_{n-1}).$$

Find a formula for the adjoint map T^*.

(l) Let $T \in \mathrm{Hom}_F(V, W)$. Show that T is injective if and only if T^* is surjective.

Chapter 6

Inner product spaces and the spectral theorem

We now specialize some of the results of the previous chapter to situations where more can be said. We will require all our vector spaces (V, φ) to satisfy that φ is positive definite. In particular, we will restrict to the case that $\mathbb{Q} \subset F \subset \mathbb{R}$ in the case that φ is a symmetric bilinear form and $F \subset \mathbb{C}$ if φ is Hermitian. One can restrict to $F = \mathbb{R}$ or $F = \mathbb{C}$ in this chapter and not much will be lost unless one really feels the need to work over other fields.

In this chapter we will recover some of the things one sees in undergraduate linear algebra such as the Gram-Schmidt process. We will end with the Spectral Theorem, which tells when one can choose a "nice" basis of a linear map that is also "nice" with respect to the bilinear form on the vector space.

6.1 Inner product spaces

We begin by defining the spaces that we will work with in this chapter.

Definition 6.1.1. (a) Let (V, φ) be a vector space with φ a positive definite symmetric bilinear form. We say φ is an *inner product* on V and say (V, φ) is an *inner product space.*

(b) Let (V, φ) be a vector space with φ a positive definite Hermitian form. We say φ is an *inner product* on V and say (V, φ) is an *inner product space.*

Some of the most familiar examples given in the last chapter turn out to be inner products.

Example 6.1.2. Let $A \in \mathrm{Mat}_n(\mathbb{R})$ with ${}^tA = A$ and A positive definite or $A \in \mathrm{Mat}_n(\mathbb{C})$ with ${}^tA = \overline{A}$ and A positive definite. Then the form φ_A is an inner product. Note this example recovers the usual dot product on $\mathbb{R}^n$ by setting $A = 1_n$.

Example 6.1.3. Let $V = C^0([0,1],\mathbb{R})$ be the space of continuous functions from $[0,1]$ to $\mathbb{R}$. Define

$$\varphi(f,g) = \int_0^1 f(x)g(x)dx.$$

We have seen before this is a symmetric bilinear form. Moreover, note that

$$\varphi(f,f) = \int_0^1 f(x)^2 dx > 0$$

for all $f \in V$ with $f \neq 0$. Thus, φ is an inner product.

Example 6.1.4. Let $V = C^0([0,1],\mathbb{C})$ be the vector space of continuous functions from $[0,1]$ to $\mathbb{C}$. Define

$$\varphi(f,g) = \int_0^1 f(x)\overline{g(x)}dx.$$

As in the previous example, one has this is an inner product.

Exercise 6.1.5. Let (V,φ) be an inner product space. Show that for any subspace $W \subset V$ one has $\varphi|_W$ is nondegenerate. Conversely, if φ is a form on V so that $\varphi|_W$ is nondegenerate for each subspace $W \subset V$, then either φ or $-\varphi$ is an inner product on V.

One of the nicest things about an inner product is that it allows us to define a norm on the vector space, i.e., we can define a notion of distance on V.

Definition 6.1.6. Let (V,φ) be an inner product space. The *norm* of a vector $v \in V$, denoted $||v||$, is defined by

$$||v|| = \sqrt{\varphi(v,v)}.$$

The previous definition recovers the definition of the length of a vector given in multivariable calculus by considering $V = \mathbb{R}^n$ and φ the usual dot product.

The following lemma gives some of the basic properties of norms. These are all familiar properties from calculus class for the norm of a vector.

Lemma 6.1.7. *Let (V,φ) be an inner product space.*

(a) We have $||cv|| = |c|||v||$ for all $c \in F$, $v \in V$.

(b) We have $||v|| \geq 0$ and $||v|| = 0$ if and only if $v = 0$.

(c) (Cauchy-Schwartz inequality) We have $|\varphi(v,w)| \leq ||v||||w||$. We have equality if and only if $\{v,w\}$ is linearly dependent.

(d) (Triangle inequality) We have $||v+w|| \leq ||v|| + ||w||$ for all $v, w \in V$. This is an equality if and only if $w = 0$ or $v = cw$ for $c \in F_{\geq 0}$ where $F_{\geq 0} = F \cap \mathbb{R}_{\geq 0}$.

Proof. The proofs of the first two statements are left as exercises. We now prove the Cauchy-Schwartz inequality. If $\{v,w\}$ is linearly dependent, then up to reordering either $w = 0$ or $w = cv$ for some $c \in F$. In either case it is clear we have equality. Now assume $\{v,w\}$ is linearly independent, i.e., for any $c \in F$ we have $u = v - cw \neq 0$. Observe we have

$$\begin{aligned} 0 &\leq ||u||^2 \\ &= \varphi(u,u) \\ &= ||v||^2 + \varphi(-cw, v) + \varphi(v, -cw) + |c|^2||w||^2 \\ &= ||v||^2 - c\varphi(w,v) - \overline{c}\varphi(v,w) + |c|^2||w||^2 \end{aligned}$$

for any $c \in F$. Set $c = \frac{\varphi(v,w)}{\varphi(w,w)}$. This gives

$$0 \leq ||v||^2 - \frac{\varphi(v,w)\overline{\varphi(v,w)}}{||w||^2} - \frac{\overline{\varphi(v,w)}\varphi(v,w)}{||w||^2} + \left|\frac{\varphi(v,w)}{\varphi(w,w)}\right|^2 ||w||^2$$

i.e., $0 \leq ||v||^2 - \frac{|\varphi(v,w)|^2}{||w||^2}$. Thus, $|\varphi(v,w)| \leq ||v||||w||$ as claimed. Moreover, note that we get a strict inequality here since $\{v,w\}$ is linearly independent, we cannot have equality.

We now turn our attention to the triangle inequality. Observe for any $v, w \in V$ we have

$$\begin{aligned} ||v+w||^2 &= \varphi(v+w, v+w) \\ &= ||v||^2 + \varphi(v,w) + \varphi(w,v) + ||w||^2 \\ &= ||v||^2 + \varphi(v,w) + \overline{\varphi(v,w)} + ||w||^2 \\ &= ||v||^2 + 2\Re(\varphi(v,w)) + ||w||^2 \\ &\leq ||v||^2 + 2|\varphi(v,w)| + ||w||^2 \\ &\leq ||v||^2 + 2||v||||w|| + ||w||^2 \\ &= (||v|| + ||w||)^2. \end{aligned}$$

This gives the triangle inequality. It remains to deal with the case when $||v+w||^2 = (||v|| + ||w||)^2$. If this is the case, then both inequalities above must be equalities. The second inequality is equality if and only if $w = 0$ (note the first inequality is also an equality if $w = 0$) or if $w \neq 0$ and $w = cv$ for some $c \in F$. We have

$$\varphi(v,w) + \varphi(w,v)\varphi(cw,w) + \varphi(w,cw) = (c+\overline{c})||w||^2.$$

Thus the first inequality is an equality if and only if $c \in F_{\geq 0}$. □

The fact that we are restricting ourselves to characteristic 0 fields and φ being symmetric bilinear form or a Hermitian form immediately gives that there is a basis of V consisting of orthogonal vectors via Theorem 5.2.23. Moreover, if we take F to be $\mathbb{R}$ or $\mathbb{C}$ then we can combine the facts that the form is positive definite with Sylvestor's Law of Inertia to conclude that V has an orthonormal basis as well. We now give a few easy lemmas that deal with orthogonality of vectors.

Lemma 6.1.8. *Let $\mathcal{B} = \{v_i\}$ be an orthogonal set of nonzero vectors in an inner product space (V, φ). If $v \in V$ can be written as $v = \sum_j c_j v_j$ for $c_j \in F$, then $c_j = \frac{\varphi(v,v_j)}{||v_j||^2}$. If $\mathcal{B}$ is orthonormal, then $c_j = \varphi(v, v_j)$.*

Proof. Observe we have

$$\begin{aligned}\varphi(v, v_j) &= \varphi\left(\sum_i c_i v_i, v_j\right)\\ &= \sum_i c_i \varphi(v_i, v_j)\\ &= c_j \varphi(v_j, v_j).\end{aligned}$$

This gives $c_j = \frac{\varphi(v,v_j)}{\varphi(v_j,v_j)}$. If $\mathcal{B}$ happens to be orthonormal then $\varphi(v_j, v_j) = 1$. □

We can use the previous result to talk about projection of vectors onto subspaces. Let $W \subset V$ be a subspace and let v be any vector in V. We can project v onto a unique vector v_W in W and onto a unique vector $v_{W^\perp}$ in $W^\perp$ so that $v = v_W + v_{W^\perp}$. To see this we just use that $V = W \perp W^\perp$. We can use the previous result to write down explicitly what v_W and $v_{W^\perp}$ are in terms of an orthogonal basis. Let $\mathcal{B}_W = \{v_1, \dots, v_m\}$ be an orthogonal basis of W and $\mathcal{B} = \{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$ be an orthogonal basis of V. We have

$$v = \sum_{j=1}^{m} \frac{\varphi(v, v_j)}{\varphi(v_j, v_j)} v_j + \sum_{j=m+1}^{n} \frac{\varphi(v, v_j)}{\varphi(v_j, v_j)} v_j.$$

It is now clear that if we set

$$v_W = \sum_{j=1}^{m} \frac{\varphi(v, v_j)}{\varphi(v_j, v_j)} v_j$$

and

$$v_{W^\perp} = \sum_{j=m+1}^{n} \frac{\varphi(v, v_j)}{\varphi(v_j, v_j)} v_j,$$

then we have the claim. In this case one usually writes $\operatorname{proj}_W v$ for v_W and $\operatorname{proj}_{W^\perp} v$ for $v_{W^\perp}$. If one takes $V = \mathbb{R}^n$ and φ to be the usual dot product, this recovers vector projection studied in multivariable calculus.

Lemma 6.1.9. *Let $W \subset V$ be a subspace and $v \in V$. Then for all $w \in W$ with $w \neq \operatorname{proj}_W v$ we have*

$$||v - \operatorname{proj}_W v|| < ||v - w||.$$

Proof. Observe we have

$$v - w = (v - \operatorname{proj}_W v) + (\operatorname{proj}_W v - w).$$

We have $\operatorname{proj}_W v - w \in W$ and $(v - \operatorname{proj}_W v) \in W^\perp$. Thus, the triangle inequality gives

$$||v - w||^2 = ||v - \operatorname{proj}_W v||^2 + ||\operatorname{proj}_W v - w||^2.$$

Since $w \neq \operatorname{proj}_W v$, we have $||\operatorname{proj}_W v - w||^2 > 0$, which gives the result. □

Corollary 6.1.10. *Let $\mathcal{B} = \{v_i\}$ be a set of nonzero orthogonal vectors in an inner product space (V, φ). Then $\mathcal{B}$ is linearly independent.*

Proof. Suppose there exists $c_j \in F$ so that $0 = \sum_j c_j v_j$. The previous result gives $c_j = \frac{\varphi(0, v_j)}{||v_j||^2} = 0$. Thus, we have $\mathcal{B}$ is linearly independent. □

Lemma 6.1.11. *Let $\mathcal{B} = \{v_i\}$ be a set of nonzero orthogonal vectors in an inner product space (V, φ). Let $v \in V$ and suppose v can be written as $v = \sum_j c_j v_j$. Then*

$$||v||^2 = \sum_j |c_j|^2 ||v_j||^2.$$

If $\mathcal{B}$ is orthonormal, then

$$||v||^2 = \sum_j |c_j|^2.$$

Proof. We have

$$\begin{aligned} ||v||^2 &= \varphi(v, v) \\ &= \varphi\left(\sum_i c_i v_i, \sum_j c_j v_j\right) \\ &= \sum_{i,j} c_i \overline{c}_j \varphi(v_i, v_j) \\ &= \sum_j |c_j|^2 \varphi(v_j, v_j) \\ &= \sum_j |c_j|^2 ||v_j||^2. \end{aligned}$$

This gives the result for $\mathcal{B}$ orthogonal. The orthonormal case follows immediately upon observing $||v_j|| = 1$ for all j. □

Corollary 6.1.12. *Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a set of nonzero orthogonal vectors in V. Then for any vector $v \in V$ we have*

$$\sum_{j=1}^{n} \frac{|\varphi(v, v)|^2}{||v_j||^2} \leq ||v||^2$$

with equality if and only if $v = \displaystyle\sum_{j=1}^{n} \frac{\varphi(v, v_j)}{||v_j||^2} v_j$.

Proof. Set $w = \displaystyle\sum_{j=1}^{n} \left(\frac{\varphi(v, v_j)}{||v_j||^2} \right) v_j$ and $u = v - w$. Note,

$$\begin{aligned} \varphi(w, v_i) &= \sum_{j=1}^{n} \left(\frac{\varphi(v, v_j)}{||v_j||^2} \right) \varphi(v_j, v_i) \\ &= \varphi(v, v_i), \end{aligned}$$

and so $\varphi(u, v_i) = \varphi(v, v_i) - \varphi(w, v_i) = 0$ for $i = 1, \ldots, n$. Thus,

$$\begin{aligned} ||v||^2 &= \varphi(v, v) \\ &= \varphi(u + w, u + w) \\ &= \varphi(u, u) + \varphi(u, w) + \varphi(w, u) + \varphi(w, w) \\ &= ||u||^2 + ||w||^2 \end{aligned}$$

since $\varphi(u, w) = 0$ as $\varphi(u, v_i) = 0$ for all i. Hence,

$$\begin{aligned} \sum_{j=1}^{n} \frac{|\varphi(v, v_j)|^2}{||v_j||^2} &= ||w||^2 \\ &\leq ||v||^2, \end{aligned}$$

as claimed. We obtain equality if and only if $u = 0$, i.e., v is of the form $\displaystyle\sum_{j=1}^{n} \left(\frac{\varphi(v, v_j)}{||v_j||^2} \right) v_j$. □

Our next step is to prove the traditional Gram-Schmidt process from undergraduate linear algebra. Upon completing this, we give a brief outline of how one can do this in a more general setting.

Theorem 6.1.13. *(Gram-Schmidt) Let W be a finite dimensional subspace of V with $\dim_F W = k$. Let $\mathcal{B} = \{v_1, \ldots, v_k\}$ be a basis of W. Then there is an orthogonal basis $\mathcal{C} = \{w_1, \ldots, w_k\}$ of W so that $\operatorname{span}_F\{v_1, \ldots, v_l\} = \operatorname{span}_F\{w_1, \ldots, w_l\}$ for $l = 1, \ldots, k$. Moreover, if F contains $\mathbb{R}$ we can choose $\mathcal{C}$ to be orthonormal.*

Proof. The fact that an orthogonal basis exists follows Theorem 5.2.23. If $\mathbb{R} \subset F$ we obtain an orthonormal basis via Sylvestor's Law of Inertia and the fact that φ is positive definite. To construct the basis we work inductively. Set $x_1 = v_1$ and for $j \geq 1$ we set

$$x_i = v_i - \sum_{j<i} \frac{\varphi(v_i, x_j)}{||x_j||^2} x_j.$$

One can easily check this gives an orthogonal basis that satisfies $\operatorname{span}_F\{v_1, \dots, v_l\} = \operatorname{span}_F\{x_1, \dots, x_l\}$ for $l = 1, \dots, k$. If $\mathbb{R} \subset F$, we set $w_j = x_j/||x_j||$ and obtain the result. □

Let (V, φ) be a real vector space with φ symmetric. (We do not require that φ be positive definite here!) Let $\mathcal{B} = \{v_1, \dots, v_r\}$ be a basis of $\ker \varphi$. We can write $V = \ker \varphi \perp W$ and $\varphi|_W$ is non-degenerate. Let W^+ be the maximal subspace so that $\varphi|_{W^+}$ is positive definite and W^- the maximal subspace so that $\varphi|_{W^-}$ is negative definite. Then we have $V = \ker \varphi \perp W^+ \perp W^-$. Since $\varphi|_{W^+}$ is positive definite, we can apply Gram-Schmidt to get the desired basis of W^+. Similarly, since φ_{W^-} is negative definite, we can apply Gram-Schmidt to $-\varphi$ to get the desired basis of W^-. This gives the matrix

$$[\varphi]_{\mathcal{C}} = \begin{pmatrix} 0_r & & \\ & 1_p & \\ & & -1_q \end{pmatrix}.$$

Note the difficulty in this case is in calculating $\ker \varphi$, W^+, and W^-. This issue is not present for positive definite forms.

Proposition 6.1.14. *Let (V, φ) and (W, ψ) be finite dimensional inner product spaces and let $T \in \operatorname{Hom}_F(V, W)$. Then we have:*

(a) $\ker T^* = (\operatorname{Im} T)^\perp$

(b) $\ker T = (\operatorname{Im} T^*)^\perp$

(c) $\operatorname{Im} T^* = (\ker T)^\perp$

(d) $\operatorname{Im} T = (\ker T^*)^\perp$

Proof. 1) Let $w \in W$. We have $w \in \ker T^*$ if and only if $T^*(w) = 0$. Since φ is nondegenerate, $T^*(w) = 0$ if and only if $\varphi(v, T^*(w)) = 0$ for all $v \in V$. However, this is equivalent to $\psi(T(v), w) = 0$ for all $v \in V$ by the definition of the adjoint map. Finally, this is equivalent to $w \in (\operatorname{Im} T)^\perp$.

2) This follows exactly as 1) upon interchanging T and T^*.

3) Observe we have $V = \ker T \perp (\ker T)^\perp = (\operatorname{Im} T^*)^\perp \perp (\ker T)^\perp$ by 2). Moreover, we have $V = \operatorname{Im} T^* \perp (\operatorname{Im} T^*)^\perp$. Combining these gives

$\operatorname{Im} T^* = (\ker T)^\perp$.

4) This follows exactly as in 3) upon interchanging T and T^*. □

Lemma 6.1.15. *Let (V, φ) be a finite dimensional inner product space and let $T \in \operatorname{Hom}_F(V, V)$. Then we have*

$$\dim_F \ker T = \dim_F \ker T^*.$$

Proof. The previous proposition gives that $\operatorname{Im} T^* = (\ker T)^\perp$. Observe we have

$$\begin{aligned}\dim_F V &= \dim_F \ker T + \dim_F (\ker T)^\perp \\ &= \dim_F \ker T + \dim_F \operatorname{Im} T^*.\end{aligned}$$

On the other hand, we also have $\dim_F V = \dim_F \ker T^* + \dim_F \operatorname{Im} T^*$. Equating these and canceling $\dim_F \operatorname{Im} T^*$ gives the result. □

Example 6.1.16. Suppose one is given a collection of points in $\mathbb{R}^2$, say $p_1 = (x_1, y_1), \ldots, p_n = (x_n, y_n)$. We can consider the line $y = c_1 x + c_0$ that "best fits" these points. If the points all were on the line we would have a consistent set of linear equations

$$\begin{aligned} y_1 &= c_1 x_1 + c_0 \\ y_2 &= c_1 x_2 + c_0 \\ &\vdots \\ y_n &= c_1 x_n + c_0. \end{aligned}$$

This can be represented in matrix form by say there is a solution to the equation $Xc = Y$ where $X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$, $c = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$, and $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$. In general the points will not be collinear, so we do not have a solution to this equation. However, we can ask for the values of c_0 and c_1 that minimize the distance $||Xc - Y||$. We call the solution of this minimization problem the least square regression line $y = c_0 + c_1 x$.

We now rephrase this a bit. As $X \in \operatorname{Mat}_{n,2}(\mathbb{R})$, we view X as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^n$. The goal is to find a vector $Xc \in \operatorname{Im}(X)$ that is as close as possible to Y. We know from Lemma 6.1.9 that this is given by the projection of Y onto the subspace $\operatorname{Im}(X)$. Thus, $Xc = \operatorname{proj}_{\operatorname{Im}(X)}(Y)$. As above, we have $Xc - Y = \operatorname{proj}_{\operatorname{Im}(X)} Y - Y$ is orthogonal to $\operatorname{Im}(X)$, i.e., $Xc - Y \in \operatorname{Im}(X)^\perp = \ker(X^*) = \ker({}^tX)$. Thus,

$$0 = {}^tX(Xc - Y)$$

i.e., we have

$$({}^tXX)c = {}^tXY.$$

So finding the best fit line boils down to solving this system of linear equations.

We can generalize the previous example as follows.

Example 6.1.17. Let $A \in \mathrm{Mat}_{m,n}(\mathbb{C})$ with $m \geq n$ and $b \in \mathbb{C}^m$. In this case one is interested in solving the equation $Ax = b$ for some $x \in \mathbb{C}^n$. Of course, in general one will not be able to find a solution so one has to do a best approximation, i.e., find the vector x that minimizes the length of the error vector $Ax - b$. Assume that A has rank n, i.e., it has n linearly independent columns. We will see there is a unique x that minimizes the error vector.

The analysis goes essentially exactly as in the previous example. The first thing to show is that since A has rank n, A^*A is invertible where we recall the adjoint A^* is given here by the conjugate transpose of A. Fix $z \in \mathbb{C}^n$ and observe that since z is arbitrary, if we show that $A^*Az = 0$ implies that $z = 0$, we have A^*A is invertible since it is necessarily injective and maps $\mathbb{C}^n$ to $\mathbb{C}^n$. Suppose that $A^*Az = 0$. Then we have

$$\begin{aligned}||Az|| &= \varphi(Az, Az)\\ &= (Az)^*Az\\ &= z^*A^*Az\\ &= 0.\end{aligned}$$

Thus, we must have $Az = 0$. If z is not 0, we can write the equation $Az = 0$ as an equation in the column vectors of A, giving a linear dependence among them unless $z = 0$. Thus, we have $z = 0$ and so A^*A is invertible. Using this, we have $x = (A^*A)^{-1}A^*b$ as in the previous example.

It remains to show this x minimizes this error vector and it is the unique such vector. This follows from what we did in the previous example if one follows the same argument and uniqueness follows from the uniqueness of the projection vector, but we give a direct proof here for clarity. Let $y \in \mathbb{C}^n$. We have

$$\begin{aligned}\varphi(Ax - b, A(y - x)) &= (y - x)^*A^*(Ax - b)\\ &= (y - x)^*(A^*Ax - A^*b)\\ &= 0.\end{aligned}$$

Thus, the vectors $Ax - b$ and $A(y - x)$ are orthogonal. We have via the Pythagorean theorem

$$\begin{aligned}||Ay - b||^2 &= ||A(y - x) + (Ax - b)||^2\\ &= ||A(y - x)||^2 + ||Ax - b||^2\\ &\geq ||Ax - b||^2.\end{aligned}$$

This shows of all vectors, x minimizes $||Ax - b||^2$. Moreover, we have equality above if and only if $||A(y - x)|| = 0$, i.e., $x = y$. Thus, the solution is unique.

We end this section with a couple of elementary results on the minimal and characteristic polynomials as well as the Jordan canonical form of T^*.

Corollary 6.1.18. *Let (V, φ) be a finite dimensional inner product space over F let and $T \in \mathrm{Hom}_F(V, V)$. Suppose $c_T(x)$ factors into linear terms over F, i.e., T has Jordan canonical form over F. Then T^* has Jordan canonical form over F and the Jordan canonical form of T^* is given by conjugating the diagonals of the Jordan canonical form of T.*

Proof. Assuming the roots of $c_T(x)$ lie in F, the dimensions of the spaces $E_\lambda^k(T) = \ker(T - \lambda\,\mathrm{id})^k$ for λ an eigenvalue of T determine the Jordan canonical form of T. (Note we include the linear map T in the notation to avoid confusion.) Recall from Exercise 5.3.8 that we have $((T - \lambda\,\mathrm{id})^k)^* = (T^* - \overline{\lambda}\,\mathrm{id})^k$. We now apply the previous lemma to obtain

$$\begin{aligned}\dim_F E_{\overline{\lambda}}^k(T^*) &= \dim_F \ker(T^* - \overline{\lambda}\,\mathrm{id})^k \\ &= \dim_F \ker((T - \lambda\,\mathrm{id})^k)^* \\ &= \dim_F \ker(T - \lambda\,\mathrm{id})^k \\ &= \dim_F E_\lambda^k(T).\end{aligned}$$

Since the dimensions of these spaces are equal, we have the Jordan canonical form of T^* exists and is obtained by conjugating the diagonals of the Jordan canonical form of T. □

Corollary 6.1.19. *Let (V, φ) be a finite dimensional inner product space. Let $T \in \mathrm{Hom}_F(V, V)$. Then*

(a) $m_{T^*}(x) = \overline{m_T(x)}$;

(b) $c_{T^*}(x) = \overline{c_T(x)}$.

Proof. Observe that Exercise 5.3.8 gives that $m_{T^*}(x) = \overline{m_T(x)}$ immediately. For the second result it is enough to work over $\mathbb{C}$, and then one can apply the previous result. □

6.2 The spectral theorem

In this final section we give one of the more important results of the course. The spectral theorem tells us when we can choose a nice basis for a linear map that is also nice with respect to the inner products, i.e., is orthonormal. Unfortunately this theorem does not apply to all linear maps, so we begin by defining the linear maps we will be considering. Throughout this section we fix (V, φ) to be an inner product space.

Definition 6.2.1. Let $T \in \mathrm{Hom}_F(V, V)$. We say T is *normal* if T^* exists and $T \circ T^* = T^* \circ T$. We say T is *self-adjoint* if T^* exists and $T = T^*$.

Lemma 6.2.2. *Let $T \in \mathrm{Hom}_F(V, V)$. Then $\varphi(T(v), T(w)) = \varphi(v, w)$ for all $v, w \in V$ if and only if $||T(v)|| = ||v||$ for all $v \in V$. Furthermore, If $||T(v)|| = ||v||$ for all $v \in V$, then T is an injection. If V is finite dimensional this gives an isomorphism.*

Proof. Suppose $\varphi(T(v), T(w)) = \varphi(v, w)$ for all $v, w \in V$. This gives

$$\begin{aligned} ||T(v)||^2 &= \varphi(T(v), T(v)) \\ &= \varphi(v, v) \\ &= ||v||^2, \end{aligned}$$

i.e., $||T(v)|| = ||v||$ for all $v \in V$.

Conversely, suppose $||T(v)|| = ||v||$ for all $v \in V$. Then we have the following identities:

(a) If φ is a symmetric bilinear form, then

$$\varphi(v, w) = \frac{1}{4}||v + w||^2 - \frac{1}{4}||v - w||^2.$$

(b) If φ is a Hermitian form, then

$$\varphi(v, w) = \frac{1}{4}||v + w||^2 - \frac{i}{4}||v + iw||^2 - \frac{1}{4}||v - w||^2 - \frac{i}{4}||v - iw||^2.$$

These two identities are enough to give the result. For example, if φ is a symmetric bilinear form we have

$$\begin{aligned} \varphi(T(v), T(w)) &= \frac{1}{4}||T(v) + T(w)||^2 - \frac{1}{4}||T(v) - T(w)||^2 \\ &= \frac{1}{4}||T(v + w)||^2 - \frac{1}{4}||T(v - w)||^2 \\ &= \frac{1}{4}||v + w||^2 - \frac{1}{4}||v - w||^2 \\ &= \varphi(v, w) \end{aligned}$$

for all $v, w \in V$. □

Corollary 6.2.3. *Let $T \in \mathrm{Hom}_F(V, V)$ be an isometry. Then T has an adjoint and $T^* = T^{-1}$.*

Proof. Note that by definition if T is an isometry then T is an isomorphism, which implies T^{-1} exists. Now we calculate

$$\begin{aligned} \varphi(v, T^{-1}(w)) &= \varphi(T(v), T(T^{-1}(w))) \\ &= \varphi(T(v), w) \end{aligned}$$

for all $v, w \in V$. This fact, combined with the uniqueness of the adjoint map gives $T^* = T^{-1}$. □

Corollary 6.2.4. *Let $T \in \operatorname{Hom}_F(V,V)$.*

(a) If T is self-adjoint, then T is normal.

(b) If T is an isometry, then T is normal.

Proof. If T is self-adjoint, the only thing we need to check for T to be normal is that T commutes with its adjoint. However, self-adjoint gives $T = T^*$ so this is obvious. If T is an isometry we have just seen that T^* exists and is equal to T^{-1}. Again, the fact that T commutes with its adjoint is clear. □

Recall that for a nondegenerate symmetric bilinear form the isometry group is denoted $\mathrm{O}(\varphi)$ and elements of this group are said to be *orthogonal*. If φ is a nondegenerate Hermitian form we denoted the isometry group by $\mathrm{U}(\varphi)$ and elements of this group are said to be *unitary*. We say a matrix $P \in \mathrm{GL}_n(F)$ is *orthogonal* if ${}^tP = P^{-1}$ and we say P is *unitary* if ${}^t\overline{P} = P^{-1}$. In the case $F = \mathbb{R}$ or $F = \mathbb{C}$, the statement P is orthogonal is equivalent to $P \in \mathrm{O}_n(\mathbb{R})$ and P being unitary is equivalent to $P \in \mathrm{U}_n(\mathbb{C})$.

Corollary 6.2.5. *Let $T \in \operatorname{Hom}_F(V,V)$. Let $\mathcal{C}$ be an orthonormal basis and $M = [T]_{\mathcal{C}}$. Then*

(a) If (V,φ) is a real vector space, then

  i. if T is self-adjoint, then M is symmetric (${}^tM = M$).

  ii. if T is orthogonal, then M is orthogonal.

(b) If (V,φ) is a complex vector space. Then

  i. If T is self-adjoint, then M is Hermitian (${}^tM = \overline{M}$).

  ii. If T is unitary, then M is unitary.

Proof. Recall that for $\mathcal{C}$ an orthonormal basis we have

$$[T^*]_{\mathcal{C}} = {}^t\overline{[T]_{\mathcal{C}}}$$

via Corollary 5.3.6. If T is self-adjoint, this gives

$$M = [T]_{\mathcal{C}} = [T^*]_{\mathcal{C}} = {}^tM,$$

i.e., M is symmetric. The statement that T being self-adjoint in the complex case implies M is Hermitian follows along the exact same lines. The statements in regards to orthogonal and unitary were given before when discussing isometry groups in the previous chapter. □

One should also note the reverse direction in the above result. For example, if M is a symmetric matrix, then the associated map T_A is certainly self-adjoint.

The following example is the fundamental example to keep in mind when working with normal and self-adjoint linear maps.

Example 6.2.6. Let $\lambda_1, \dots, \lambda_m \in F$. Let $W_1, \dots, W_m$ be nonzero subspaces of V such that $V = W_1 \perp \cdots \perp W_m$. Define $T \in \operatorname{Hom}_F(V, V)$ as follows. Let $\mathcal{B}_i = \{w_i^1, \dots, w_i^k\}$ be a basis for W_i. Set $T(w_i^j) = \lambda_i w_i^j$ for $j = 1, \dots, k$. This gives W_i is the λ_i-eigenspace for T, i.e., $T(v) = \lambda_i v$ for all $v \in W_i$.

For any $v \in V$ we can write $v = v_1 + \cdots + v_m$ for some $v_i \in W_i$. We have

$$T(v) = \lambda_1 v_1 + \cdots \lambda_m v_m$$

and

$$T^*(v) = \overline{\lambda_1} v_1 + \cdots + \overline{\lambda_m} v_m.$$

Thus

$$(T^* \circ T)(v) = |\lambda_1|^2 v_1 + \cdots + |\lambda_m|^2 v_m = (T \circ T^*)(v).$$

This gives that T is a normal map. Moreover, we see immediately that T is self-adjoint if and only if $\lambda_i = \overline{\lambda_i}$ for all i.

In the following proposition we collect some of the results we've already seen, as well as give some new ones that will be of use.

Proposition 6.2.7. *Let $T \in \operatorname{Hom}_F(V, V)$ be normal. Then T^* is normal. Furthermore,*

(a) We have $p(T)$ is normal for all $p(x) \in F[x]$. If T is self-adjoint, then $p(T)$ is self-adjoint.

(b) We have $||T(v)|| = ||T^*(v)||$ for all v and $\ker T = \ker T^*$.

(c) We have $\ker T = (\operatorname{Im} T)^\perp$ and $\ker T^* = (\operatorname{Im} T^*)^\perp$.

(d) If $T^2(v) = 0$, then $T(v) = 0$.

(e) If $v \in V$ is an eigenvector of T with eigenvalue λ, then v is an eigenvector of T^* with eigenvalue $\overline{\lambda}$.

(f) Eigenspaces of distinct eigenvalues are orthogonal.

Proof. We have already seen that for φ symmetric or Hermitian that if T has an adjoint, so does T^* and $(T^*)^* = T$. Since T is assumed to be normal it has an adjoint and

$$\begin{aligned} T^*(T^*)^* &= T^*T \\ &= TT^* \\ &= (T^*)^*T^*. \end{aligned}$$

Thus, T^* is normal.

We have already seen 1) and 5).

For 2) we have

$$\begin{aligned}
||T(v)||^2 &= \varphi(T(v), T(v)) \\
&= \varphi(v, T^*T(v)) \\
&= \varphi(v, TT^*(v)) \\
&= \varphi(v, (T^*)^*T^*(v)) \\
&= \varphi(T^*(v), T^*(v)) \\
&= ||T^*(v)||^2.
\end{aligned}$$

The statement about kernels now follows immediately.

We now prove 3). Note we have just seen that $v \in \ker T$ if and only if $v \in \ker T^*$. We have $v \in \ker T = \ker T^*$ is equivalent to $\varphi(T^*(v), w) = 0$ for every $w \in V$. However, since $\varphi(T^*(v), w) = \varphi(v, T(w))$, this gives that $v \in \ker T$ if and only if $\varphi(v, T(w)) = 0$ for every $w \in V$, i.e., if and only if $v \in (\operatorname{Im} T)^{\perp}$. One gets the other equality by switching T and T^*.

Assume $T^2(v) = 0$ and let $w = T(v)$. Note that $T(w) = T^2(v) = 0$, so $w \in \ker T$. However, $w \in \operatorname{Im} T$ as well, so we have $w \in \ker T \cap \operatorname{Im} T = 0$ by 3). This gives part 4).

We now prove 6). Let v_1 be an eigenvector of T with eigenvalue λ_1 and v_2 with eigenvalue λ_2 and assume $\lambda_1 \neq \lambda_2$. Set $S = T - \lambda_1 I$, so $S(v_1) = 0$. We have

$$\begin{aligned}
0 &= \varphi(S(v_1), v_2) \\
&= \varphi(v_1, S^*(v_2)) \\
&= \varphi(v_1, (T^* - \overline{\lambda_1})(v_2)) \\
&= \varphi(v_1, (\overline{\lambda_2} - \overline{\lambda_1})v_2) \\
&= (\lambda_1 - \lambda_2)\varphi(v_1, v_2).
\end{aligned}$$

However, by assumption we have $\lambda_2 - \lambda_1 \neq 0$ so it must be that $\varphi(v_1, v_2) = 0$, i.e., v_1 and v_2 are orthogonal. □

Exercise 6.2.8. Let (V, φ) be finite dimensional and $T \in \operatorname{Hom}_F(V, V)$ be normal. Show $\operatorname{Im} T = \operatorname{Im} T^*$.

Proposition 6.2.9. *Let (V, φ) be a finite dimensional inner product space and let $T \in \operatorname{Hom}_F(V, V)$ be normal. The minimal polynomial of T is a product of distinct irreducible factors. If V is a $\mathbb{C}$-vector space or V is an $\mathbb{R}$-vector space and T is self-adjoint, then every factor is linear.*

Proof. Let $p(x)$ be an irreducible factor of $m_T(x)$. Suppose $p^2 | m_T$. Thus, there is a vector v so that $p^2(T)(v) = 0$, but $p(T)(v) \neq 0$. Let $S = p(T)$. Then S is normal and $S^2(v) = 0$, but $S(v) \neq 0$. This is a contradiction to 4) in the previous proposition, so $p(x)$ exactly divides $m_T(x)$.

For the second statement if V is a $\mathbb{C}$ vector space there is nothing to prove. Assume V is an $\mathbb{R}$-vector space and T is self-adjoint. We can factor $m_T(x)$ into linear and quadratic factors as this is true for any polynomial over $\mathbb{R}$. Let $p(x) = x^2 + bx + c$ be irreducible over $\mathbb{R}$ and assume $p(x) \mid m_T(x)$. Note that $p(x)$ irreducible over $\mathbb{R}$ gives $b^2 - 4c < 0$. Let $v \in V$ with $v \neq 0$ such that $p(T)(v) = 0$. Write $p(x) = (x + \frac{b}{2})^2 + d^2$ with $d = \sqrt{\frac{4c-b^2}{4}} \in \mathbb{R}$. Set $S = T + \frac{b}{2}I$. Then $(S^2 + d^2I)(v) = 0$, i.e., $S^2v = -d^2v$. Since we are assuming T is self-adjoint we obtain S is self-adjoint because $\frac{b}{2} \in \mathbb{R}$. Thus, we have

$$\begin{aligned} 0 &< \varphi(S(v), S(v)) \\ &= \varphi(S^*S(v), v) \\ &= \varphi(S^2(v), v) \\ &= -d^2\varphi(v, v). \end{aligned}$$

This is clearly a contradiction, so it must be that $p(x)$ is reducible over $\mathbb{R}$. □

Theorem 6.2.10 (Spectral Theorem)**.** *(a) Let (V, φ) be a finite dimensional $\mathbb{C}$-vector space and let $T \in \mathrm{Hom}_F(V, V)$ be normal. Then V has an orthonormal basis that is an eigenbasis for T.*

(b) Let V be a finite dimensional $\mathbb{R}$-vector space and let $T \in \mathrm{Hom}_F(V, V)$ be self-adjoint. Then there is an orthonormal basis of V that is an eigenbasis for T.

Proof. Use the previous result to split $m_T(x)$ into distinct linear factors and let $\lambda_1, \ldots, \lambda_k$ be the roots of $m_T(x)$. Let $E^\infty_{\lambda_i}$ be the eigenspace associated to λ_i. We have $V = E^\infty_{\lambda_1} \oplus \cdots \oplus E^\infty_{\lambda_k}$. But we saw eigenspaces of distinct eigenvalues are orthonormal. So $V = E^\infty_{\lambda_1} \perp \cdots \perp E^\infty_{\lambda_k}$. Let $\mathcal{C}_i$ be an orthonormal basis of $E^\infty_{\lambda_i}$. Then $\mathcal{C} = \cup_{i=1}^k \mathcal{C}_i$ is the desired basis. □

One can also write the Spectral theorem in the following form which will be useful in the next section. The proof is left as an exercise.

Corollary 6.2.11. *Let $T \in \mathrm{Hom}_F(V, V)$ be as in Theorem 6.2.10. Let $\lambda_1, \ldots, \lambda_r$ be the distinct eigenvalues of T. There are projection maps $\pi_1, \ldots, \pi_r$ so that*

(a) $T = \lambda_1\pi_1 + \cdots + \lambda_r\pi_r$*;*

(b) $\mathrm{id} = \pi_1 + \cdots + \pi_r$*;*

(c) $\pi_i \circ \pi_j = 0$ *if* $i \neq j$*.*

Exercise 6.2.12. Let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of T and write

$$T = \lambda_1\pi_1 + \cdots + \lambda_r\pi_r$$

as above. Show that

$$\pi_i = \prod_{j \neq i} \frac{1}{\lambda_i - \lambda_j}(T - \lambda_j \operatorname{id}).$$

We can also rephrase the Spectral theorem in terms of matrices.

Corollary 6.2.13. *(a) Let A be a Hermitian matrix. Then there is a unitary matrix P and a diagonal matrix D such that $A = PDP^{-1} = PD\,{}^t\overline{P}$.*

(b) Let A be a real symmetric matrix. Then there is a real orthogonal matrix P and real diagonal matrix D such that $A = PDP^{-1} = PD\,{}^tP$.

Proof. Let $A \in \mathrm{Mat}_n(\mathbb{R})$ be a symmetric matrix. Let $V = \mathbb{R}^n$ and let $\mathcal{E}_n$ be the standard basis of V. We have that $A = [T_A]_{\mathcal{E}_n}$. The fact that A is symmetric immediately gives T_A is a self-adjoint map. Thus, the spectral theorem implies there is an orthonormal basis $\mathcal{B}$ of V that is an eigenbasis for T_A, i.e., $[T_A]_{\mathcal{B}} = D$ is a diagonal matrix. Let Q be the change of basis matrix from $\mathcal{B}$ to $\mathcal{E}_n$. Then Q is an orthogonal matrix and $QAQ^{-1} = D$. (To see why it is orthogonal, see the discussion preceding Exercise 5.3.7.) Thus, if we set $P = Q^{-1}$ we have the result in the real symmetric case. The Hermitian case follows along the same lines. □

Example 6.2.14. Consider the matrix $A = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$. We want to find a unitary matrix P and a diagonal matrix D so that $A = PDP^{-1}$. Observe we have $c_A(x) = x^2 - 5x + 4 = (x-4)(x-1)$, so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 4$. Thus, we must have $D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$. This gives that $\mathbb{C}^2 = \ker(A - 1_2) \perp \ker(A - 41_2)$. One computes that $\ker(A - 1_2) = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix} -1-i \\ 1 \end{pmatrix}\right\}$ and $\ker(A - 41_2) = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix} 1+i \\ 2 \end{pmatrix}\right\}$. We now apply the Gram-Schmidt procedure to obtain an orthonormal basis given by $w_1 = \begin{pmatrix} \frac{-1-i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$ and $w_2 = \begin{pmatrix} \frac{1+i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$. Thus, we have

$$P = \begin{pmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix}.$$

One immediately checks that $A = PDP^{-1}$ as desired.

Example 6.2.15. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \\ 4 & 4 & -2 \end{pmatrix}$. We easily calculate that $c_A(x) = x(x-6)(x+6)$. Write $\lambda_1 = 0, \lambda_2 = 6, \lambda_3 = -6$.

Using the above exercise we have

$$\begin{aligned}
\pi_1 &= \frac{A - 6\cdot 1_3}{-6}\cdot\frac{A+6\cdot 1_3}{6} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\pi_2 &= \frac{A}{6}\cdot\frac{A+6\cdot 1_3}{12} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \\
\pi_3 &= \frac{A}{-6}\cdot\frac{A-6\cdot 1_3}{-12} = \begin{pmatrix} 1/6 & 1/6 & -1/3 \\ 1/6 & 1/6 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}.
\end{aligned}$$

Now given any $v \in \mathbb{R}^3$, we have $\pi_i(v)$ lies in the λ_i-eigenspace. For example, we have

$$\pi_3\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

and

$$A\begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -6 \end{pmatrix} = -6\begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}.$$

This allows one to quickly give an eigenbasis, and then one applies Gram-Schmidt to get an orthonormal eigenbasis to recover the orthogonal matrix P that diagonalizes A.

We saw in the previous section a way to use Gram-Schmidt to compute the signature. The following corollary allows us to compute the signature by counting eigenvalues.

Corollary 6.2.16. *Let (V, φ) be an inner product space. Set $n = \dim_F V$. Let $\mathcal{B}$ be a basis for V and set $A = [\varphi]_{\mathcal{B}}$. Then*

(a) A has n real eigenvalues (counting multiplicity);

(b) the signature of φ is (r, s) where r is the number of positive eigenvalues and s is the number of negative eigenvalues of A.

Proof. The first statement is left as a homework problem.

Let φ be a nondegenerate symmetric bilinear form. Then if we take any basis $\mathcal{B}$ of V, we have $A = [\varphi]_{\mathcal{B}}$ is a symmetric matrix. We apply Corollary 6.2.13 to see that there exists an orthonormal basis $\mathcal{C} = \{v_1, \ldots, v_n\}$ so that $D = [\varphi]_{\mathcal{C}}$ is a diagonal matrix, i.e., there is an orthogonal matrix P so that $A = PDP^{-1}$. The diagonal entries of D are precisely the eigenvalues of A. Upon reordering $\mathcal{C}$ we can assume that the first r diagonal entries of D are positive and the remaining $s = n - r$ are negative. Let $W_1 = \operatorname{span}_F\{v_1, \ldots, v_r\}$ and $W_2 = \operatorname{span}_F\{v_{r+1}, \ldots, v_n\}$. We have

$V = W_1 \perp W_2$. Observe that $\varphi|_{W_1}$ is positive definite and $\varphi|_{W_2}$ is negative definite. Thus, the signature of φ is exactly $(\dim_F W_1, \dim_F W_2) = (r, s)$, as claimed. □

Recall we saw before that if S and T are diagonalizable, they are simultaneously diagonalizable if and only if S and T commute. In fact, it was noted this result can be extended to a countable collection of diagonalizable maps. This, along with the spectral theorem, immediately gives the following result.

Corollary 6.2.17. *Let $\{T_i\}$ be a countable collection of normal or self-adjoint maps. Then $\{T_i\}$ is simultaneously diagonalizable if and only if the T_i all commute.*

We have the following elementary calculation of when an isometry with a minimal polynomial that splits completely is an isometry.

Corollary 6.2.18. *Let $T \in \operatorname{Hom}_F(V, V)$ be normal. Suppose $m_T(x)$ is a product of linear factors over F. Then T is an isometry if and only if $|\lambda| = 1$ for every eigenvalue λ of T.*

Proof. Let T be an isometry and let v be an eigenvector with eigenvalue λ. Then

$$\begin{aligned}\varphi(v, v) &= \varphi(T(v), T(v)) \\ &= \varphi(\lambda v, \lambda v) \\ &= \lambda\overline{\lambda}\varphi(v, v) \\ &= |\lambda|^2\varphi(v, v).\end{aligned}$$

Thus, $|\lambda| = 1$.

Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of orthogonal eigenvectors with eigenvalues $\lambda_1, \ldots, \lambda_n$. Assume $|\lambda_i| = 1$ for all i. Pick any $v \in V$ and write $v = \sum a_i v_i$

for some $a_i \in F$. We have

$$\begin{aligned}
\varphi(T(v), T(v)) &= \varphi\left(\sum a_i T(v_i), \sum a_j T(v_j)\right) \\
&= \sum_{i,j} \varphi(a_i \lambda_i v_i, a_j \lambda_j v_j) \\
&= \sum_{i,j} a_i \lambda_i \overline{a_j}\,\overline{\lambda_j} \varphi(v_i, v_j) \\
&= \sum_j a_j \overline{a_j} |\lambda_j|^2 \varphi(v_j, v_j) \\
&= \sum_j a_j \overline{a_j} \varphi(v_j, v_j) \\
&= \varphi\left(\sum_j a_j v_j, \sum_j a_j v_j\right) \\
&= \varphi(v, v).
\end{aligned}$$

Thus, T is an isometry as desired. □

We now have the very nice result that if we have a normal map then it is diagonalizable with respect to an orthonormal basis. Of course, not all maps are normal so it is natural to ask if anything can be said in the more general case. We close this section with a couple of results in that direction.

Theorem 6.2.19 (Schur's Theorem)**.** *Let (V, φ) be a finite dimensional inner product space and let $T \in \operatorname{Hom}_F(V, V)$. Then V has an orthonormal basis $\mathcal{C}$ so that $[T]_{\mathcal{C}}$ is upper triangular if and only if the minimal polynomial $m_T(x)$ is a product of linear factors.*

Proof. Suppose there is an orthonormal basis $\mathcal{C}$ such that $[T]_{\mathcal{C}}$ is upper triangular. Then $c_T(x) = \prod(x - \lambda_i)$ for λ_i the diagonal entries. Since $m_T(x) \mid c_T(x)$ we immediately obtain $m_T(x)$ splits into a product of linear factors.

Suppose $m_T(x)$ factors as a product of linear factors. Let W be a T-invariant subspace. We claim $W^\perp$ is a T^*-invariant subspace. To see this, observe that for any $w \in W, y \in W^\perp$ we have

$$0 = \varphi(T(w), y) = \varphi(w, T^*(y)).$$

Thus, $T^*(y) \in W^\perp$, i.e., $W^\perp$ is T^*-invariant as claimed. We now use induction on the dimension of V. If $\dim_F V = 1$ the result is trivial. Suppose the result is true for any vector space of dimension less than n. Let $\dim_F V = n$. Note that since $m_T(x)$ splits into linear factors, so does $m_{T^*}(x)$ because $m_{T^*}(x) = \overline{m_T(x)}$. Then T^* has an eigenvector, say

v_n. Scale this so $||v_n|| = 1$. Let $W = \mathrm{span}_F\{v_n\}$. Using the fact that W is T^*-invariant, $W^\perp$ is an $n-1$ dimensional subspace of V that is a $(T^*)^*$-invariant subspace, i.e., a T-invariant subspace. Set $S = T\mid_{W^\perp}$. Now $m_S \mid m_T$ so m_S splits into linear factors. Applying the induction hypothesis we get an orthonormal basis $\mathcal{C}_1 = \{v_1, \dots, v_{n-1}\}$ such that $[S]_{\mathcal{C}_1}$ is upper triangular. Thus, setting $\mathcal{C} = \mathcal{C}_1 \cup \{v_n\}$, we have $[T]_{\mathcal{C}}$ is upper triangular as desired. □

In fact, it turns out we can characterize whether a map is normal by whether it can be diagonalized.

Theorem 6.2.20. *Let (V, φ) be a finite dimensional inner product space and $T \in \mathrm{Hom}_F(V, V)$. Let $\mathcal{C}$ be any orthonormal basis of V with $[T]_{\mathcal{C}}$ upper triangular. Then T is normal if and only if $[T]_{\mathcal{C}}$ is diagonal.*

Proof. Certainly if $[T]_{\mathcal{C}}$ is diagonal then T is normal. So suppose T is normal and set $E = [T]_{\mathcal{C}}$. Let $\mathcal{C}_1$ be a basis so that $[T]_{\mathcal{C}_1} = D$ is diagonal. Such a basis exists by the Spectral Theorem. Set $P = [T]_{\mathcal{C}_1}^{\mathcal{C}}$ so $E = PDP^{-1}$. Since $\mathcal{C}$ and $\mathcal{C}_1$ are both orthonormal, we have P is orthogonal in the real case and unitary in the Hermitian case, i.e., if $F = \mathbb{R}$, then ${}^tP = P^{-1}$ and if $F = \mathbb{C}$, then ${}^tP = \overline{P}^{-1}$. Using this, we have if $F = \mathbb{R}$ then

$$\begin{aligned} {}^tE &= {}^t(PDP^{-1}) \\ &= {}^t(PD\,{}^tP) \\ &= PD\,{}^tP \\ &= PDP^{-1} \\ &= E. \end{aligned}$$

Since E is assumed to be upper triangular, the only way it can equal its transpose is if it is actually diagonal. The same argument works in the Hermitian case as well. □

6.3 Polar decomposition and the Singular Value Theorem

Recall that given a complex number z, one can write $z = x+iy$ for $x, y \in \mathbb{R}$ or $z = re^{i\theta}$ for $r \in \mathbb{R}_{\geq 0}$ and $\theta \in [0, 2\pi)$. One can ask if the same thing can be done for a matrix $A \in \mathrm{Mat}_n(\mathbb{C})$. In the case of writing $z = x + iy$, the analogy was given in Problem 16e in Section 5.4 via the Hermitian decomposition of a matrix. To see this, recall that one showed $A = H + S$ for H a Hermitian matrix and S a skew-Hermitian matrix. Furthermore, recall that one can write $S = iH_2$ for H_2 a Hermitian matrix, so this equation can be written as $A = H_1 + iH_2$ for H_i Hermitian matrices.

(One does not have that the H_i have entries in $\mathbb{R}$ so don't read too much into the analogy.) Naturally, we would like to give a version of the polar decomposition as well. That will be one of the main goals of this section. Once we have the polar decomposition of a complex matrix, we will use this to prove the singular value decomposition theorem.

We require all the vector spaces to be finite dimensional inner product spaces in this section.

We begin with some more results on self-adjoint linear maps. First, observe that given a linear map $T \in \mathrm{Hom}_F(V,V)$, the map T^*T is always a self-adjoint linear map. This is because we have for any $v, w \in V$

$$\begin{aligned}\varphi((T^*T)(v), w) &= \varphi(T(v), T(w)) \\ &= \varphi(v, (T^*T)(w)).\end{aligned}$$

Similarly, we have TT^* is self-adjoint. This allows us to apply the Spectral Theorem to these maps and conclude they are both diagonalizable. This will be very important for our applications. Moreover, since they are self-adjoint one has that the eigenvalues are all non-negative real numbers. Thus, we can write the eigenvalues of T^*T as $\mu_1^2, \ldots, \mu_r^2$ for some $\mu_i > 0$. Moreover, T^*T and TT^* actually have the same eigenvalues. (This is true in general. Given $T : V \to W$ and $S : W \to V$, TS and ST have the same eigenvalues.)

Definition 6.3.1. Let $T \in \mathrm{Hom}_F(V,V)$. The positive square roots of the eigenvalues of T^*T are called the *singular values* of T.

Definition 6.3.2. Let $T \in \mathrm{Hom}_F(V,V)$. We say T is *positive* if T is self-adjoint and $\varphi(T(v), v) > 0$ for all $v \neq 0$.

The positive linear maps act as our generalization of positive numbers. In particular, we know a complex number z is positive if and only if we can write $z = w\overline{w}$ for some $w \in \mathbb{C}$. We have the following result.

Theorem 6.3.3. *Let $T \in \mathrm{Hom}_F(V,V)$. Then T is positive if and only if there is an invertible linear map $S \in \mathrm{Hom}_F(V,V)$ so that $T = S^*S$.*

Proof. First, suppose there exists such an S. We have $T^* = (S^*S)^* = S^*S = T$, so T is self-adjoint. Observe that $\varphi(T(v), v) = \varphi(S^*S(v), v) = \varphi(S(v), S(v)) \geq 0$. Since S is invertible, if $v \neq 0$ then $S(v) \neq 0$ and so $\varphi(T(v), v) > 0$ and so T is positive.

Now assume T is positive. We have via the homework problems that $\psi(v, w) = \varphi(T(v), w)$ is an inner product on V. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be an orthonormal basis with respect to φ and $\mathcal{C} = \{w_1, \ldots, w_n\}$ an orthonormal basis with respect to ψ. Then we have

$$\psi(w_i, w_j) = \delta_{ij} = \varphi(v_i, v_j).$$

Define S to be the unique linear map defined by $S(w_j) = v_j$. This is invertible since $\mathcal{B}$ and $\mathcal{C}$ are bases. Observe that

$$\psi(w_i, w_j) = \varphi(S(w_i), S(w_j)) = \varphi(v_i, v_j).$$

Let $w, w' \in V$ and write $w = \sum a_i w_i$, $w' = \sum b_i w_i$. Using that ψ is an inner product we see

$$\psi(w, w') = \varphi(S(w), S(w')).$$

The definition of ψ gives $\varphi(T(w), w') = \psi(w, w') = \varphi(S(w), S(w')) = \varphi(S^*S(w), w')$ for all $w, w' \in V$. Thus, $T = S^*S$. □

If $F = \mathbb{C}$ we can drop the requirement that T be self-adjoint from the definition of positive. This is because if $\varphi(T(v), v) > 0$ for all $v \neq 0$ in this case then T is automatically self-adjoint. This follows from the following result.

Lemma 6.3.4. *Let V be a complex inner product space and $T \in \operatorname{Hom}_F(V, V)$. If $\varphi(T(v), v) \in \mathbb{R}$ for all $v \in V$, then T is self-adjoint.*

Proof. Let $v, w \in V$. Observe we have

$$\varphi(T(v + w), v + w) = \varphi(T(v), v) + \varphi(T(v), w) + \varphi(T(w), v) + \varphi(T(w), w).$$

We are assuming $\varphi(T(v + w), v + w), \varphi(T(v), v)$, and $\varphi(T(w), w)$ are all real, so we must have $\varphi(T(v), w) + \varphi(T(w), v)$ is real as well. Now run the same argument with the vector $v + iw$ and we have

$$\varphi(T(v+iw), v+iw) = \varphi(T(v), v) - i\varphi(T(v), w) + i\varphi(T(w), v) + \varphi(T(w), w).$$

As above, this gives $-i\varphi(T(v), w) + i\varphi(T(v), w)$ is real. Since these numbers are real, they are equal to their complex conjugates:

$$\begin{aligned} \varphi(T(v), w) + \varphi(T(w), v) &= \overline{\varphi(T(v), w)} + \overline{\varphi(T(w), v)} \\ &= \varphi(w, T(v)) + \varphi(v, T(w)) \end{aligned}$$

and

$$\begin{aligned} -i\varphi(T(v), w) + i\varphi(T(v), w) &= \overline{-i\varphi(T(v), w)} + \overline{i\varphi(T(v), w)} \\ &= i\varphi(w, T(v)) - i\varphi(w, T(v)). \end{aligned}$$

We now multiply the second set of equations by i and add the result to the first set of equations and obtain

$$2\varphi(T(v), w) = 2\varphi(w, T(v)),$$

i.e., T is self-adjoint. □

The previous result is not true for real inner product spaces as clearly we have $\varphi(T(v), v) \in \mathbb{R}$ for any $v \in V$ automatically. In general, if T is self-adjoint we have $\varphi(T(v), v) = \varphi(v, T(v)) = \overline{\varphi(T(v), v)}$ so $\varphi(T(v), v)$ is real. Thus, for a complex inner product space we see that T is self-adjoint if and only if $\varphi(T(v), v) \in \mathbb{R}$ for all $v \in V$ and T is positive if and only if $\varphi(T(v), v) > 0$ for all $v \neq 0$.

Theorem 6.3.5. *Let $T \in \operatorname{Hom}_F(V, V)$ be a normal map on a complex inner product space. Then T is self-adjoint, positive, or unitary according to if its eigenvalues are real, positive, or absolute value 1.*

Proof. The result on unitary maps was given in the homework of the previous chapter. The result on self-adjoint maps follows immediately from the fact that $\varphi(T(v), v) \in \mathbb{R}$ if T is self-adjoint. It only remains to prove the result for positive maps. Using the Spectral Theorem write $T = \lambda_1\pi_1 + \cdots + \lambda_r\pi_r$. Then we have for $v \in V$ that

$$\begin{aligned}
\varphi(T(v), v) &= \varphi\left(\sum_{i=1}^{r} \lambda_i \pi_i(v), \sum_{j=1}^{r} \pi_j(v)\right) \\
&= \sum_{i,j=1}^{r} \lambda_i \varphi(\pi_i(v), \pi_j(v)) \\
&= \sum_{j=1}^{r} \lambda_j ||\pi_j(v)||^2.
\end{aligned}$$

Thus, we see that $\varphi(T(v), v) > 0$ is satisfied if and only if $\lambda_j > 0$ for each j. □

Definition 6.3.6. Let $T \in \operatorname{Hom}_F(V, V)$. We say T is *non-negative* if T is self-adjoint and $\varphi(T(v), v) \geq 0$ for all $v \in V$.

As above, if $F = \mathbb{C}$ then we can drop the assumption that T is self-adjoint.

Lemma 6.3.7. *Let $T \in \operatorname{Hom}_F(V, V)$ be a non-negative linear map. There is a unique non-negative linear map $S \in \operatorname{Hom}_F(V, V)$ so that $T = S^2$.*

Proof. We can apply the Spectral Theorem to write $T = \lambda_1\pi_1 + \cdots + \lambda_r\pi_r$. Since T is assumed to be non-negative, we have $\lambda_i \geq 0$ for all i. Set $S = \sqrt{\lambda_1}\pi_1 + \cdots + \sqrt{\lambda_r}\pi_r$. It is clear we have $S^2 = T$. Now let R be any other non-negative linear map with $R^2 = T$. Let $R = d_1\pi_1 + \cdots + d_r\pi_r$ be the spectral resolution of R. Since R is assumed to be non-negative, $d_i \geq 0$ for all i. Since $R^2 = T$ we have

$$T = d_1^2\pi_1 + \cdots + d_r^2\pi_r.$$

This decomposition satisfies the conditions of being a spectral resolution of T, and so we must have $d_i^2 = \lambda_i$ for each i. Since $d_i \geq 0$, we get $d_i = \sqrt{\lambda_i}$ for each i and so $R = S$ as claimed. □

Exercise 6.3.8. Given T and S as in the last theorem, the eigenvalues of S are given by $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_r}$ and for each i we have $E^{\infty}_{\lambda_i}(T) = E^{\infty}_{\sqrt{\lambda_i}}(S)$.

Exercise 6.3.9. Prove that if $T \in \mathrm{Hom}_F(V,V)$ is a positive self-adjoint linear map, then for any $m \geq 2$ there is a unique positive self-adjoint linear map $S \in \mathrm{Hom}_F(V,V)$ so that $T = S^m$.

We now state the Polar Decomposition Theorem. One should think of the non-negative linear map as the "r" and the orthogonal/unitary map as the "$e^{i\theta}$".

Theorem 6.3.10. *Let $T \in \mathrm{Hom}_F(V,V)$. There are two non-negative linear maps $S_1, S_2 \in \mathrm{Hom}_F(V,V)$ and a linear map $U \in \mathrm{Hom}_F(V,V)$ (R is orthogonal if $F = \mathbb{R}$ and unitary if $F = \mathbb{C}$) so that*

$$T = US_1 = S_2U.$$

Moreover, if $\dim_F \mathrm{Im}(T) = r$, then S_1 and S_2 have the same positive eigenvalues $\mu_1, \ldots, \mu_r$, which are the singular values of T. If T is invertible then U, S_1, and S_2 are unique. If T is normal, $S_1 = S_2$.

Proof. Suppose that we have $T = US_1$. Then $T^* = (US_1)^* = S_1^*U^* = S_1U^*$ because S_1 is self-adjoint. Thus, $T^*T = S_1U^*US_1 = S_1^2$ because U is orthogonal or unitary. Thus, S_1 is uniquely determined by T because it is the unique square root of T^*T as given above. A similar argument works for S_2 except one gets it is the square-root of TT^*.

We now need to prove the existence of U and S_1. Set S_1 to be the square-root of T^*T. Similarly, define S_2 to be the square-root of TT^*. This immediately gives that S_1 and S_2 are unique and have the same positive eigenvalues. We also have that if T is normal then $S_1 = S_2$. Moreover, Note that if T is invertible, S_1 is invertible as well because

$$\begin{aligned} \varphi(S_1(v), S_1(v)) &= \varphi(S_1^2(v), v) \\ &= \varphi(T^*T(v), v) \\ &= \varphi(T(v), T(v)). \end{aligned}$$

In this case, set $U = TS_1^{-1}$. One defines S_2 similarly. It only remains to prove that U is orthogonal or unitary. We have $U^* = (TS_1^{-1})^* = S_1^{-1}T^*$. Thus we have

$$\begin{aligned} UU^* &= TS_1^{-1}S_1^{-1}T^* \\ &= T(S_1^{-2})^2T^* \\ &= T(S_1^2)^{-1}T^* \\ &= T(T^*T)^{-1}T^* \\ &= TT^{-1}(T^*)^{-1}T^* \\ &= \mathrm{id}\,. \end{aligned}$$

Thus, if T is invertible we have the result.

We now must deal with the case when T is not invertible. This is considerably more work to define U. We begin by defining U on the image of S_1. Let $w \in W = \mathrm{Im}(S_1)$, say $w = S_1(v)$. We want to define U so that $US_1(v) = T(v)$. Thus, we set $U(w) = T(v)$. We must check this is well-defined. Let $w_1 = S_1(v_1) = S_1(v_2)$; we need to show that $T(v_1) = T(v_2)$. We saw above that $||S_1(v)||^2 = ||T(v)||^2$ for all v. Set $v = v_1 - v_2$. Then we have $S_1(v) = 0$ if and only if $T(v) = 0$. This gives that U is well-defined. It now remains to define U on $W^\perp$. Observe that T and S_1 have the same kernel, so the dimension of the image of S_1 is equal to the dimension of the image of T. Thus, $W^\perp$ has the same dimension as the orthogonal complement of the image of T. Thus, we have an isomorphism of inner product spaces $U_0 : W^\perp \to \mathrm{Im}(T)^\perp$. Define U to be equal to U_0 on $W^\perp$. We make this a bit clearer. Let $v \in V$. We can uniquely write $v = w_1 + w_2$ with $w_1 \in W$ and $w_2 \in W^\perp$. Write $w_1 = S_1(v_1)$. Define

$$U(v) = T(v_1) + U_0(w_2).$$

It is clear that U is a well-defined linear map. We now check it is orthogonal/unitary. We have

$$\begin{aligned}\varphi(U(v), U(v)) &= \varphi(T(v_1) + U_0(w_2), T(v_1) + U_0(w_2)) \\ &= \varphi(T(v_1), T(v_1)) + \varphi(U_0(w_2), U_0(w_2)) \\ &= \varphi(S_1(v_1), S_1(v_1)) + \varphi(w_2, w_2) \\ &= \varphi(v, v)\end{aligned}$$

where we have used that $||T(v_1)|| = ||S_1(v_1)||$ and U_0 is orthogonal/unitary because it is an isomorphism of inner product spaces. Thus, U is orthogonal/unitary. This gives $US_1(v) = T(v)$ for each v.

One can run the same argument with S_2. □

We now rephrase this in terms of matrices.

Corollary 6.3.11. *(a) Let $A \in \mathrm{Mat}_n(\mathbb{R})$. There is a matrix $U \in \mathrm{O}_n(\mathbb{R})$ and a non-negative symmetric matrix S so that*

$$A = US.$$

Moreover, if A is invertible then U and S are unique.

(b) Let $A \in \mathrm{Mat}_n(\mathbb{C})$. There is a matrix $U \in \mathrm{U}_n(\mathbb{C})$ and a non-negative Hermitian matrix S so that

$$A = US.$$

Moreover, if A is invertible then U and S are unique.

We also obtain as an immediate corollary the Singular Value Decomposition Theorem (SVD).

Theorem 6.3.12. *Let $T \in \operatorname{Hom}_F(V, V)$ with (V, φ) a inner product space of dimension n. There are orthonormal bases $\mathcal{B} = \{u_1, \ldots, u_n\}$ and $\mathcal{C} = \{v_1, \ldots, v_n\}$ such that if $r = \dim_F \operatorname{Im}(T)$, we have*

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}$$

where $\mu_1, \ldots, \mu_r$ are the singular values of T and $\mu_{r+1} = \cdots = \mu_n = 0$. Moreover, $u_1, \ldots, u_n$ are the eigenvectors of T^*T, $v_1, \ldots, v_n$ are the eigenvectors of TT^*, and $T(u_i) = \mu_i v_i$ for $1 \leq i \leq n$.

We can rephrase the SVD in terms of matrices.

Theorem 6.3.13. *(a) Let $A \in \operatorname{Mat}_n(\mathbb{R})$. There are two matrices $U, V \in \mathrm{O}_n(\mathbb{R})$ and a diagonal matrix D so that $A = VD\,{}^tU$ where D is a diagonal matrix with the positive square roots of the eigenvalues of ${}^tAA$ on the diagonal. The columns of U are the eigenvectors of ${}^tAA$ and the columns of V are the eigenvectors of $A\,{}^tA$.*

(b) Let $A \in \operatorname{Mat}_n(\mathbb{R})$. There are two matrices $U, V \in \mathrm{O}_n(\mathbb{R})$ and a diagonal matrix D so that $A = VD\,\overline{{}^tU}$ where D is a diagonal matrix with the positive square roots of the eigenvalues of $\overline{{}^tA}A$ on the diagonal. The columns of U are the eigenvectors of $\overline{{}^tA}A$ and the columns of V are the eigenvectors of $A\,\overline{{}^tA}$.

We briefly indicate how to go from the polar decomposition to the SVD and vice versa. Suppose $A = U_1S$ with U_1 being orthogonal/unitary and S positive symmetric/Hermitian. We can use the Spectral Theorem to write $S = U_2DU_2^*$ with D a positive diagonal matrix and U_2 orthogonal/unitary. Thus,

$$A = U_1U_2DU_2^*.$$

The SVD is given by setting $V = U_1U_2$ and $U = U_2$. Now suppose we have $A = VDU^*$ is the SVD of A. Set $R = VU^*$ and $S = UDU^*$. Then we have R is orthogonal/unitary, S is positive symmetric/Hermitian and

$$\begin{aligned} RS &= VU^*UDU^* \\ &= VDU^* \\ &= A. \end{aligned}$$

We find the singular value decomposition of a matrix now.

Example 6.3.14. Let $A = \begin{pmatrix} 5 & 5 \\ -1 & 7 \end{pmatrix}$. We want to find U, V and D so that $A = UD\,{}^t\!V$ with U, V orthogonal and D diagonal. First, we observe that the two equations we need to work with are

$$ {}^t\!AA = V\,{}^t\!DD\,{}^t\!V $$

and

$$ AV = UD. $$

The first equation is just the diagonalization of ${}^t\!AA$. We have ${}^t\!AA = \begin{pmatrix} 26 & 18 \\ 18 & 74 \end{pmatrix}$. The eigenvalues of this are 20 and 80 as $c_{{}^t\!AA}(x) = (x-20)(x-80)$. To find V, we need to find an orthonormal basis for the eigenspaces of ${}^t\!AA$. We have

$$ {}^t\!AA - 20 \cdot 1_2 = \begin{pmatrix} 6 & 18 \\ 18 & 54 \end{pmatrix}. $$

A basis for the kernel of this space is given by $v_1 = \begin{pmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$. Similarly, a basis for the eigenspace associated to the eigenvector 80 is given by $v_2 = \begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}$. These form an orthonormal basis, so

$$ V = \begin{pmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{pmatrix}. $$

The singular values of A are given by $\sqrt{20}$ and $\sqrt{80}$, so $D = \begin{pmatrix} \sqrt{20} & 0 \\ 0 & \sqrt{80} \end{pmatrix}$. We now just write

$$ U = AVD^{-1} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. $$

We now have all the components of the singular value decomposition of A.

One can give similar results for $A \in \mathrm{Mat}_{m,n}(F)$, but we do not pursue those here. We end by stating (without proof) the following result that relates the eigenvalues of A and the singular values of A.

Theorem 6.3.15. *Let $A \in \mathrm{Mat}_n(\mathbb{C})$ with eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and singular values $\mu_1, \dots, \mu_n \in \mathbb{R}_{>0}$ listed so that $|\lambda_1| \geq \cdots \geq |\lambda_n|$ and $\mu_1 \geq \cdots \geq \mu_n$, then*

(a) $|\lambda_1| \cdots |\lambda_n| = \mu_1 \cdots \mu_n$

(b) $|\lambda_1| \cdots |\lambda_k| \leq \mu_1 \cdots \mu_k$ *for* $k = 1, \dots, n-1$.

6.4 Problems

For these problems V and W are finite dimensional F-vector spaces.

(a) Prove the two identities used in the proof of Lemma 6.2.2. Complete the proof in the case that φ is a Hermitian form.

(b) Prove the first statement in Corollary 6.2.16.

(c) Let $V = \mathrm{Mat}_2(\mathbb{R})$.

   (a) Show that the map $\varphi : V \times V \to \mathbb{R}$ given by $(A, B) \mapsto \mathrm{Tr}(AB)$ is a bilinear form.

   (b) Determine the signature of φ.

   (c) Determine the signature of φ on the subspace $\mathfrak{sl}_2(\mathbb{R})$.

(d) Let φ be a nondegenerate bilinear form on V.

   (a) Given any $\psi \in V^\vee$, show there is a unique vector $v \in V$ so that

$$\psi(w) = \varphi(w, v)$$

   for every $w \in V$.

   (b) Find a polynomial $q \in P_2(\mathbb{R})$ so that

$$p(1/2) = \int_0^1 p(t)q(t)dt$$

   for every $p \in P_2(\mathbb{R})$.

(e) Let $V = P_2(\mathbb{R})$ and let $\varphi(f, g) = \int_{-1}^1 f(x)g(x)dx$. Find an orthonormal basis of V.

(f) (a) Show that if V is a finite dimensional vector space over F with F not of characteristic 2 and φ is a symmetric bilinear form (not necessarily nondegenerate!) then V has an orthogonal basis with respect to φ.

   (b) Let $V = \mathbb{Q}^3$. Let φ be the bilinear form represented by

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

   in the standard basis. Find an orthogonal basis of V with respect to φ.

(g) (a) Let φ be a bilinear form on a real vector space V. Show there is a symmetric form φ_1 and a skew-symmetric form φ_2 so that $\varphi = \varphi_1 + \varphi_2$.

(b) Let $A \in \mathrm{Mat}_n(\mathbb{R})$. Show that there is a unique symmetric matrix B and a unique skew-symmetric matrix C so that $A = B+C$.

(h) Let $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$. Find the projection maps π_i associated to A via the Spectral Theorem. Give an orthogonal matrix P that diagonalizes A.

(i) Show that $T \in \mathrm{Hom}_V(F, F)$ is positive if and only if $\psi(v,w) = \varphi(T(v), w)$ is an inner product.

(j) Find a polar decomposition for the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

(k) Provide a detailed proof of Theorem 6.3.12.

(l) Compute the singular value decomposition and polar decomposition of the matrix $A = \begin{pmatrix} 8+i & -12 \\ 4 & -6+1 \end{pmatrix}$.

Chapter 7

Tensor products, exterior algebras, and the determinant

In this chapter we will introduce tensor products and exterior algebras and use the exterior algebra to provide a coordinate free definition of the determinant. The first time one sees a determinant it seems very unnatural. Viewing the determinant from the perspective of exterior algebras makes it a very natural object to consider. It takes consider machinery to work up to the definition, but once it has been developed the familiar properties of the determinant essentially fall out for free. The benefit of this is that the machinery that one builds up is useful in many other contexts as well. Unfortunately, by restricting ourselves to tensor products over vector spaces we miss many of the more interesting and nontrivial properties one obtains by studying tensor products of modules.

7.1 Extension of scalars

In this section we discuss a particular example of tensor products that can be useful in many situations. Namely, given an F-vector space V and a field K that contains F, can we form a K-vector space that contains a copy of V? Before we can do this, we need to address a basic question about forming vector spaces with a given basis.

We have seen early on that given a vector space V over a field F, we can always find a basis for V. Before we can define tensor products, we need to address the very natural question of if we are given a set X, can we find a vector space that has X as a basis? In fact, one can construct such a vector space and moreover the vector space we construct satisfies a universal property.

Theorem 7.1.1. *Let F be a field and X a set. There is an F-vector space $\mathrm{VS}_F(X)$ that has X as a basis. Moreover, $\mathrm{VS}_F(X)$ satisfies the following universal property. If W is any F-vector space and $t : X \to W$ is any map of sets, there is a unique $T \in \mathrm{Hom}_F(\mathrm{VS}_F(X), W)$ so that $T(x) = t(x)$ for all $x \in X$, i.e., the following diagram commutes:*

$$\begin{tikzcd}
X \arrow[r, "incl."] \arrow[dr, "t"'] & \mathrm{VS}_F(X) \arrow[d, "T"] \\
& W
\end{tikzcd}$$

Proof. This result should not be surprising. In fact, the universal property essentially amounts to the fact that a linear map is determined by its image on a basis.

If $X = \emptyset$, set $\mathrm{VS}_F(X) = \{0\}$ and we are done. If $X \neq \emptyset$, let

$$\mathrm{VS}_F(X) = \left\{ \sum_{i \in I} a_i x_i : a_i \in F,\ a_i = 0 \text{ for all but finitely many } i \right\}$$

where I is an indexing set of the same cardinality as X. We define

$$\sum_{i \in I} a_i x_i + \sum_{i \in I} b_i x_i = \sum_{i \in I} (a_i + b_i) x_i$$

and

$$c \left(\sum_{i \in I} a_i x_i \right) = \sum_{i \in I} c a_i x_i.$$

It is easy to see these both lie in $\mathrm{VS}_F(X)$ and that $\mathrm{VS}_F(X)$ is a vector space under this addition and scalar multiplication. By definition we have X spans $\mathrm{VS}_F(X)$ and is linearly independent by construction, so X is a basis for $\mathrm{VS}_F(X)$.

It remains to show the universal property. Let $t : X \to W$. We define $T : \mathrm{VS}_F(X) \to W$ by

$$T \left(\sum_{i \in I} a_i x_i \right) = \sum_{i \in I} a_i t(x_i).$$

This gives a well-defined linear map because X is a basis. It is unique because any linear map that agrees with t on X must also agree with T on $\mathrm{VS}_F(X)$. This gives the result. □

It is important to note in the above result that we treat X as strictly a set. It may be the case that X is itself a vector space, but when we form $\mathrm{VS}_F(X)$ we ignore any structure that X may have.

Example 7.1.2. Let $F = \mathbb{R}$ and $X = \mathbb{R}$. We know that as an $\mathbb{R}$-vector space $\mathbb{R}$ is 1-dimensional with 1 as a basis. However, if we consider $\mathrm{VS}_{\mathbb{R}}(\mathbb{R})$ as an $\mathbb{R}$-vector space, this is infinite dimensional with every element of $\mathbb{R}$ as a basis element. For example, an element of $\mathrm{VS}_{\mathbb{R}}(\mathbb{R})$ is $2\cdot 3+4\cdot\pi$ where 3 and π are considered as elements of X. This sum does not simplify in $\mathrm{VS}_{\mathbb{R}}(\mathbb{R})$.

We now turn our attention to the main topic of this section, extension of scalars. Let V be an F-vector space and let K/F be an extension of fields, i.e., $F \subset K$ and K is a field. We can naturally consider K as an F-vector space as well. When studying problems over the vector space V, say looking for the Jordan canonical form of a linear transformation, the field F may not be big enough. In this case we would like to change the scalars of V so that we are working over a bigger field. We can construct such a vector space by forming the tensor product of V with K. Let $X = \{(a, v) : a \in K, v \in V\}$. As above, we consider this as just a set and forget the vector space structure on it. We form the K-vector space $\mathrm{VS}_K(X)$; elements in this space are given by $\sum c_i(a_i, v_i)$ where $c_i \in K$. This is a K-vector space, but it does not take into account the F-vector space structure of K or V at all. This means it is much too large to be useful for anything. We will cut this space down to something useful by taking the quotient by an appropriate subspace. We define a subspace $\mathrm{Rel}_K(X)$ of $\mathrm{VS}_K(X)$ by setting $\mathrm{Rel}_K(X)$ to be the K-span of the elements

(a) $(a_1 + a_2, v) - (a_1, v) - (a_2, v)$ for all $a_1, a_2 \in K$, $v \in V$;

(b) $(a, v_1 + v_2) - (a, v_1) - (a, v_2)$ for all $a \in K$, $v_1, v_2 \in V$;

(c) $(ca, v) - (a, cv)$ for all $c \in F, a \in K$ and $v \in V$;

(d) $a_1(a_2, v) - (a_1a_2, v)$ for all $a_1, a_2 \in K$, $v \in V$.

Note the Rel stands for "relations" as we are using this quotient to identify some relations we are requiring be satisfied. We now consider the quotient space $K \otimes_F V = \mathrm{VS}_K(X)/\mathrm{Rel}_K(X)$. The fact that $K \otimes_F V$ is a vector space follows immediately because we have constructed it as the quotient of two vector spaces. Note the F subscript on the $\otimes$ indicates the original field that K and V are vector spaces over. Given an element $(a, v) \in \mathrm{VS}_K(X)$, we denote the equivalence class $(a, v) + \mathrm{Rel}_K(X)$ by $a \otimes v$. Observe from the definition of $K \otimes_F V$ that we have

(a) $(a_1 + a_2) \otimes v = a_1 \otimes v + a_2 \otimes v$ for all $a_1, a_2 \in K$, $v \in V$;

(b) $a \otimes (v_1 + v_2) = a \otimes v_1 + a \otimes v_2$ for all $a \in K$, $v_1, v_2 \in V$;

(c) $ca \otimes v = a \otimes cv$ for all $c \in F, a \in K$ and $v \in V$;

(d) $a_1(a_2 \otimes v) = (a_1a_2) \otimes v$ for all $a_1, a_2 \in K$, $v \in V$.

It is very important to note that a typical element in $K \otimes_F V$ is of the form $\sum c_i(a_i \otimes v_i)$ where c_i is 0 for all but finitely many i. Note we can combine the c_i and a_i so in this case a typical element is really just $\sum a_i \otimes v_i$ where $a_i = 0$ for all but finitely many i. It is a common mistake when working with tensor products to check things for elements $a \otimes v$ without remembering this is not a typical element! One other important point is that since $K \otimes_F V$ is a quotient, the elements are all equivalence classes. So one must be very careful to check things are well-defined when working with tensor products!

We have that $0 \otimes v = c \otimes 0 = 0 \otimes 0 = 0_{K \otimes_F V}$ for all $v \in V$, $c \in K$. First, note that since $0 \in K$ is also in F, we have $0 \otimes v = 0 \otimes 0v = 0 \otimes 0$ and $c \otimes 0 = 0c \otimes 0 = 0 \otimes 0$. Now we use uniqueness of the additive identity in a vector space and the fact that $0 \otimes 0$ is clearly an additive identity in V.

Example 7.1.3. Let $K = \mathbb{C}$, $F = \mathbb{R}$ and let V be an F-vector space. We consider some elements in $\mathbb{C} \otimes_{\mathbb{R}} V$ and how they simplify. We have

$$\begin{aligned} i((2+i) \otimes v) + 6 \otimes v &= (-1+2i) \otimes v + 6 \otimes v \\ &= (5+2i) \otimes v. \end{aligned}$$

The reason we were able to combine the terms is the term coming from V, namely v, was the same in both. Similarly, we have

$$2 \otimes v_1 + 2 \otimes v_2 = 2 \otimes (v_1 + v_2).$$

One other thing we can use to simplify is to bring a scalar that is in $F = \mathbb{R}$ across the tensor. So we have

$$\begin{aligned} 2 \otimes v_1 + 7 \otimes v_2 &= 1 \otimes 2v_1 + 1 \otimes 7v_2 \\ &= 1 \otimes (2v_1 + 7v_2). \end{aligned}$$

However, if the first terms in the tensors lies in $K = \mathbb{C}$ and are unequal, and the second terms are not equal, the sum cannot be combined into one term. For instance, we have

$$\begin{aligned} (2+i) \otimes v_1 + 3i \otimes v_2 &= 2 \otimes v_1 + i \otimes v_1 + 3i \otimes v_2 \\ &= 1 \otimes 2v_1 + i \otimes v_1 + i \otimes 3v_2. \end{aligned}$$

The only way the first two terms could be combined is if $v_1 = 0$, and for the second two one needs $v_1 = 3v_2$. Which means to combine this to one term the vectors must all be 0.

Since the original goal was to extend V to be a vector space over K, it is important to check that there is actually a subspace of $K \otimes_F V$ that is isomorphic to V as an F-vector space.

Proposition 7.1.4. *Let K/F be a field extension and V an F-vector space. Then $K \otimes_F V$ contains an F-subspace isomorphic to V as an F-vector space.*

Proof. Let $\mathcal{B} = \{v_i\}$ be an F-basis of V. Define $T \in \mathrm{Hom}_F(V, K \otimes_F V)$ by setting $T(v_i) = 1 \otimes v_i$. Let W be the image of T, i.e., W is the F-span of $\{1 \otimes v_i\}$. By definition this gives W is an F-subspace of $K \otimes_F V$. Note that this is not a K-subspace of $K \otimes_F V$. It is also clear that T is a surjective linear map. It only remains to see that it is injective. Suppose $T(v) = 0$ for some $v \in V$. Then we have $1 \otimes v = 0 \otimes 0$. This is equivalent to $(1, v) - (0, 0) \in \mathrm{Rel}_K(X)$. However, from the definition of $\mathrm{Rel}_K(X)$ it is clear this is only the case if $v = 0$. Thus, T is injective and so we have the result. □

The K-vector space $K \otimes_F V$ is referred to as the *extension of scalars of V by K*. Thus, we have constructed a K-vector space that contains a copy of V as an F-subspace. The following universal property shows that we have done as good as possible, i.e., the K-vector space $K \otimes_F V$ is the smallest K-vector space that contains V as an F-subspace.

Theorem 7.1.5. *Let K/F be an extension of fields, V an F-vector space, and $\iota : V \to K \otimes_F V$ given by $\iota(v) = 1 \otimes v$. Let W be a K-vector space and $t \in \mathrm{Hom}_F(V, W)$. There is a unique $T \in \mathrm{Hom}_K(K \otimes_F V, W)$ so that $t = T \circ \iota$, i.e., the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "t"'] & K \otimes_F V \arrow[d, "T"] \\ & W. \end{tikzcd}$$

Conversely, if $T \in \mathrm{Hom}_K(K \otimes_F V, W)$ then $T \circ \iota \in \mathrm{Hom}_F(V, W)$.

Proof. Let $t \in \mathrm{Hom}_F(V, W)$. Consider the K-vector space $\mathrm{VS}_K(K \times V)$. Since W is a K-vector space, we have a map

$$\begin{aligned} K \times V &\to W \\ (c, v) &\mapsto ct(v). \end{aligned}$$

This extends to a map $T : \mathrm{VS}_K(K \times V) \to W$ by Theorem 7.1.1. Moreover, $T \in \mathrm{Hom}_K(\mathrm{VS}_K(K \times V), W)$. It is now easy to check that T is 0 when restricted to $\mathrm{Rel}_K(K \times V)$. Thus, we have $T : K \otimes_F V \to W$ given by $T(a \otimes v) = at(v)$. Observe we have

$$\begin{aligned} cT(a \otimes v) &= c(at(v)) \\ &= (ca)t(v) \\ &= T(c(a \otimes v)) \end{aligned}$$

for all $a, c \in K$ and $v \in V$. It is also easy to see that T is additive, and so $T \in \mathrm{Hom}_K(K \otimes_F V, W)$. This gives the existence of T and that the diagram commutes.

We have that $K \otimes_F V$ is given by the K-span of elements of the form $1 \otimes v$, so any K-linear map on $K \otimes_F V$ is determined by its image on these elements. Since $T(1 \otimes v) = t(v)$, we get T is uniquely determined by t. This gives the uniqueness statement.

It is now an easy exercise to check that for any $T \in \mathrm{Hom}_K(K \otimes_F V, W)$, one has $T \circ \iota \in \mathrm{Hom}_F(V, W)$. □

Example 7.1.6. Let K/F be an extension of fields. In this example we show that $K \otimes_F F \cong K$ as K-vector spaces. We have a natural inclusion map $i : F \to K$. We write $\iota : F \to K \otimes_F F$ as above. From the previous result we obtain a unique K-linear map $T : K \otimes_F F \to K$ so that the following diagram commutes

$$\begin{tikzcd}
F \arrow[r, "\iota"] \arrow[dr, "i"'] & K \otimes_F F \arrow[d, "T"] \\
& K.
\end{tikzcd}$$

Thus, we see $T(1 \otimes x) = x$. Moreover, since T is K-linear this completely determines T because for $\sum a_i \otimes x_i \in K \otimes_F F$, we have

$$\begin{aligned}
T\left(\sum a_i \otimes x_i\right) &= \sum T(a_i \otimes x_i) \\
&= \sum T(a_i(1 \otimes x_i)) \\
&= \sum a_i T(1 \otimes x_i).
\end{aligned}$$

Define $S : K \to K \otimes_F F$ by $S(y) = y \otimes 1$. We clearly have $S \in \mathrm{Hom}_K(K, K \otimes_F F)$ and we have

$$S \circ T(y \otimes 1) = S(y) = y \otimes 1$$

and

$$T \circ S(y) = T(y \otimes 1) = y.$$

Thus, we have $T^{-1} = S$ and so $K \otimes_F F \cong K$ as K-vector spaces.

Example 7.1.7. More generally, let K/F be an extension of fields and let V be an n-dimensional F-vector space. We claim that $K \otimes_F V \cong K^n$ as K-vector spaces. We being by using the universal property to obtain a K-linear map from $K \otimes_F V$ to K^n. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V and define an F-linear map $t : V \to K^n$ by $t(v_i) = e_i$ where e_i is the standard basis element of K^n. Using the universal property we obtain a K-linear map $T : K \otimes_F V \to K^n$ given by $T(1 \otimes v_i) = e_i$ for $i = 1, \ldots, n$.

Define a linear map $S : K^n \to K \otimes_F V$ by $S(e_i) = 1 \otimes v_i$. We know to define a linear map we only need to specify where it sends a basis, so this gives a well-defined K-linear map. Moreover, it is easy to see that S and T are inverse maps. Thus, $K \otimes_F V \cong K^n$. Moreover, since S is an isomorphism and $\{e_1, \dots, e_n\}$ is a basis for K^n, we see $\{1 \otimes v_1, \dots, 1 \otimes v_n\}$ is a basis of $K \otimes_F V$.

7.2 Tensor products of vector spaces

We now turn our attention to tensor products of two F-vector spaces. One motivation for defining tensor products is it gives a way to form a new vector space where one has the "product" of elements from the original vector space. We began with a fairly simple case to help motivate the more general definition. Now that we have seen a particular example of tensor products in the previous section, namely, extension of scalars, we return to the more general situation of forming the tensor product of two F-vector spaces V and W. Since the set-up is very similar to what was done in the previous section many of the details will be left to the reader.

Let V and W be F-vector spaces. (If we allow them to be K-vector spaces as well, we will recover what was done in the previous section.) Consider $X = V \times W = \{(v, w) : v \in V, w \in W\}$ and let $\mathrm{VS}_F(V \times W)$ be the associated F-vector space as above. Let $\mathrm{Rel}_F(V \times W)$ be the subspace of $\mathrm{VS}_F(V \times W)$ given by the F-span of

(a) $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$ for all $v_1, v_2 \in V$, $w \in W$;

(b) $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$ for all $v \in V$, $w_1, w_2 \in W$;

(c) $(cv, w) - (v, cw)$ for all $c \in F$, $v \in V, w \in W$;

(d) $c(v, w) - (cv, w)$ for all $c \in F$, $v \in V, w \in W$.

Then, as above, we define $V \otimes_F W = \mathrm{VS}_F(V \times W) / \mathrm{Rel}_F(V \times W)$. As before, denote the equivalence class containing (v, w) by $v \otimes w$.

We have $V \otimes_F W$ is an F-vector space with elements of the form $\sum_i c_i(v_i \otimes w_i)$. Note, we can represent any such element by combining c_i with the $v_i \otimes w_i$, so elements can be represented in the form $\sum v_i \otimes w_i$ for $v_i \in V, w_i \in W$ with only finitely many terms being nonzero. Elements of the form $v \otimes w$ are called *pure tensors*.

We would like a similar universal property to the one we gave above for $K \otimes_F V$. However, this requires a new type of map. We defined bilinear maps $\mathrm{Hom}_F(V, V; F)$ in Chapter 5. We now extend that definition.

Definition 7.2.1. Let V, W, and U be F-vector spaces. We say a map $t : V \times W \to U$ is an *F-bilinear map* (or just bilinear map if F is clear from context), and write $t \in \mathrm{Hom}_V(V, W; U)$, if it satisfies

(a) $t(cv_1 + v_2, w) = ct(v_1, w) + t(v_2, w)$ for all $c \in F$, $v_i \in V$, $w \in W$;

(b) $t(v, cw_1 + w_2) = ct(v, w_1) + t(v, w_2)$ for all $c \in F$, $v \in V$, $w_i \in W$.

The point of bilinear maps as opposed to linear maps is they treat V and W as vector spaces, but they do not use the fact that we can define a vector space structure on $V \times W$. They keep V and W separate in terms of the algebraic structure; they are linear in each variable separately. This allows us to give the appropriate universal property for $V \otimes_F W$.

Theorem 7.2.2. *Let U, V, and W be F-vector spaces. Define a map $\iota : V \times W \to V \otimes_F W$ by $\iota(v, w) = v \otimes w$. Then*

(a) $\iota \in \operatorname{Hom}_F(V, W; V \otimes_F W)$, i.e., ι is F-bilinear;

(b) if $T \in \operatorname{Hom}_F(V \otimes_F W, U)$, then $T \circ \iota \in \operatorname{Hom}_F(V, W; U)$;

(c) if $t \in \operatorname{Hom}_F(V, W; U)$, then there is a unique $T \in \operatorname{Hom}_F(V \otimes_F W, U)$ so that $t = T \circ \iota$.

Equivalently, we can write the correspondence by saying we have a bijection between $\operatorname{Hom}_F(V, W; U)$ and $\operatorname{Hom}_F(V \otimes_F W, U)$ so that the following diagram commutes:

$$\begin{tikzcd}
V \times W \arrow[r, "\iota"] \arrow[dr, "t"'] & V \otimes_F W \arrow[d, "T"] \\
 & U.
\end{tikzcd}$$

Proof. (a) This part follows immediately from the definition and properties of the tensors.

(b) We will show that $t = T \circ \iota$ is linear in the first variable; linear in the second variable is the same argument. Let $v_1, v_2 \in V$, $w \in W$, and $c \in F$. We have

$$\begin{aligned}
t(cv_1 + v_2, w) &= T \circ \iota(cv_1 + v_2, w) \\
&= T((cv_1 + v_2) \otimes w) \\
&= T(cv_1 \otimes w + v_2 \otimes w) \\
&= cT(v_1 \otimes w) + T(v_2 \otimes w) \\
&= ct(v_1, w) + t(v_2, w).
\end{aligned}$$

Thus, t is bilinear in the first variable.

(c) Let $t \in \operatorname{Hom}_F(V, W; U)$. We have that t vanishes on the elements of $\operatorname{Rel}_F(V \times W)$ by the properties of being a bilinear map. Thus, we obtain a well-defined linear map $T : V \otimes_F W \to U$ so that $T(v \otimes w) = t(v, w)$. Thus, we have a unique map $T \in \operatorname{Hom}_F(V \otimes_F W, U)$ satisfying $T \circ \iota(v, w) = T(v \otimes w) = t(v, w)$, as claimed.

□

Exercise 7.2.3. Show that $\mathrm{Hom}_F(V, W; U)$ is isomorphic to $\mathrm{Hom}_F(V \otimes_F W, U)$ as F-vector spaces.

We now illustrate how this universal property can be used to prove basic properties about tensor products. It is extremely powerful because it allows one to define a bilinear map on $V \times W$ and obtain a linear map on $V \otimes_F W$. The reason this is so nice is to define a map directly on $V \otimes_F W$ one must also check the map is well-defined, which can be very tedious.

Corollary 7.2.4. *Let V and W be F-vector spaces with bases $\mathcal{B} = \{v_1, \ldots, v_m\}$ and $\mathcal{C} = \{w_1, \ldots, w_n\}$ respectively. Then $\{v_i \otimes w_j\}_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$ is a basis for $V \otimes_F W$. In particular, $\dim_F(V \otimes_F W) = \dim_F V \dim_F W$.*

Proof. We show this result by showing that $V \otimes_F W \cong \mathrm{Mat}_{m,n}(F) \cong F^{mn}$. Define a map $t : V \times W \to F^{mn}$ by $t((v_i, w_j)) = e_{i,j}$. First, observe this is enough to define a bilinear map. Given $v \in V$ and $w \in W$, write $v = \sum_{i=1}^{m} a_i v_i$ and $w = \sum_{j=1}^{m} b_j w_j$. Then

$$\begin{aligned} t(v, w) &= t\left(\sum_{i=1}^{m} a_i v_i, \sum_{j=1}^{n} b_j w_j\right) \\ &= \sum_{i=1}^{m} a_i t\left(v_i, \sum_{j=1}^{n} b_j w_j\right) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j t(v_i, w_j). \end{aligned}$$

Thus, just as it was enough to define a linear map on a basis, it is enough to specify the values of a bilinear map on elements (v_i, w_j) for $v_i \in \mathcal{B}$ and $w_j \in \mathcal{C}$. We now apply the universal property to obtain an F-linear map $T : V \otimes_F W \to \mathrm{Mat}_{m,n}(F)$ that satisfies $T(v_i \otimes w_j) = e_{i,j}$. We can define an F-linear map $S : \mathrm{Mat}_{m,n}(F) \to V \otimes_F W$ by $S(e_{i,j}) = v_i \otimes w_j$. It is clear that S and T are inverse maps, so we obtain $V \otimes_F W \cong \mathrm{Mat}_{m,n}(F) \cong F^{mn}$. Moreover, since $\{e_{i,j}\}$ forms a basis for $\mathrm{Mat}_{m,n}(F)$ and S is an isomorphism, we have that $\{v_i \otimes w_j\}$ is a basis of $V \otimes_F W$. □

Example 7.2.5. The vector space $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{R}^4$. A basis for $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is given by $\{1 \otimes 1, 1 \otimes i, i \otimes 1, i \otimes i\}$.

The following results follow immediately from this corollary.

Corollary 7.2.6. *Let U, V and W be F-vector spaces. We have*

(a) $V \otimes_F W \cong W \otimes_F V$;

(b) $(U \otimes_F V) \otimes_F W \cong U \otimes_F (V \otimes_F W)$.

Given F-vector spaces U, V and W, the next theorem follows immediately from Corollary 7.2.4. However, we give a direct proof. The benefits of this are it shows the spirit of how such things would be proven for modules, and it also allows us to easily see what the isomorphism is and show it is unique.

Theorem 7.2.7. *Let U, V and W be F-vector spaces. There is a unique isomorphism*

$$\begin{aligned}(U \oplus V) \otimes_F W &\xrightarrow{\simeq} (U \otimes_F W) \oplus (V \otimes_F W)\\ (u,v) \otimes w &\mapsto (u \otimes w, v \otimes w).\end{aligned}$$

Proof. We will once again make use of the universal property to define the appropriate maps. Define

$$\begin{aligned}(U \oplus V) \times W &\longrightarrow (U \otimes_F W) \oplus (V \otimes_F W)\\ ((u,v),w) &\mapsto (u \otimes w, v \otimes w).\end{aligned}$$

It is easy to see this map is bilinear, so the universal property gives a unique linear map

$$\begin{aligned}T : (U \oplus V) \otimes_F W &\longrightarrow (U \otimes_F W) \oplus (V \otimes_F W)\\ (u,v) \otimes w &\mapsto (u \otimes w, v \otimes w).\end{aligned}$$

It now remains to define an inverse map. We begin by defining maps

$$\begin{aligned}U \times W &\longrightarrow (U \oplus V) \otimes_F W\\ (u,w) &\mapsto (u,0) \otimes w\end{aligned}$$

and

$$\begin{aligned}V \times W &\longrightarrow (U \oplus V) \otimes_F W\\ (v,w) &\mapsto (0,v) \otimes w.\end{aligned}$$

The universal property applied to each of these maps gives

$$\begin{aligned}S_1 : U \otimes_F W &\longrightarrow (U \oplus V) \otimes_F W\\ u \otimes w &\mapsto (u,0) \otimes w\end{aligned}$$

and

$$\begin{aligned}S_2 : V \otimes_F W &\longrightarrow (U \oplus V) \otimes_F W\\ v \otimes w &\mapsto (0,v) \otimes w.\end{aligned}$$

Combining these gives a linear map

$$\begin{aligned}S : (U \otimes_F W) \oplus (V \otimes_F W) &\longrightarrow (U \oplus V) \otimes_F W\\ (u \otimes w_1, v \otimes w_2) &\mapsto (u,0) \otimes w_1 + (0,v) \otimes w_2.\end{aligned}$$

It now only remains to check that these are inverse maps. Since these are linear maps it is enough to check it on pure tensors. We have

$$\begin{aligned} S \circ T((u,v) \otimes w) &= S(u \otimes w, v \otimes w) \\ &= (u,0) \otimes w + (0,v) \otimes w \\ &= (u,v) \otimes w \end{aligned}$$

and

$$\begin{aligned} T \circ S(u \otimes w_1, v \otimes w_2) &= T((u,0) \otimes w_1 + (0,v) \otimes w_2) \\ &= T((u,0) \otimes w_1) + T((0,v) \otimes w_2) \\ &= (u \otimes w_1, 0 \otimes w_1) + (0 \otimes w_2, v \otimes w_2) \\ &= (u \otimes w_1, 0) + (0, v \otimes w_2) \\ &= (u \otimes w_1, v \otimes w_2). \end{aligned}$$

Thus, we have the desired isomorphism. □

We can use the tensor product to give a coordinate-free construction of the trace map. We begin with the following lemma

Lemma 7.2.8. *Let V be a finite dimensional F vector space. Then $V \otimes V^\vee \cong \mathrm{Hom}_F(V,V)$.*

Proof. It is clear since the dimensions of the spaces are the same that they are isomorphic, but for our purposes we need to know the specific map giving the isomorphism. We define a map t

$$\begin{aligned} V \times V^\vee &\to \mathrm{Hom}_F(V,V) \\ (v,\varphi) &\mapsto (w \mapsto \varphi(w)v). \end{aligned}$$

It is easy to check this is a bilinear map, so the universal property gives a map $\mathcal{T} : V \otimes_F V^\vee \to \mathrm{Hom}_F(V,V)$ so that $\mathcal{T}(v \otimes \varphi)(w) = \varphi(w)v$. It remains to check this map is an isomorphism. Since the dimensions of the spaces are the same, it is enough to show that $\mathcal{T}$ is injective. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for V and $\{v_1^\vee, \dots, v_n^\vee\}$ a dual basis for $V^\vee$. Suppose that $\mathcal{T}\left(\sum_{i=1}^n a_{i,j}(v_i \otimes v_j^\vee)\right) = 0$ in $\mathrm{Hom}_F(V,V)$. Thus, for v_m, we must have

$$\begin{aligned} 0 &= \mathcal{T}\left(\sum_{i,j=1}^n a_{i,j}(v_i \otimes v_j^\vee)\right)(v_m) \\ &= \sum_{i,j=1}^n a_{i,j} v_j^\vee(v_m) v_i \\ &= \sum_{i=1}^n a_{i,m} v_i. \end{aligned}$$

However, since $\mathcal{B}$ is a basis, this gives $a_{i,m} = 0$ for all $i = 1, \ldots, n$. Since m was arbitrary, this gives $a_{i,j} = 0$ for all i, j, i.e., $\mathcal{T}$ is injective. Thus, we have $\mathcal{T} : V \otimes_F V^\vee \to \operatorname{Hom}_F(V, V)$ is an isomorphism. □

In $\operatorname{Hom}_F(V, V)$ one can compose maps (recall this corresponds to matrix multiplication.) Since $V \otimes_F V^\vee \cong \operatorname{Hom}_F(V, V)$, there is a well-defined map $(V \otimes_F V^\vee) \times (V \otimes_F V^\vee) \to (V \otimes_F V^\vee)$ that corresponds to the composition of linear maps on V. We claim this map is given by sending $(v \otimes \varphi) \times (w \otimes \psi)$ to $\varphi(w)v \otimes \psi$. Denote this map by Ψ. We need to show the following diagram commutes:

$$\begin{tikzcd}
(V \otimes_F V^\vee) \times (V \otimes_F V^\vee) \arrow[r, "\Psi"] \arrow[d, "\mathcal{T}\times\mathcal{T}"'] & V \otimes V^\vee \arrow[d, "\mathcal{T}"] \\
\operatorname{Hom}_F(V, V) \times \operatorname{Hom}_F(V, V) \arrow[r, "\text{comp}"] & \operatorname{Hom}_F(V, V).
\end{tikzcd}$$

Let $(v \otimes \varphi) \times (w \otimes \psi) \in (V \otimes_F V^\vee) \times (V \otimes_F V^\vee)$. We compute the image of this in $\operatorname{Hom}_F(V, V)$ by going in each direction of the diagram. If we first apply $\mathcal{T} \times \mathcal{T}$ and then composition we obtain for each $x \in V$

$$\begin{aligned}
\mathcal{T}(v \otimes \varphi) \circ \mathcal{T}(w \otimes \psi)(x) &= \mathcal{T}(v \otimes \varphi)(\mathcal{T}(w \otimes \psi)(x)) \\
&= \mathcal{T}(v \otimes \varphi)(\psi(x)w) \\
&= \psi(x)\mathcal{T}(v \otimes \varphi)(w) \\
&= \psi(x)\varphi(w)v.
\end{aligned}$$

Going in the other direction we obtain

$$\begin{aligned}
\mathcal{T}(\Psi((v \otimes \varphi) \times (w \otimes \psi)))(x) &= \mathcal{T}(\varphi(w)(v \otimes \psi))(x) \\
&= \varphi(w)\mathcal{T}(v \otimes \psi)(x) \\
&= \varphi(w)\psi(x)v.
\end{aligned}$$

Since the diagram commutes, we have that Ψ is the map corresponding to composition of functions.

We now apply these results to the trace map. Let $T \in \operatorname{Hom}_F(V, V)$. The trace map from undergraduate linear algebra is defined relative to a choice of basis. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V and $A = (a_{ij}) = [T]_{\mathcal{B}}$ the associated matrix. Let $\mathcal{B}^\vee = \{v_1^\vee, \ldots, v_n^\vee\}$ be the dual basis. It is straightforward to check that

$$a_{ij} = v_i^\vee(T(v_j))$$

for all $1 \leq i, j \leq n$. In particular, if we use Tr to denote the familiar trace from undergraduate linear algebra, then

$$\operatorname{Tr}(A) = \sum_{i=1}^{n} v_i^\vee(T(v_j)).$$

This definition depends upon first choosing coordinates on V. We now give a coordinate free definition of the trace and show it agrees with this definition upon choosing a basis. Consider the linear map $V \otimes_F V^\vee \to F$ induced from the bilinear map $V \times V^\vee \to F$ given by $(v, \varphi) \mapsto \varphi(v)$. Since $V \otimes_F V^\vee \cong \mathrm{Hom}_F(V, V)$, this gives a map $\mathrm{tr} : \mathrm{Hom}_F(V, V) \to F$. Note that this map does not depend upon any choice of basis. We have that the elements $v_k \otimes v_l^\vee$ form a basis for $V \otimes_F V^\vee$, and so the elements $\mathcal{T}(v_k \otimes v_l^\vee)$ form a basis of $\mathrm{Hom}_F(V, V)$. We compute Tr on these elements with respect to the basis $\mathcal{B}$.

$$\begin{aligned}
\mathrm{Tr}(\mathcal{T}(v_k \otimes v_l^\vee)) &= \sum_{i=1}^n v_i^\vee(\mathcal{T}(v_k \otimes v_l^\vee)(v_i)) \\
&= \sum_{i=1}^n v_i^\vee(v_l^\vee(v_i)v_k) \\
&= \sum_{i=1}^n v_l^\vee(v_i)v_i^\vee(v_k) \\
&= v_l^\vee(v_k) \\
&= \begin{cases} 1 & k = l \\ 0 & k \neq l. \end{cases}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathrm{tr}(v_k \otimes v_l^\vee) &= v_l^\vee(v_k) \\
&= \begin{cases} 1 & k = l \\ 0 & k \neq l. \end{cases}
\end{aligned}$$

Since these two maps agree on basis elements, they are the same. We will use Tr to denote the trace map on $V \otimes V^\vee$ and $\mathrm{Hom}_F(V, V)$ as well since we have seen they specialize to the undergraduate definition upon choosing coordinates. This coordinate-free definition allows us to easily prove basic properties of the trace map. For instance, the construction we constructed the coordinate-free trace to be a linear map, so for $c \in F$ and $A, B \in \mathrm{Mat}_n(F)$, we have

$$\mathrm{Tr}(cA + B) = c\,\mathrm{Tr}(A) + \mathrm{Tr}(B).$$

Corollary 7.2.9. *Let $A, B \in \mathrm{Mat}_n(F)$. Then $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.*

Proof. We prove this using the coordinate-free definition; the result in terms of matrices follows immediately upon choosing coordinates. Observe that the maps

$$\begin{aligned}
\mathrm{Hom}_F(V, V) \times \mathrm{Hom}_F(V, V) &\to F \\
(A, B) &\mapsto \mathrm{Tr}(AB)
\end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_F(V,V)\times\mathrm{Hom}_F(V,V)&\to F\\(A,B)&\mapsto\mathrm{Tr}(BA)\end{aligned}$$

are both bilinear forms. Thus, it is enough to show they agree on pure tensors $v\otimes\varphi\in V\otimes V^\vee$. Let $v\otimes\varphi$ and $w\otimes\psi$ be two such pure tensors. We must show that

$$\mathrm{Tr}(\mathcal{T}(v\otimes\varphi)\circ\mathcal{T}(w\otimes\psi))=\mathrm{Tr}(\mathcal{T}(w\otimes\psi)\circ\mathcal{T}(v\otimes\varphi)).$$

Recall we showed above that composition of the linear maps $\mathcal{T}(v\otimes\varphi)\circ\mathcal{T}(w\otimes\psi)$ corresponds under the isomorphism identifying $V\otimes_F V^\vee$ with $\mathrm{Hom}_F(V,V)$ to $\varphi(w)v\otimes\psi$. Using this, we have

$$\begin{aligned}\mathrm{Tr}(\mathcal{T}(v\otimes\varphi)\circ\mathcal{T}(w\otimes\psi))&=\mathrm{Tr}(\varphi(w)v\otimes\psi)\\&=\varphi(w)\,\mathrm{Tr}(v\otimes\psi)\\&=\varphi(w)\psi(v)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Tr}(\mathcal{T}(w\otimes\psi)\circ\mathcal{T}(v\otimes\varphi))&=\mathrm{Tr}(\psi(v)w\otimes\varphi)\\&=\psi(v)\,\mathrm{Tr}(w\otimes\varphi)\\&=\psi(v)\varphi(w).\end{aligned}$$

Thus, we have the result. □

Before we can introduce the exterior product of a vector space, we need to consider multilinear maps and tensor products of finitely many vector spaces. In particular, we saw above that given F-vector spaces U,V and W, we have $U\otimes_F(V\otimes_F W)\cong(U\otimes_F V)\otimes_F W$ as F-vector spaces. Thus, it makes sense to just write $U\otimes_F V\otimes_F W$. By induction, given F-vector spaces $V_1,\dots,V_n$, it makes sense to write $V\otimes_F V_2\otimes_F\cdots\otimes_F V_n$. We will be particularly interested in the case when $V_i=V$ for all i. In this case we write $V^{\otimes n}$ for $V\otimes_F\cdots\otimes_F V$. If there is any chance of confusion we write $V^{\otimes_F n}$. We now define multilinear maps; these are the appropriate maps to consider in this context.

Definition 7.2.10. Let $V_1,\dots,V_n$ and W be F-vector spaces. A map

$$t:V_1\times\cdots\times V_n\to W$$

is said to be *multilinear* if t is linear in each variable separately. We denote the set of multilinear maps by $\mathrm{Hom}_F(V_1,\dots,V_n;W)$.

Exercise 7.2.11. Show that $\mathrm{Hom}_F(V_1,\dots,V_n;W)$ is an F-vector space.

For tensor products of several vector spaces we have a universal property as well; bilinear maps are just replaced by multilinear maps.

Theorem 7.2.12. *Let $V_1, \dots, V_n$ be F-vector spaces. Define*

$$\begin{aligned} \iota : V_1 \times \cdots \times V_n &\longrightarrow V_1 \otimes_F \cdots \otimes_F V_n \\ (v_1, \dots, v_n) &\mapsto v_1 \otimes \cdots \otimes v_n. \end{aligned}$$

Then we have

(a) For every $T \in \mathrm{Hom}_F(V_1 \otimes_F \cdots \otimes_F V_n; W)$ the map $T \circ \iota \in \mathrm{Hom}_F(V_1, \dots, V_n; W)$.

(b) For every $t \in \mathrm{Hom}_F(V_1, \dots, V_n; W)$ there is a unique $T \in \mathrm{Hom}_F(V_1 \otimes_F \cdots \otimes_F V_n, W)$ so that $t = T \circ \iota$.

The proof of this theorem is left as an exercise as it follows from the same type of arguments used in the case $n = 2$. Note the theorem can be restated by saying there is a bijection between $\mathrm{Hom}_F(V_1, \dots, V_n; W)$ and $\mathrm{Hom}_F(V_1 \otimes_F \cdots \otimes_F V_n, W)$ so that the following diagram commutes

$$\begin{tikzcd} V_1 \times \cdots \times V_n \arrow[r, "\iota"] \arrow[rd, "t"'] & V_1 \otimes_F \cdots \otimes_F V_n \arrow[d, "T"] \\ & W. \end{tikzcd}$$

Corollary 7.2.13. *Let $V_1, \dots V_k$ be F-vector spaces of dimension $n_1, \dots, n_k$. Let $\mathcal{B}_i = \{v_1^i, \dots, v_{n_i}^i\}$ be a basis of V_i. Show that $\{v_{i_1}^1 \otimes \cdots \otimes v_{i_k}^k\}$ is a basis for $V_1 \otimes_F \cdots \otimes_F V_k$. In particular, show that*

$$\dim_F(V_1 \otimes_F \cdots \otimes_F V_k) = \prod_{j=1}^{k} \dim_F V_i.$$

Proof. See homework problems. □

7.3 Alternating forms, exterior powers, and the determinant

In this section we will define exterior powers of a vector space and see how studying these gives the correct definition of the determinant.

Let V be a finite dimensional F-vector space and let k be a positive integer. We begin by defining the k^{th} exterior power of V.

Definition 7.3.1. Let $\mathcal{A}_k(V)$ of $V^{\otimes k}$ be the subspace spanned by $v_1 \otimes \cdots \otimes v_k$ where $v_i = v_j$ for some $i \neq j$. The k^{th} *exterior power of* V is the quotient space

$$\Lambda^k(V) = V^{\otimes k} / \mathcal{A}_k(V).$$

We denote elements of $\Lambda^k(V)$ by $v_1 \wedge \cdots \wedge v_k = v_1 \otimes \cdots \otimes v_k + \mathcal{A}_k(V)$. Note that we have $v_1 \wedge \cdots \wedge v_k = 0$ if $v_i = v_j$ for any $i \neq j$. Thus, for $v, w \in V$ we have

$$\begin{aligned} 0 &= (v+w) \wedge (v+w) \\ &= v \wedge v + v \wedge w + w \wedge v + w \wedge w \\ &= v \wedge w + w \wedge v. \end{aligned}$$

This gives that in $\Lambda^2(V)$ we have $v \wedge w = -w \wedge v$. This can be used for higher degree exterior powers as well. Namely, we have

$$v_1 \wedge \cdots \wedge v_i \wedge v_{i+1} \wedge \cdots \wedge v_k = -v_1 \wedge \cdots v_{i+1} \wedge v_i \wedge \cdots \wedge v_k.$$

This will be very useful when proving the universal property for exterior powers. Before we can state that, we need to define the appropriate maps.

Definition 7.3.2. Let V and W be F-vector spaces and let $t \in \mathrm{Hom}_F(V, \dots, V; W)$. We say t is *alternating* if $t(v_1, \dots, v_k) = 0$ whenever $v_i = v_{i+1}$ for some $1 \leq i \leq k-1$. We denote the set of alternating maps by $\mathrm{Alt}_F^k(V; W)$. We set $\mathrm{Alt}_F^0(V; W) = F$.

Note that in the case that $W = F$ and $k = 2$, an alternating map is just a skew-symmetric bilinear form.

Exercise 7.3.3. (a) Show that $\mathrm{Alt}_F^k(V; W)$ is an F-subspace of $\mathrm{Hom}_F(V, \dots, V; W)$.

(b) Show that $\mathrm{Alt}_F^1(V; F) = \mathrm{Hom}_F(V, F) = V^\vee$.

(c) If $\dim_F V = n$, show $\mathrm{Alt}_F^k(V; F) = 0$ for all $k > n$.

Theorem 7.3.4. *Let V and W be F-vector spaces and k a positive integer. Define a map $\iota : V \times \cdots \times V \to \Lambda^k(V)$ by $\iota(v_1, \dots, v_k) = v_1 \wedge \cdots \wedge v_k$. Then*

(a) $\iota \in \mathrm{Alt}_F^k(V; \Lambda^k(V))$, i.e., ι is alternating;

(b) if $T \in \mathrm{Hom}_F(\Lambda^k(V), W)$, then $T \circ \iota \in \mathrm{Alt}_F^k(V \times \cdots \times V; W)$;

(c) if $t \in \mathrm{Alt}_F^k(V \times \cdots \times V; W)$, then there is a unique $T \in \mathrm{Hom}_F(\Lambda^k(V), W)$ so that $t = T \circ \iota$.

Equivalently, we can write the correspondence by saying we have a bijection between $\mathrm{Alt}^k_F(V \times \cdots \times V; W)$ *and* $\mathrm{Hom}_F(\Lambda^k(V), W)$ *so that the following diagram commutes:*

$$\begin{tikzcd}
V \times \cdots \times V \arrow[r, "\iota"] \arrow[dr, "t"'] & \mathrm{Alt}^k_F(V) \arrow[d, "T"] \\
 & W.
\end{tikzcd}$$

Proof. This essentially follows from Theorem 7.2.12. To see ι is alternating, observe that it is the composition of the multilinear map used in the universal property of the tensor product composed with projection onto the quotient. This gives ι is multilinear. It vanishes on elements of the form $(v_1, \ldots, v_k)$ with $v_i = v_j$ for $i \neq j$ by the definition of the exterior power. This gives the first statement.

We now show $t = T \circ \iota$ is alternating. Let $c \in F$ and $v_1, \ldots, v_k, v_1' \in V$. We have

$$\begin{aligned}
t(av_1 + v_1', v_2, \ldots, v_k) &= T((av_1 + v_1') \wedge v_2 \wedge \cdots \wedge v_k) \\
&= T(av_1 \wedge v_2 \wedge \cdots \wedge v_k + v_1' \wedge v_2 \wedge \cdots \wedge v_k) \\
&= aT(v_1 \wedge v_2 \wedge \cdots \wedge v_k) + T(v_1' \wedge v_2 \wedge \cdots \wedge v_k) \\
&= at(v_1 \wedge v_2 \wedge \cdots \wedge v_k) + t(v_1' \wedge v_2 \wedge \cdots \wedge v_k).
\end{aligned}$$

This shows t is multilinear. To see it is also alternating, one just uses ι is alternating.

Suppose now we have $t \in \mathrm{Alt}^k_F(V \times \cdots \times V; W)$. Since the alternating forms are a subset of the multilinear forms, we obtain a linear map $S : V^{\otimes k} \to W$ so that $S(v_1 \otimes \cdots \otimes v_k) = t(v_1, \ldots, v_k)$. As $\Lambda^k(V)$ is a quotient of $V^{\otimes k}$, we obtain a linear map $T : \Lambda^k(V) \to W$ given by composing S with the projection map, i.e., $T(v_1 \wedge \cdots \wedge v_i) = t(v_1, \ldots, v_n)$. It is now easy to check this map is unique because it agrees with t and that the diagram above commutes. □

Example 7.3.5. Let V be an F-vector space with $\dim_F V = 1$. Thus, given any nonzero $v \in V$, the set $\{v\}$ forms a basis for V. Given any positive integer k, we can write elements of $\Lambda^k(V)$ as finite sums of elements of the form $a_1 v \wedge \cdots \wedge a_k v$. However, we know $a_1 v \wedge \cdots \wedge a_k v = (a_1 \cdots a_k) v \wedge \cdots \wedge v$ and since $v \wedge v = 0$, we have $a_1 v \wedge \cdots \wedge a_k v = 0$ for $k \geq 2$. Thus, we have $\Lambda^0(V) = F$, $\Lambda^1(V) \cong V$, and $\Lambda^k(V) = 0$ for $k \geq 2$.

Example 7.3.6. Let V be a 2-dimensional F-vector space and let $\mathcal{B} = \{v_1, v_2\}$ be a basis. Let k be a positive integer. Elements of $\Lambda^k(V)$ consist of finite sums of the form $(a_1 v_1 + b_1 v_2) \wedge \cdots \wedge (a_k v_1 + b_k v_2)$ for $a_i, b_i \in F$.

If $k \geq 3$, this sum is 0 because each term will have a term $v_i \wedge v_j \wedge v_l$, and since the dimension is 2 one of these terms will be a repeat and so 0. If $k = 2$, a typical element can be written in the form

$$\begin{aligned}(av_1 + bv_2) \wedge (cv_1 + dv_2) &= adv_1 \wedge v_2 + bcv_2 \wedge v_1 \\ &= (ad - bc)v_1 \wedge v_2.\end{aligned}$$

Thus, in this case we have $\Lambda^0(V) = F$, $\Lambda^1(V) \cong V$, and $\Lambda^2(V) \cong F(v_1 \wedge v_2) \cong F$.

Theorem 7.3.7. *Let V be an n-dimensional F-vector space. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V. For $k \leq n$ the the vectors $\{v_{i_1} \wedge \cdots \wedge v_{i_k} : 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$ form a basis of $\Lambda^k(V)$. For $k > n$ we have $\Lambda^k(V) = 0$. In particular, $\dim_F \Lambda^k(V) = \binom{n}{k}$.*

Proof. This follows easily from the fact that $\{v_{i_1} \otimes \cdots \otimes v_{i_k} : 1 \leq i_j \leq n\}$ is a basis for $V^{\otimes k}$. Since $\Lambda^k(V)$ is a quotient of $V^{\otimes k}$ by $\mathcal{A}_k(V)$, to find a basis it only amounts to finding a basis of $\mathcal{A}_k(V)$. However, we clearly have any element $v_{i_1} \otimes \cdots \otimes v_{i_k}$ with $i_{j_1} = i_{j_2}$ for some $j_1 \neq j_2$ lies in $\mathcal{A}_k(V)$. Moreover, we know the elements $v_{i_1} \wedge \cdots \wedge v_{i_k}$ can be reordered by introducing a negative sign. This gives the result. □

Exercise 7.3.8. Prove the above theorem directly from Theorem 7.3.4.

As was observed in the previous examples for particular cases, this theorem shows in general that for an n-dimensional F-vector space one has $\Lambda^n(V)$ is of dimension 1 over F and has as a basis $v_1 \wedge \cdots \wedge v_n$ for $\{v_1, \ldots, v_n\}$ a basis of V. This is the key point in defining the determinant of a linear map.

Let $T \in \operatorname{Hom}_F(V, W)$ with V and W finite dimensional vector spaces. The map T induces a map

$$\begin{aligned}T^{\otimes k} : V^{\otimes k} &\to W^{\otimes k} \\ v_1 \otimes \cdots \otimes v_k &\mapsto T(v_1) \otimes \cdots \otimes T(v_k).\end{aligned}$$

It is easy to see that the generators of $\mathcal{A}_k(V)$ are sent to generators of $\mathcal{A}_k(W)$, so this descends to a map

$$\begin{aligned}\Lambda^k(T) : \Lambda^k(V) &\to \Lambda^k(W) \\ v_1 \wedge \cdots \wedge v_k &\mapsto T(v_1) \wedge \cdots \wedge T(v_k).\end{aligned}$$

We now restrict to the case that $V = W$ and $\dim_F V = n$. Since $\Lambda^n(V)$ is 1-dimensional over F, we have $\Lambda^n(T)$ is just multiplication by a constant. This leads to the definition of the determinant of a map.

Definition 7.3.9. Let V be an F-vector space with $\dim_F V = n$. Let $T \in \operatorname{Hom}_F(V, V)$. We define the *determinant of* T, denoted $\det(T)$, to be the constant so that $\Lambda^n(T)(v) = (\det(T))v$ for all $v \in \Lambda^n(V)$. Given a matrix $A \in \operatorname{Mat}_n(F)$, we define the determinant of A to be the determinant of the associated linear map T_A.

One should note that it is clear from this definition of the determinant that there is no dependence on the choice of a basis of V as none was used in defining the determinant.

Lemma 7.3.10. *Let* $S, T \in \operatorname{Hom}_F(V, V)$. *Then* $\det(T \circ S) = \det(T)\det(S)$.

Proof. Let $v_1 \wedge \cdots \wedge v_n \in \Lambda^n(V)$ be any nonzero element (so a basis). We have

$$\begin{aligned}
\det(T \circ S) v_1 \wedge \cdots \wedge v_n &= \Lambda^n(T \circ S)(v_1 \wedge \cdots \wedge v_n) \\
&= T \circ S(v_1) \wedge \cdots \wedge T \circ S(v_n) \\
&= T(S(v_1)) \wedge \cdots \wedge T(S(v_n)) \\
&= \Lambda^n(T)(S(v_1) \wedge \cdots \wedge S(v_n)) \\
&= \Lambda^n(T)(\Lambda^n(S)(v_1 \wedge \cdots \wedge v_n)) \\
&= \Lambda^n(T)(\det(S) v_1 \wedge \cdots \wedge v_n) \\
&= \det(S)\Lambda^n(T)(v_1 \wedge \cdots \wedge v_n) \\
&= \det(S)\det(T) v_1 \wedge \cdots \wedge v_n.
\end{aligned}$$

This gives the result. □

Of course, for this to be useful we want to show this is the same determinant that was defined in undergraduate linear algebra class. Before showing this in general we check the case V has dimension 2.

Example 7.3.11. Let $V = F^2$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let T_A be the associated linear map. Thus, we have

$$\begin{aligned}
\Lambda^2(T)(e_1 \wedge e_2) &= T(e_1) \wedge T(e_2) \\
&= (ae_1 + ce_2) \wedge (be_1 + de_2) \\
&= abe_1 \wedge e_1 + ade_1 \wedge e_2 + cbe_2 \wedge e_1 + cde_2 \wedge e_2 \\
&= ade_1 \wedge e_2 + bce_2 \wedge e_1 \\
&= (ad - bc) e_1 \wedge e_2.
\end{aligned}$$

Thus, we see $\det(A) = ad - bc$, as one expects from undergraduate linear algebra.

Exercise 7.3.12. Let $A \in \operatorname{Mat}_3(F)$. Show that the definition of $\det(A)$ given here matches the definition from undergraduate linear algebra.

We now want to prove in general that given a matrix $A \in \mathrm{Mat}_n(F)$ one has $\det(A)$ is the same as the value from undergraduate linear algebra. Note that we can view A as an element of $F^n \times \cdots \times F^n$ with each column of A a vector in F^n. Thus, we can view

$$\det : F^n \times \cdots \times F^n \to F.$$

Theorem 7.3.13. *The determinant function is in* $\mathrm{Alt}^n(F^n, F)$ *and satisfies* $\det(1_n) = 1$.

Proof. We begin by checking bilinearity in the first variable; the other variables follow from the same argument. Write

$$\begin{aligned} w_1 &= a_{11}e_1 + \cdots + a_{n1}e_n \\ &\vdots \\ w_n &= a_{n1}e_1 + \cdots + a_{nn}e_n \end{aligned}$$

and $w = b_{11}e_1 + \cdots + b_{n1}e_n$ for $e_1, \ldots, e_n$ the standard basis of F^n. We want to show that for any $c \in F$ we have

$$\det(w_1 + cw, w_2, \ldots, w_n) = \det(w_1, w_2, \ldots, w_n) + c\det(w, w_2, \ldots, w_n).$$

To do this, we need to translate this into a statement about linear maps. Define $T_1 : F^n \to F^n$ by $T_1(e_i) = w_i$, so

$$[T_1]_{\mathcal{E}_n} = A_1 = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix};$$

$T_2 : F^n \to F^n$ by $T_2(e_1) = w$ and $T_2(e_j) = w_j$ for $2 \leq j \leq n$ so

$$[T_2]_{\mathcal{E}_n} = A_2 = \begin{pmatrix} b_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ b_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix};$$

$T_3 : F^n \to F^n$ by $T_3(e_1) = w_1 + cw$ and $T_3(e_j) = w_j$ for $2 \leq j \leq n$ so

$$[T_3]_{\mathcal{E}_n} = A_3 = \begin{pmatrix} a_{11} + cb_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} + cb_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Now observe we have

$$\begin{aligned} \det(w_1, w_2, \ldots, w_n) &= \det(T_1) \\ \det(w, w_2, \ldots, w_n) &= \det(T_2) \\ \det(w_1 + cw, w_2, \ldots, w_n) &= \det(T_3). \end{aligned}$$

Thus, we just need to show that $\det(T_3) = \det(T_1) + c\det(T_2)$. We have

$$\begin{aligned}\det(T_3)e_1 \wedge \cdots \wedge e_n &= \Lambda^n(T_3)(e_1 \wedge e_2 \wedge \cdots \wedge e_n)\\ &= T_3(e_1) \wedge T(e_2) \wedge \cdots \wedge T(e_n)\\ &= (w_1 + cw) \wedge w_2 \wedge \cdots \wedge w_n\\ &= w_1 \wedge \cdots \wedge w_n + cw \wedge w_2 \wedge \cdots \wedge w_n\\ &= T_1(e_1) \wedge \cdots \wedge T_1(e_n) + cT_2(e_1) \wedge \cdots \wedge T_2(e_n)\\ &= \Lambda^n(T_1)(e_1 \wedge \cdots \wedge e_n) + c\Lambda^n(T_2)(e_1 \wedge \cdots \wedge e_n)\\ &= \det(T_1)(e_1 \wedge \cdots \wedge e_n) + c\det(T_2)(e_1 \wedge \cdots \wedge e_n)\\ &= (\det(T_1) + c\det(T_2))(e_1 \wedge \cdots \wedge e_n).\end{aligned}$$

This gives $\det \in \mathrm{Hom}_F(F^n, \ldots, F^n; F)$. Next we show that det is alternating. Suppose that $w_i = w_j$ for some $i \neq j$. Then we have

$$\begin{aligned}\det(w_1, \ldots, w_n) &= \Lambda^n(T_1)(e_1 \wedge \cdots \wedge e_n)\\ &= T_1(e_1) \wedge \cdots \wedge T_1(e_n)\\ &= w_1 \wedge \cdots \wedge w_n\\ &= 0.\end{aligned}$$

Thus,

$$\det(w_1, \ldots, w_n) = \det(T_1) = 0.$$

This gives $\det \in \mathrm{Alt}^n(F^n; F)$. It is easy to see that $\det 1_n = 1$. □

Not only is det in $\mathrm{Alt}^n(F^n; F)$ satisfying $\det(1_n) = 1$, it is the only such map. This takes a little work to prove, but it is the key to seeing this is actually the same map as the undergraduate version. This will follow immediately from observing the undergraduate definition certainly takes 1_n to 1 and is in $\mathrm{Alt}^n(F^n; F)$. To see the undergraduate definition is in $\mathrm{Alt}^n(F^n; F)$, one only needs to recall that the determinant is preserved under elementary column operations. We need a few result before proving the uniqueness.

Let n be a positive integer and let S_n denote the collection of bijections from $\{1, \ldots, n\}$ to itself. It is easy to see this is a group under composition of functions. Given a tuple $(x_1, \ldots, x_n)$, we define

$$\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Note this tuple could be a tuple of anything: variables, vectors, numbers, etc. Define

$$\Delta = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

For example, if $n = 3$ we have

$$\Delta = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

We have

$$\sigma\Delta = \prod_{1\leq i<j\leq n} (x_{\sigma(i)} - x_{\sigma(j)}).$$

Since σ just permutes the values of $\{1, \dots, n\}$, we have $\sigma\Delta = \pm\Delta$ where the $\pm$ depends on σ. We define $\operatorname{sign}(\sigma) \in \{\pm 1\}$ by

$$\sigma\Delta = \operatorname{sign}(\sigma)\Delta.$$

Lemma 7.3.14. *Let $t \in \operatorname{Alt}^k(V, F)$. Then*

$$t(v_1, \dots, v_i, v_{i+1}, \dots, v_k) = -t(v_1, \dots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \dots, v_k).$$

Proof. Define

$$\psi(x, y) = t(v_1, \dots, v_{i-1}, x, y, v_{i+1}, \dots, v_n)$$

for fixed $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n$. It is enough to show that $\psi(x, y) = -\psi(y, x)$. Since $t \in \operatorname{Alt}^k(V; F)$ we have

$$\psi(x + y, x + y) = 0.$$

Expanding this and noting that $\psi(x, x) = \psi(y, y) = 0$ we obtain

$$\psi(x, y) = -\psi(y, x).$$

□

We leave the proof of the following lemma to the homework problems.

Lemma 7.3.15. *Let $t \in \operatorname{Alt}^k(V; F)$.*

(a) For each $\sigma \in S_k$,

$$t(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sign}(\sigma) t(v_1, \dots, v_k).$$

(b) If $v_i = v_j$ for any $i \neq j$, then $t(v_1, \dots, v_k) = 0$.

(c) If v_i is replaced by $v_i + cv_j$ for any $i \neq j$, $c \in F$, then the value of t is unchanged.

The following proposition is the key step needed to show that det is unique.

Proposition 7.3.16. *Let $t \in \operatorname{Alt}^k(V; F)$. Assume for some $v_1, \dots, v_n \in V$, $w_1, \dots, w_n \in V$, and $a_{ij} \in F$ we have*

$$\begin{aligned} w_1 &= a_{11}v_1 + \cdots + a_{n1}v_n \\ &\vdots \\ w_n &= a_{1n}v_1 + \cdots + a_{nn}v_n. \end{aligned}$$

Then

$$t(w_1, \dots, w_n) = \sum_{\sigma\in S_n} \operatorname{sign}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} t(v_1, \dots, v_n).$$

Proof. We expand $t(w_1, \dots, w_n)$ using multilinearity. Note that any of the terms with equal entries vanish since t is alternating so we are left with terms of the form

$$a_{i_1 1} \cdots a_{i_n n} t(v_{i_1}, \dots, v_{i_n})$$

where the $i_1, \dots, i_n$ run over $1, \dots, n$ and are all distinct. Such tuples are in bijective correspondence with S_n, so we can write each such term as

$$a_{\sigma(1)1} \cdots a_{\sigma(n)n} t(v_{\sigma(1)}, \dots, v_{\sigma(n)})$$

for a unique $\sigma \in S_n$. This gives the result. □

Corollary 7.3.17. *The determinant function defined above is the unique map in* $\mathrm{Alt}^n(F^n; F)$ *that satisfies* $\det(1_n) = 1$.

Proof. We have already seen that det satisfies the conditions; it only remains to show it is the unique function that does so. Let $\mathcal{E}_n = \{e_1, \dots, e_n\}$ be the standard basis of F^n. As above, we can identify elements of $\mathrm{Mat}_n(F)$ with $F^n \times \cdots \times F^n$ via column vectors. Thus, the identity matrix is identified with $(e_1, \dots, e_n)$. Let $A \in \mathrm{Mat}_n(F)$ with column vectors $v_1, \dots, v_n$, i.e.,

$$\begin{aligned} v_1 &= a_{11}e_1 + \cdots + a_{n1}e_n \\ &\vdots \\ v_n &= a_{1n}e_1 + \cdots + a_{nn}e_n. \end{aligned}$$

We now apply the previous proposition to see

$$\begin{aligned} \det(A) &= \det(v_1, \dots, v_n) \\ &= \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} \det(e_1, \dots, e_n) \\ &= \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n}. \end{aligned}$$

Moreover, given any $t \in \mathrm{Alt}^n(F^n; F)$ with $t(1_n) = 1$ we see

$$t(v_1, \dots, v_n) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n}.$$

Thus, we have the result. □

We finish the section with another coordinate-free construction of the trace of a linear map. Let $T \in \mathrm{Hom}_F(V, V)$ with V an n-dimensional F-vector space. Define a map

$$\begin{aligned} \varphi_T &: \Lambda^n(V) \to \Lambda^n(V) \\ v_1 \wedge \cdots \wedge v_n &\mapsto \sum_{j=1}^{n} v_1 \wedge \cdots \wedge v_{j-1} \wedge T(v_j) \wedge v_{j+1} \wedge \cdots \wedge v_n. \end{aligned}$$

We claim this is well-defined and F-linear. It is clear that if it is well-defined then it is F-linear. To show it is well-defined we must show that $\varphi_T(v_1 \wedge \cdots \wedge v_n) = 0$ if $v_i = v_j$ for some $i \neq j$. For simplicity we consider the case that $v_1 = v_2$. Then we have

$$\begin{aligned}\varphi_T(v \wedge v \wedge v_3 \cdots \wedge v_n) &= \sum_{j=1}^{n} v \wedge v \wedge \cdots \wedge v_{j-1} \wedge T(v_j) \wedge v_{j+1} \wedge \cdots \wedge v_n \\ &= T(v) \wedge v \wedge v_3 \wedge \cdots \wedge v_n + v \wedge T(v) \wedge v_3 \wedge \cdots \wedge v_n \\ &= T(v) \wedge v \wedge v_3 \wedge \cdots \wedge v_n - T(v) \wedge v \wedge v_3 \wedge \cdots \wedge v_n \\ &= 0.\end{aligned}$$

Thus, we see φ_T is a well-defined F-linear map. Since $\Lambda^n(V)$ is 1-dimensional, there exists a constant $\mathrm{tr}(T) \in F$ so that for any $v_1, \ldots, v_n \in V$ we have

$$\varphi_T(v_1 \wedge \cdots \wedge v_n) = \mathrm{tr}(T) v_1 \wedge \cdots \wedge v_n.$$

Definition 7.3.18. Let V be an n-dimensional F-vector space. Let $T \in \mathrm{Hom}_F(V, V)$. Define the *trace of* T to be the element of F so that

$$\varphi_T = \mathrm{tr}(T) \, \mathrm{id}_{\Lambda^n(V)} .$$

It is easy to see that $\mathrm{tr}(T)$ is linear, i.e., $\mathrm{tr}(cS + T) = c\,\mathrm{tr}(S) + \mathrm{tr}(T)$ for $c \in F$ and $S, T \in \mathrm{Hom}_F(V, V)$.

It only remains to show this new coordinate-free definition of trace agrees with the familiar definition of trace for a matrix $A = (a_{ij}) \in \mathrm{Mat}_n(F)$. Let $v_1, \ldots, v_n$ denote the column vectors of A and let T_A be the linear map associated to A, i.e., $T_A(e_j) = vj = \displaystyle\sum_{i=1}^{n} a_{ij} e_i$. Then we have for any $1 \leq j \leq n$

$$\begin{aligned}e_1 \wedge \cdots \wedge e_{j-1} \wedge T_A(e_j) \wedge e_{j+1} \wedge \cdots e_n &= e_1 \wedge \cdots \wedge e_{j-1} \wedge \sum_{i=1}^{n} a_{ij} e_i \wedge e_{j+1} \wedge \cdots e_n \\ &= e_1 \wedge \cdots \wedge e_{j-1} \wedge a_{jj} e_j \wedge e_{j+1} \wedge \cdots e_n \\ &= a_{jj} e_1 \wedge \cdots \wedge e_n.\end{aligned}$$

Thus,

$$\begin{aligned}\varphi_{T_A}(e_1 \wedge \cdots \wedge e_n) &= \sum_{j=1}^{n} e_1 \wedge \cdots \wedge e_{j-1} \wedge T_A(e_j) \wedge e_{j+1} \wedge \cdots \wedge e_n \\ &= \sum_{j=1}^{n} a_{jj} e_1 \wedge \cdots \wedge e_n.\end{aligned}$$

Thus, $\mathrm{tr}(T_A) = \sum_{j=1}^{n} a_{jj}$, which agrees with the undergraduate definition. Set $\mathrm{tr}(A) = \mathrm{tr}(T_A)$. Since tr is a coordinate-free definition, we obtain immediately that if A and B are similar matrices that $\mathrm{tr}(A) = \mathrm{tr}(B)$, which is not so obvious if one defines the trace in terms of the diagonal entries of a matrix.

7.4 Tensor products and exterior powers of modules

In this section we give a brief survey of tensor product and exterior powers of modules over a commutative ring with identity. One can consider more general tensor products, but this is sufficient for our purposes. As this mirrors the presentation for vector spaces in many ways, we skip the motivation and proceed straight to the constructions and results. We will also give a correct construction of the characteristic polynomial of a linear map. It is actually essential to consider tensor products of modules to give such a construction; the world of vector spaces is not enough to do this properly. We will end with a proof of the Cayley-Hamilton theorem using exterior products that mirrors what a student first wants to do, i.e., to say $c_T(T) = 0$ because one just plugs in T for x. Of course, we must first make sense of what it even means to plug in T for x. Gratitude for this proof of the Cayley-Hamilton theorem via exterior algebras goes to Paul Garrett; his abstract algebra notes are where I learned this argument.

Let R be a commutative ring with identity. We will assume this throughout this section. Let M and N be R-modules and let $\mathrm{Fr}(M \times N)$ be the free abelian group on $M \times N$. Let $\mathrm{Rel}_R(M \times N)$ be the subgroup of $\mathrm{Fr}(M \times N)$ generated by the following elements:

(a) $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$ for $m_1, m_2 \in M$, $n \in N$,

(b) $(m, n_1 + n_2) - (m, n_1) - (m, n_2)$ for $m \in M$, $n_1, n_2 \in N$,

(c) $(rm, n) - (m, rn)$ for $m \in M, n \in N$ and $r \in R$.

We set $M \otimes_R N = \mathrm{Fr}(M \times N)/\mathrm{Rel}_R(M \times N)$ and refer to this abelian group as the *tensor product of M and N over R*. We denote the image of (m, n) in $M \otimes_R N$ by $m \otimes n$ and refer to such an element as a *pure tensor*. Note that one immediately obtains that $M \otimes_R N$ is an R-module via the action $r \cdot (m \otimes n) = rm \otimes n$. Moreover, if S is another commutative ring with identity and M is an S-module, then $M \otimes_R N$ is an S-module via $s \cdot (m \otimes n) = sm \otimes n$. As in the case of tensor products of vector spaces, we need the correct maps here as well.

Definition 7.4.1. Let M, N and P be R-modules. We say a map $\varphi : M \times N \to P$ is *R-bilinear* or just *bilinear* if it is linear in each variable separately. We denote the space of R-bilinear forms by $\mathrm{Hom}_R(M, N; P)$.

Note this is exactly the same definition as in the case of vector spaces with the only difference being the linearity is as an R-module map.

We have the following universal property. We omit the proof as it follows in exactly the same manner as the corresponding result for vector spaces.

Theorem 7.4.2. *Let $M, N,$ and P be R-modules. Define a map $\iota : M \times N \to M \otimes_R N$ by $\iota(m, n) = m \otimes n$. Then* *(a)* $\iota \in \mathrm{Hom}_R(M, N; M \otimes_R N)$, *i.e.,* ι *is R-bilinear;*

(b) if $T \in \mathrm{Hom}_R(M \otimes_R N, P)$, *then* $T \circ \iota \in \mathrm{Hom}_R(M, N; P)$*;*

(c) if $t \in \mathrm{Hom}_R(M, N; P)$, *then there is a unique* $T \in \mathrm{Hom}_R(M \otimes_R N, P)$ *so that* $t = T \circ \iota$.

Equivalently, we can write the correspondence by saying we have a bijection between $\mathrm{Hom}_R(M, N; P)$ *and* $\mathrm{Hom}_R(M \otimes_R N, P)$ *so that the following diagram commutes:*

$$\begin{tikzcd}
M \times N \arrow[r, "\iota"] \arrow[dr, "t"'] & M \otimes_R N \arrow[d, "T"] \\
& P.
\end{tikzcd}$$

There is one notable difference. In the case of vector spaces one has that ι is an injective map; for general modules this is not the case. For instance, consider the following example.

Example 7.4.3. Consider the space $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $m \otimes q \in \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$. We have

$$\begin{aligned} m \otimes q &= m \otimes \frac{qn}{n} \\ &= nm \otimes \frac{q}{n} \\ &= 0 \otimes \frac{q}{n} \\ &= 0 \end{aligned}$$

where we have used $n \in \mathbb{Z}$ so it can be moved across the tensor product. Since all pure tensors are 0, we must have $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. However, $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Q}$ is not the 0 set, so ι in this case cannot be injective; the entire domain is in the kernel!

By expanding to modules there are more interesting examples. We will cover several, but here is one we can easily do with no further information.

Example 7.4.4. Let m and n be positive integers with $\gcd(m, n) = 1$. We consider the space $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. We claim this is 0. Let $a \otimes b \in \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. Since $\gcd(m, n) = 1$, there exists $s, t \in \mathbb{Z}$ so that

$1 = ms + nt$. We have

$$\begin{aligned} a \otimes b &= 1 \cdot (a \otimes b) \\ &= (ms + nt) \cdot (a \otimes b) \\ &= ms(a \otimes b) + nt(a \otimes b) \\ &= (msa) \otimes b + (nta \otimes b) \\ &= 0 \otimes b + a \otimes (ntb) \\ &= a \otimes 0 \\ &= 0. \end{aligned}$$

Since all the pure tensors vanish, the entire space must be 0.

We now establish some of the basic properties. These were given for vector spaces before, but for modules we cannot immediately just resort to bases since our modules may not have bases.

Proposition 7.4.5. *Let M, N, and P be R-modules. We have*

- *(a)* $M \otimes_R N \cong N \otimes_R M$,
- *(b)* $(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P)$,
- *(c)* $(M \oplus N) \otimes_R P \cong (M \otimes_R P) \oplus (N \otimes_R P)$.

Proof. We leave the proof of the first statement as an exercise. The proof of Theorem 7.2.7 given above is the same proof one uses for modules so the third statement follows immediately from just changing notation in that proof. We prove the second claim. Let $p \in P$. Define a map

$$\begin{aligned} M \times N &\to M \otimes_R (M \otimes_R P) \\ (m, n) &\mapsto m \otimes (n \otimes p). \end{aligned}$$

It is easy to check that this is a bilinear map. (Note p is fixed!) Thus, we can apply the universal property to this map to obtain an R-linear map

$$\begin{aligned} M \otimes_R N &\to M \otimes_R (N \otimes_R P) \\ m \otimes n &\mapsto m \otimes (n \otimes p). \end{aligned}$$

Now consider the map

$$\begin{aligned} (M \otimes_R N) \times P &\to M \otimes_R (N \otimes_R P) \\ (m \otimes n, p) &\mapsto m \otimes (n \otimes p). \end{aligned}$$

This is well-defined and easily seen to be a bilinear map. Thus, we apply the universal property again to obtain an R-linear map

$$\begin{aligned} (M \otimes_R N) \otimes_R P &\to M \otimes_R (N \otimes_R P) \\ (m \otimes n) \otimes p &\mapsto m \otimes (n \otimes p). \end{aligned}$$

Now that we have an R-linear map form $(M\otimes_R N)\otimes_R P \to M\otimes_R(N\otimes_R P)$, we use the exact same process to get the inverse map. Namely, we begin by fixing a $m \in M$ and define a bilinear map

$$\begin{aligned} N \times P &\to (M \otimes_R N) \otimes_R P \\ (n, p) &\mapsto (m \otimes n) \otimes p. \end{aligned}$$

As above, one applies the universal property to obtain an R-linear map

$$\begin{aligned} N \otimes_R P &\to (M \otimes_R N) \otimes_R P \\ n \otimes p &\mapsto (m \otimes n) \otimes p. \end{aligned}$$

Now consider the bilinear map

$$\begin{aligned} M \times (N \otimes_R P) &\to (M \otimes_R N) \otimes_R P \\ (m, n \otimes p) &\mapsto (m \otimes n) \otimes p. \end{aligned}$$

We again apply the universal property to obtain the R-linear map

$$\begin{aligned} M \otimes_R (N \otimes_R P) &\to (M \otimes_R N) \otimes_R P \\ m \otimes (n \otimes p) &\mapsto (m \otimes n) \otimes p. \end{aligned}$$

It is clear this is the inverse map to the one constructed above, so we have the isomorphism. □

These basic results allow us to conclude several interesting results. For instance, via induction we have the following.

Corollary 7.4.6. *Let M and N be a free R-modules of dimension m and n respectively. Then*

$$M \otimes_R N \cong R^{mn}.$$

In particular, if $\{x_1, \ldots, x_m\}$ is a basis for M and $\{y_1, \ldots, y_n\}$ is a basis for N, then $\{x_i \otimes y_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis of $M \otimes_R N$.

Note that the corresponding result for vector spaces was proven early and used to conclude essentially all the results above tensor products of vector spaces. In this case we cannot do that. For instance, since our modules in general do not have bases, we certainly don't get bases of the tensor products in general. We do have one result along those lines that can be helpful.

Proposition 7.4.7. *Let $\{x_i\}_{i\in I}$ be a set of generators of an R-module M and $\{y_j\}_{j\in J}$ a set of generators of an R-module N. Then $\{x_i \otimes y_j : i \in I, j \in J\}$ is a set of generators of $M \otimes_R N$.*

Proof. Note it is enough to show each pure tensor is generated by this set. Let $m \otimes n \in M \otimes_R N$. Write $m = \sum_{i\in I} a_i x_i$ and $n = \sum_{j\in J} b_j y_j$. Then we have $m \otimes n = \sum_{i\in I} \sum_{j\in J} a_i b_j (x_i \otimes y_j)$. □

Example 7.4.8. Let $m, n \in \mathbb{Z}$ with $d = \gcd(m,n)$. Consider the space $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. We know that $1+m\mathbb{Z}$ generates $\mathbb{Z}/m\mathbb{Z}$ and $1+n\mathbb{Z}$ generates $\mathbb{Z}/n\mathbb{Z}$, so $1 \otimes 1 = (1+m\mathbb{Z}) \otimes (1+n\mathbb{Z})$ must generate $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. The same argument given above when $d=1$ shows that d annihilates all the pure tensors in $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$; thus it must annihilate $1 \otimes 1$. This gives that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is necessarily a quotient of $\mathbb{Z}/d\mathbb{Z}$. Define a map

$$\begin{aligned} \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} &\to \mathbb{Z}/d\mathbb{Z} \\ (a,b) &\mapsto ab. \end{aligned}$$

First, since $d \mid m$ and $d \mid n$ one can easily check this map is well-defined. It is also easy to check it is bilinear, so the universal property gives a $\mathbb{Z}$-linear map (i.e., a group homomorphism)

$$\begin{aligned} \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} &\to \mathbb{Z}/d\mathbb{Z} \\ a \otimes b &\mapsto ab. \end{aligned}$$

In particular, we have this maps $1 \otimes 1$ to $1 + d\mathbb{Z}$. Since $1 + d\mathbb{Z}$ generates $\mathbb{Z}/d\mathbb{Z}$, we have it has exact order d. Thus, it must be that $1 \otimes 1$ has order at least d. However, above we showed it had order at most d. Thus, $1 \otimes 1$ has exact order d and so we have an isomorphism.

Proposition 7.4.5 also shows it makes sense to write $M \otimes_R N \otimes_R P$. More generally, using induction one can show it makes sense to write $M_1 \otimes_R \cdots \otimes_R M_n$ for $M_1, \dots, M_n$ a collection of R-modules. One defines multilinear maps exactly as was done for vector spaces replacing F-linear with R-linear. One then obtains the same universal property for $M_1 \otimes_R \cdots \otimes_R M_n$ as for vector spaces upon making the adjustment from F to R.

Theorem 7.4.9. *Let $M_1, \dots, M_n$, and N be R-modules. Define*

$$\begin{aligned} \iota : M_1 \times \cdots \times M_n &\longrightarrow M_1 \otimes_R \cdots \otimes_R M_n \\ (m_1, \dots, m_n) &\mapsto m_1 \otimes \cdots \otimes m_n. \end{aligned}$$

Then we have

(a) For every $T \in \operatorname{Hom}_R(M_1 \otimes_R \cdots \otimes_R M_n; N)$ the map $T \circ \iota \in \operatorname{Hom}_R(M_1, \dots, M_n; N)$.

(b) For every $t \in \operatorname{Hom}_R(M_1, \dots, M_n; N)$ there is a unique $T \in \operatorname{Hom}_R(M_1 \otimes_R \cdots \otimes_R M_n, N)$ so that $t = T \circ \iota$.

If $M_1, \dots, M_n$ are free R-modules with bases $\mathcal{B}_i = \{x_1^{(i)}, \dots, x_{n_i}^{(i)}\}$, then $M_1 \otimes_R \cdots \otimes_R M_n$ is a free R-module with basis $\{x_{j_1}^{(1)} \otimes \cdots \otimes x_{j_n}^{(n)} : x_{j_i}^{(i)} \in \mathcal{B}_i\}$. In the case that $M_1 = \cdots = M_n$, we write $M^{\otimes n} = M_1 \otimes_R \cdots \otimes_R M_n$.

We can now form the space of exterior powers just as before. Again let $\mathcal{A}_k(M)$ be the subspace of $M^{\otimes k}$ generated by elements of the form

$m_1 \otimes \cdots \otimes m_k$ with $m_i = m_j$ for some $i \neq j$. Set $\Lambda^k_R(M) = M^{\otimes k}/\mathcal{A}_k(M)$. We write

$$m_1 \wedge \cdots \wedge m_n = m_1 \otimes \cdots \otimes m_n + \mathcal{A}_k(M).$$

The same basic properties hold for exterior powers in this case, as the interested reader can easily verify. For instance, one has the following theorem.

Theorem 7.4.10. *Let M be a free R-module of dimension n. Let $\mathcal{B} = \{m_1, \ldots, m_n\}$ be a basis of M. For $k \leq n$ the the vectors $\{m_{i_1} \wedge \cdots \wedge m_{i_k} : 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$ form a basis of $\Lambda^k_R(M)$. For $k > n$ we have $\Lambda^k_R(M) = 0$. In particular, $\dim_R \Lambda^k_R(M) = \binom{n}{k}$.*

As in the case of vector apaces, this gives $\Lambda^n_R(M)$ has dimension 1. Given $T \in \mathrm{Hom}_R(M, M)$, we get an induced map $\Lambda^k_R(T) : \Lambda^k_R(M) \to \Lambda^k_R(M)$. Again, if $k = n$ this leads to the definition of $\det(T)$ just as before. All of the same proofs as in the previous section go through unchanged so we do not repeat them here.

We are now in a position to give a correct definition of the characteristic polynomial of a linear map. Let V be an n-dimensional F-vector space and let $T \in \mathrm{Hom}_F(V, V)$. Consider the set $V \otimes_F F[x]$. This is a $F[x]$-module via the action of $F[x]$ on the right, i.e., for $v \otimes f(x) \in V \otimes_F F[x]$ and $g(x) \in F[x]$, we set $g(x) \cdot v \otimes f(x) = v \otimes g(x)f(x)$. In fact, $V \otimes_F F[x]$ is a free $F[x]$-module of rank n with basis given by $v_1 \otimes 1, \ldots, v_n \otimes 1$ if $\{v_1, \ldots, v_n\}$ is a basis of V. We also have that $V \otimes_F F[x]$ is a $F[T]$-module via $T \cdot v \otimes f(x) = T(v) \otimes f(x)$. Thus, $V \otimes_F F[x]$ is a $F[T] \otimes_F F[x]$-module. We have $1 \otimes x - T \otimes 1 \in F[T] \otimes_F F[x]$. We can view this element as an element of $\mathrm{Hom}_{F[T]\otimes_F F[x]}(V \otimes_F F[x], V \otimes_F F[x])$ by identifying it with the multiplication by $1 \otimes x - T \otimes 1$ map. In particular, since this is an $F[T] \otimes_F F[x]$-linear map, it is certainly $F[x]$-linear. Thus, we have an induced $F[x]$-linear map

$$\Lambda^n_{F[x]}(1 \otimes x - T \otimes 1) : \Lambda^n_{F[x]}(V \otimes_F F[x]) \to \Lambda^n_{F[x]}(V \otimes_F F[x]).$$

Since $V \otimes_F F[x]$ is a free $F[x]$-module of rank n, we have $\Lambda^n_{F[x]}(V \otimes_F F[x])$ is a free rank 1 $F[x]$-module. Thus, we must have an element $c_T(x) \in F[x]$ so that

$$\Lambda^n_{F[x]}(1 \otimes x - T \otimes 1) = c_T(x) 1_{V \otimes_F F[x]}.$$

This element $c_T(x)$ is the characteristic polynomial of T. Note, in shorthand it can be written as

$$c_T(x) = \det(1 \otimes x - T \otimes 1).$$

Recall that the Cayley-Hamilton theorem gives that $c_T(T) = 0$. We give another proof of this result using exterior algebras. In the course of the proof we will also see the correct way to interpret this statement. Consider the $F[x]$-bilinear map

$$\begin{aligned}\langle\, ,\, \rangle : \Lambda^{n-1}_{F[x]}(V \otimes_F F[x]) \times \Lambda^1_{F[x]}(V \otimes_F F[x]) &\longrightarrow \Lambda^n_{F[x]}(V \otimes_F F[x]) \\ (x_1 \wedge \cdots \wedge x_{n-1}, x_n) &\mapsto x_1 \wedge \cdots \wedge x_{n-1} \wedge x_n.\end{aligned}$$

To ease notation, write $A = 1 \otimes x - T \otimes 1$. We have

$$\begin{aligned}\langle \Lambda^{n-1}_{F[x]}(A)(x_1 \wedge \cdots \wedge x_n), A(x_n)\rangle &= \Lambda^{n-1}_{F[x]}(x_1 \wedge \cdots \wedge x_{n-1}) \wedge A(x_n) \\ &= A(x_1) \wedge \cdots \wedge A(x_n) \\ &= \Lambda^n_{F[x]}(A)(x_1 \wedge \cdots \wedge x_n) \\ &= c_T(x)x_1 \wedge \cdots \wedge x_n.\end{aligned}$$

Let A^{adj} be the adjoint of $\Lambda^{n-1}_{F[x]}(A)$ with respect to this pairing. Note this is not the adjoint of A here! Note that since the pairing is $F[x]$-bilinear a priori we only have A^{adj} is $F[x]$-linear, not $F[T] \otimes_F F[x]$-linear. We have

$$\begin{aligned}\langle x_1 \wedge \cdots \wedge x_n, c_T(x)x_n\rangle &= c_T(x)x_1 \wedge \cdots \wedge x_n \\ &= \langle \Lambda^{n-1}_{F[x]}(x_1 \wedge \cdots \wedge x_{n-1}), A(x_n)\rangle \\ &= \langle x_1 \wedge \cdots \wedge x_{n-1}, A^{\text{adj}} \circ A(x_n)\rangle.\end{aligned}$$

Thus, we see that $A^{\text{adj}} \circ A = c_T(x)1_{V \otimes_F F[x]}$. We need to show that A^{adj} commutes with A as then we will have A^{adj} is $F[T] \otimes_F F[x]$-linear. Before we show why this is true, we show how it gives the Cayley-Hamilton theorem. To ease notation write $M = V \otimes_F F[x]$, $R = F[T] \otimes_F F[x]$, and $I = AR$. If A^{adj} commutes with A, then necessarily we have A^{adj} is actually an R-linear map. Thus, we have $A^{\text{adj}} \in \text{Hom}_R(M, M)$. Moreover, one has that A^{adj} descends to an R/I-linear map from M/IM to M/IM where we recall that

$$IM = \left\{\sum_{\text{finite}} a_i m_i : a_i \in I, m_i \in M\right\}$$

is a submodule of M. The point is that one must have A^{adj} commutes with A for the map $A^{\text{adj}} : M/IM \to M/IM$ to be well-defined. Descending to M/IM is what is meant by substituting T in for x as in M/IM the terms $T \otimes 1$ is identified with $x \otimes 1$. Thus, we want to consider the image of $c_T(x)$ acting on M/IM, i.e., $c_T(x)1_{M/IM}$. Write $\overline{A}$ for the map from M/IM to M/IM induced by A and likewise for A^{adj}. We have

$$\overline{A^{\text{adj}}} \circ \overline{A} = c_T(x)1_{M/IM}.$$

However, we know $\overline{A} = 0_{M/IM}$ by construction ($I = AR$), so we have $c_T(x)1_{M/IM} = 0_{M/IM}$, as desired. Now we just observe that the composition

$$V \longrightarrow M \longrightarrow M/IM$$

is an isomorphism if $F[T]$-modules, so certainly an isomorphism of F-vector spaces. Thus, we see the corresponding map on V given by $c_T(x)1_{M/IM}$ is 0; this map is denoted by $c_T(T)$.

It only remains to prove that A^{adj} and A commute. To see this, we extend scalars to $F(x)$. Thus, viewing everything now over $V \otimes_F F(x)$ we have

$$A^{\text{adj}} \circ A = c_T(x)1_{V\otimes_F F(x)}.$$

Since $c_T(x)$ is a monic polynomial, it is invertible in $F(x)$. However, we know that $c_T(x)$ is the determinant of A, so that means A is invertible over $V \otimes_F F(x)$. Thus, we can write $A^{\text{adj}} = c_T(x)A^{-1}$ over $V \otimes_F F(x)$. This shows that A^{adj} commutes with A over $V \otimes_F F(x)$, so necessarily over $V \otimes_F F[x]$ as well.

7.5 Problems

For these problems V and W are finite dimensional F-vector spaces.

(a) Let V and W be F-vector spaces. Prove using the universal property that $V \otimes_F W \cong W \otimes_F V$.

(b) Let V, W, and U be F-vector spaces. Prove that the space of bilinear forms $\mathrm{Hom}_F(V, W; U)$ is an F-vector space. Moreover, prove that $\mathrm{Hom}_F(V, W; U) \cong \mathrm{Hom}_F(V \otimes_F W, U)$.

(c) Let V_1, V_2, W_1, and W_2 be F-vector spaces. Let $T_1 \in \mathrm{Hom}_F(V_1, W_1)$ and $T_2 \in \mathrm{Hom}_F(V_2, W_2)$. Prove there is a unique F-linear map $T_1 \otimes T_2$ from $V_1 \otimes_F V_2$ to $W_1 \otimes_F W_2$ satisfying $(T_1 \otimes T_2)(v_1 \otimes v_2) = T_1(v_1) \otimes T_2(v_2)$.

(d) Let $V_1 = W_1 = \mathbb{R}^3$ both with basis $\mathcal{A} = \{x_1, x_2, x_3\}$ and $V_2 = W_2 = \mathbb{R}^2$ with basis $\mathcal{B} = \{y_1, y_2\}$. Let $T_1(ax_1+bx_2+cx_3) = cx_1+2ax_2-3bx_3$ and $T_2(ay_1 + by_2) = (a + 3b)y_1 + (4b - 2a)y_2$. Let $\mathcal{C}$ be the basis for $V_1 \otimes_{\mathbb{R}} V_2$ and $W_1 \otimes_{\mathbb{R}} W_2$ given by

$$\mathcal{C} = \{x_1 \otimes y_1, x_1 \otimes y_2, x_2 \otimes y_1, x_2 \otimes y_2, x_3 \otimes y_1, x_3 \otimes y_2\}.$$

Compute the matrix $[T_1 \otimes T_2]_{\mathcal{C}}^{\mathcal{C}}$.

(e) Let $V = P_5(\mathbb{Q})$ and $W = \mathbb{Q}(\sqrt{2}) = \{a+b\sqrt{2} : a, b \in \mathbb{Q}\}$. Give a basis for $V \otimes_{\mathbb{Q}} W$.

(f) Show that the cross product, defined in multivariable calculus, is a bilinear form from $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$.

(g) (a) Let $\varphi \in V^\vee$ and $\psi \in W^\vee$. Define a map

$$\begin{aligned} B_{\varphi,\psi} : VxW &\to F \\ (v, w) &\mapsto \varphi(v)\psi(w). \end{aligned}$$

Show that $B_{\varphi,\psi}$ is a bilinear form.

(b) Prove that there is a natural isomorphism between $(V \otimes_F W)^\vee$ and $V^\vee \otimes_F W^\vee$. (Note a natural isomorphism means it does not depend on a choice of basis.)

(h) Let V and W be F-vector spaces. Prove that $\mathrm{Alt}^k(V, W)$ is an F-subspace of $\mathrm{Hom}_F(V, .., V; W)$.

(i) Use the definition to compute the determinant of a 3 by 3 matrix over a field F. Check your result agrees with the definition that was given in undergraduate linear algebra.

(j) Let V be an F-vector space. Show that $\Lambda^k(V)^\vee \cong \mathrm{Alt}^k(V)$ as F-vector spaces.

(k) Let $T \in \mathrm{Hom}_F(V, W)$. Give a detailed proof that T induces a well-defined linear map $\Lambda^k(T) : \Lambda^k(V) \to \Lambda^k(W)$ given by

$$\Lambda^k(T)(v_1 \wedge \cdots \wedge v_k) = T(v_1) \wedge \cdots \wedge T(v_k)$$

for each $k \geq 0$.

(l) Use the definition to prove that if $A \in \mathrm{GL}_n(F)$, then $\det(A^{-1}) = \det(A)^{-1}$.

(m) Prove that $v \wedge v_1 \wedge v_2 \wedge \cdots \wedge v_k = (-1)^k (v_1 \wedge v_2 \wedge \cdots \wedge v_k \wedge v)$.

Appendix A

Groups, rings, and fields: a quick recap

It is expected that students reading these notes have some experience with abstract algebra. For instance, an undergraduate course in abstract algebra is more than sufficient. A good source for those beginning in abstract algebra is [2] and a more advanced resource is [1]. We do not attempt to give a comprehensive treatment here; we present only the basic definitions and some examples to refresh the reader's memory.

Definition A.0.1. Let G be a nonempty set and $\star : G \times G \to G$ a binary operation. We call $(G, \star)$ a *group* if it satisfies the following properties:

- for all g_1, g_2, and $g_3 \in G$ we have $(g_1 \star g_2) \star g_3 = g_1 \star (g_2 \star g_3)$;
- there exists an element $e_G \in G$, referred to as the identity element of G, so that $g \star e_G = g = e_G \star g$ for all $g \in G$;
- for each $g \in G$ there exists an element g^{-1}, referred to as the inverse of g, so that $g \star g^{-1} = e_G = g^{-1} \star g$.

We say a group is *abelian* if for all $g_1, g_2 \in G$ we have $g_1 \star g_2 = g_2 \star g_1$.

Often the operation $\star$ will be clear from context so we will write our groups as G instead of $(G, \star)$.

Exercise A.0.2. Let G be a group. Show that inverses are unique, i.e., if given $g \in G$ there exists $h \in G$ so that $g \star h = e_G = h \star g$, then $h = g^{-1}$.

Definition A.0.3. Let $(G, \star)$ be a group and $H \subset G$. We say H is a *subgroup* of $(G, \star)$ if $(H, \star)$ is a group. One generally denotes H is a subgroup of G by $H \leq G$.

Example A.0.4. Let $G = \mathbb{C}$ and $\star = +$. It is easy to check that this gives an abelian group. In fact, it is easy to see that $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$. Note the operation is addition for all of these.

Example A.0.5. Let $G = \mathbb{Z}$ and let $\star$ be multiplication. This does not give a group as $2 \in \mathbb{Z}$ but the multiplicative inverse of 2, namely, $1/2$ is not in $\mathbb{Z}$.

Example A.0.6. Let $G = \mathbb{R}^{\times} = \{x \in \mathbb{R} : x \neq 0\}$ with $\star$ being multiplication. This gives an abelian group.

Example A.0.7. Let X be a set of n elements. Let S_n denote the set of bijections from X to X, i.e., the permutations of X. This forms a group under composition of functions.

Example A.0.8. Let $\mathrm{Mat}_n(\mathbb{R})$ denote the n by n matrices with entries in $\mathbb{R}$. This forms a group under matrix addition but not under matrix multiplication as the 0 matrix, the matrix with all 0's as entries, does not have a multiplicative inverse.

Example A.0.9. Let $\mathrm{GL}_n(\mathbb{R})$ be the subset of $\mathrm{Mat}_n(\mathbb{R})$ consisting of invertible matrices. This gives a group under matrix multiplication. Let $\mathrm{SL}_n(\mathbb{R})$ be the subset of $\mathrm{GL}_n(\mathbb{R})$ consisting of matrices with determinant 1. This is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.

Example A.0.10. Let n be a positive integer and consider the set $H = n\mathbb{Z} = \{nm : m \in \mathbb{Z}\}$. Note that $H \leq (\mathbb{Z}, +)$. We can form a quotient group in this case as follows. We define an equivalence relation on $\mathbb{Z}$ by setting $a \sim b$ if $n|(a-b)$. One should check this is in fact an equivalence relation. This partitions $\mathbb{Z}$ into equivalence classes. For example, given $a \in \mathbb{Z}$, the equivalence class containing a is given by $[a]_n = \{b \in \mathbb{Z} : n|(a-b)\}$. We write $a \equiv b \pmod{n}$ if a and b lie in the same equivalence class, i.e., if $[a]_n = [b]_n$. Note that for each $a \in \mathbb{Z}$, there is an element $r \in \{0, 1, \ldots, n-1\}$ so that $[a]_n = [r]_n$. This follows immediately from the division algorithm: write $a = nq + r$ with $0 \leq r \leq n-1$. Write

$$\mathbb{Z}/n\mathbb{Z} = \{[0]_n, [1]_n, \ldots, [n-1]_n\}.$$

We define addition on this set by setting $[a]_n + [b]_n = [a+b]_n$. One can easily check this is well-defined and makes $\mathbb{Z}/n\mathbb{Z}$ into an abelian group with n elements. We drop the subscript n on the equivalence classes when it is clear from context.

Consider the case that $n = 4$. We have $\mathbb{Z}/4\mathbb{Z} = \{[0], [1], [2], [3]\}$. An addition table is given here:

+	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

Example A.0.11. Let $n \in \mathbb{Z}$ and consider the subset of $\mathbb{Z}/n\mathbb{Z}$ consisting of those $[a]$ so that $\gcd(a, n) = 1$. We denote this set by $(\mathbb{Z}/n\mathbb{Z})^\times$. Recall that if $\gcd(a, n) = 1$ then there exists $x, y \in \mathbb{Z}$ so that $ax + ny = 1$. This means that $n \mid (ax - 1)$, i.e., $ax \equiv 1 \pmod{n}$. If $[a], [b] \in (\mathbb{Z}/n\mathbb{Z})^\times$, then $[ab] \in (\mathbb{Z}/n\mathbb{Z})^\times$ as well. One can now check that if we set $[a] \star [b] = [ab]$, then this makes $(\mathbb{Z}/n\mathbb{Z})^\times$ into an abelian group. Note that $(\mathbb{Z}/n\mathbb{Z})^\times$ is not a subgroup of $\mathbb{Z}/n\mathbb{Z}$ even though it is a subset; they do not have the same operations.

Exercise A.0.12. **(a)** How many elements are in $(\mathbb{Z}/n\mathbb{Z})^\times$?
(b) Give multiplication tables for $(\mathbb{Z}/4\mathbb{Z})^\times$ and $(\mathbb{Z}/5\mathbb{Z})^\times$. Do you notice anything interesting about these tables?

As is always the case, the relevant maps to study are the ones that preserve the structure being studied. For groups these are group homomorphisms.

Definition A.0.13. Let $(G, \star_G)$ and $(H, \star_H)$ be groups. A map $\phi : G \to H$ is a *group homomorphism* if it satisfies $\phi(g_1 \star_G g_2) = \phi(g_1) \star_H \phi(g_2)$ for all $g_1, g_2 \in G$. We define the *kernel* of the map ϕ by

$$\ker \phi = \{g \in G : \phi(g) = e_H\}.$$

Exercise A.0.14. **(a)** Show that $\ker \phi \leq G$.
(b) Show that if $\phi : G \to H$ is a group homomorphism, then ϕ is injective if and only if $\ker \phi = \{e_G\}$.

Example A.0.15. Let $n \in \mathbb{Z}$ be a positive integer and define $\phi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ by $\phi(a) = [a]_n$. This is easily seen to be a surjective group homomorphism. Moreover, one has that $\ker \phi = n\mathbb{Z}$.

Example A.0.16. Let $\phi : \mathbb{R} \to \mathbb{C}^\times$ be defined by $\phi(x) = e^{2\pi i x}$. Given $x, y \in \mathbb{R}$ we have $\phi(x + y) = e^{2\pi i(x+y)} = e^{2\pi i x}e^{2\pi i y} = \phi(x)\phi(y)$. Thus, ϕ is a group homomorphism. The image of this group homomorphism is $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, i.e., the unit circle. The kernel is given by $\mathbb{Z}$. For those that remember the first isomorphism theorem, this gives $\mathbb{R}/\mathbb{Z} \cong S^1$.

Example A.0.17. The determinant map gives a group homomorphism from $\mathrm{GL}_n(\mathbb{R})$ to $\mathbb{R}^\times$ with kernel $\mathrm{SL}_n(\mathbb{R})$.

As one may have observed, the first examples of groups given were familiar objects such as $\mathbb{Z}$ and $\mathbb{R}$ with only one operation considered. It is natural to consider both operations for such objects, which brings us to the definition of a ring.

Definition A.0.18. A *ring* is a nonempty set R with two binary operations, $+$ and $\cdot$, satisfying the following properties

- R is an abelian group under the operation $+$ (the additive identity is denoted 0_R);

- there is an element 1_R that satisfies $r \cdot 1_R = r = 1_R \cdot r$ for all $r \in R$;
- for all $r_1, r_2, r_3 \in R$ one has $(r_1 \cdot r_2) \cdot r_3 = r_1 \cdot (r_2 \cdot r_3)$;
- for all $r_1, r_2, r_3 \in R$ one has $(r_1 + r_2) \cdot r_3 = r_1 \cdot r_3 + r_2 \cdot r_3$;
- for all $r_1, r_2, r_3 \in R$ one has $r_1 \cdot (r_2 + r_3) = r_1 \cdot r_2 + r_1 \cdot r_3$.

We say R is *commutative* if $r_1 \cdot r_2 = r_2 \cdot r_1$ for all $r_1, r_2 \in R$.

It is often the case that one writes $r_1 \cdot r_2$ as $r_1 r_2$ to ease notation. We will follow that convention here when there is no fear of confusion.

Note that what we have defined as a ring is often referred to as a *ring with identity.* This is because in some cases one would like to consider rings without requiring that there be an element 1_R. We will not be interested in such rings so in these notes “ring” always means “ring with identity”.

Example A.0.19. Many of the examples given above are also rings. The groups $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ are all commutative rings when considered with the operations $+$ and $\cdot$.

Example A.0.20. Let R be a ring. The group $\mathrm{Mat}_n(R)$ is a ring under the operations of matrix addition and multiplication. However, if $n \geq 2$ this is not a commutative ring.

Example A.0.21. Let R be a ring and let $R[x]$ denote the set of polynomials with coefficients in R. The set $R[x]$ is a ring under polynomial addition and multiplication.

Given a ring R, a *subring* of R is a nonempty set S that is a ring under the same operations. However, it turns out that the concept of ideals is more important than the notion of subrings.

Definition A.0.22. Let R be a ring and $I \subset R$ be a subring. We say I is an *ideal* if $ra \in I$ and $ar \in I$ for all $r \in R$ and $a \in I$.

Note this is a stronger condition than being a subring. For example, $\mathbb{Z}$ is a subring of $\mathbb{R}$ but it is not an ideal.

Example A.0.23. Consider the ring $\mathbb{Z}$. For any integer n we have $n\mathbb{Z}$ is an ideal in $\mathbb{Z}$. In fact, every ideal can be written in this form. Let I be an ideal in $\mathbb{Z}$. If $I = \mathbb{Z}$ then clearly we have $I = 1\mathbb{Z}$. Suppose I is a proper subset of $\mathbb{Z}$ and $I \neq 0$. Let I^+ be the subset of I consisting of positive elements. Since $I \neq 0$ this is a nonempty set because if $m \in I$, either $m \in I^+$ or $-m \in I^+$. Since I^+ is a set of positive integers, it has a minimal element; call this minimal element n. We claim $I = n\mathbb{Z}$. Let $m \in I$. Then we can write $m = nq + r$ for some $0 \leq r \leq n - 1$. Since $m \in I$ and $n \in I$, we have $r = m - nq \in I$. However, this contradicts n being the least positive integer in I unless $r = 0$. Thus, $m \in n\mathbb{Z}$ and so $I = n\mathbb{Z}$. We call an ideal generated by a single element a *principal ideal.* A ring where all the ideals are principal is called a *principal ideal domain.* Thus, $\mathbb{Z}$ is a principal ideal domain.

Exercise A.0.24. Let F be a field. Mimic the proof that $\mathbb{Z}$ is a principal ideal domain to show $F[x]$ is a principal ideal domain.

Example A.0.25. Consider the polynomial ring $R[x]$ with R a ring. Let $f(x) \in R[x]$ be any nonzero polynomial. We define an equivalence relation on $R[x]$ much as we did on $\mathbb{Z}$ when defining $\mathbb{Z}/n\mathbb{Z}$, namely, we say $g(x) \sim h(x)$ if $f(x) \mid (g(x) - h(x))$. We write $g(x) \equiv h(x) \pmod{f(x)}$ if $g(x) \equiv h(x)$. This equivalence relation partitions $R[x]$ into equivalence classes. Given $g(x) \in R[x]$, we let $[g(x)]_{f(x)}$ denote the equivalence class containing $g(x)$, i.e., $[g(x)]_{f(x)} = \{h(x) \in R[x] : h(x) \equiv g(x) \pmod{f(x)}\}$. We drop the subscript $f(x)$ when it is clear from context. We denote the set of equivalence classes by $R[x]/(f(x)) = \{[g(x)] : g(x) \in R[x]\}$. Note that $R[x]/(f(x))$ need not be a finite set. In this case one can use the division algorithm to see that one can always choose a representative $r(x)$ for the equivalence class with $\deg r(x) < \deg f(x)$. We define addition and multiplication on $R[x]/(f(x))$ by $[g(x)] + [h(x)] = [g(x) + h(x)]$ and $[g(x)] \cdot [h(x)] = [g(x)h(x)]$. This makes $R[x]/(f(x))$ into a ring. This ring is commutative if R is commutative.

Exercise A.0.26. Consider the ring $\mathbb{R}[x]/(x^2 + 1)$. Show that one has

$$\mathbb{R}[x]/(x^2 + 1) = \{[ax + b] : a, b \in \mathbb{R}\}.$$

What is $[x]^2$ equal to in this ring? What familiar ring does this look like?

Exercise A.0.27. Let R be a ring and $I \subset R$ an ideal. Define a relation $\sim$ on R by setting $a \sim b$ if $a - b \in I$. Show this is an equivalence relation. For $a \in R$, let $a + I$ denote the equivalence class containing a, i.e., $a + I = \{a + i : i \in I\}$. (Note this is also sometimes denoted $[a]$.) Let R/I denote the set of these equivalence classes. Define $(a + I) + (b + I) = (a + b) + I$ and $(a + I)(b + I) = ab + I$. Show this makes R/I into a ring. Rectify this with the notation $\mathbb{Z}/n\mathbb{Z}$ and $R[x]/(f(x))$ used above.

While rings are extremely important if one wants to consider the more general results of linear algebra that are phrased in terms of modules, for the main results in these notes we stick with vector spaces. Thus, we are mainly interested in fields.

Definition A.0.28. A commutative ring R is said to be a *field* if every nonzero element in R has a multiplicative inverse.

Exercise A.0.29. Show the only ideals in a field F are 0 and F.

Example A.0.30. The rings $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are all fields, but $\mathbb{Z}$ is not a field.

Example A.0.31. Let n be a positive integer and recall the group $\mathbb{Z}/n\mathbb{Z}$ with the operation given by $+$. Given $[a], [b] \in \mathbb{Z}/n\mathbb{Z}$, we define $[a] \cdot [b] = [ab]$. One can easily check this makes $\mathbb{Z}/n\mathbb{Z}$ into a commutative ring. Now consider the case when $n = p$ is a prime. We still clearly have $\mathbb{Z}/p\mathbb{Z}$ is a commutative ring, but in this case we have more. Let $[a] \in \mathbb{Z}/p\mathbb{Z}$ with $[a] \neq [0]$, i.e., $p \nmid a$. Since p is a prime this means that $\gcd(a, p) = 1$, so there exists $x, y \in \mathbb{Z}$ so that $ax+by = 1$, i.e., $ax \equiv 1 \pmod{p}$. This means that $[a][x] = [1]$, i.e., $[a]$ has a multiplicative inverse. Thus, $\mathbb{Z}/p\mathbb{Z}$ is a field. (Note this does not work for general n; look back at your multiplication table for $\mathbb{Z}/4\mathbb{Z}$.) When we wish to think of $\mathbb{Z}/p\mathbb{Z}$ as a field we denote it by $\mathbb{F}_p$.

Exercise A.0.32. Let F be a field and $f(x) \in F[x]$ an irreducible polynomial. Show that $F[x]/(f(x))$ is a field. (Hint: mimic the proof that $\mathbb{Z}/p\mathbb{Z}$ is a field.)

Exercise A.0.33. Consider the ring $\mathbb{C}[x]/(x^2 + 1)$. Is this a field? Prove your answer.

Definition A.0.34. Let R and S be rings. A map $\phi : R \to S$ is a *ring homomorphism* if it satisfies

- $\phi(r_1 +_R r_2) = \phi(r_1) +_S \phi(r_2)$ for all $r_1, r_2 \in R$;
- $\phi(r_1 \cdot_R r_2) = \phi(r_1) \cdot_S \phi(r_2)$ for all $r_1, r_2 \in R$.

We define the *kernel* of ϕ by

$$\ker \phi = \{r \in R : \phi(r) = 0_S\}.$$

Exercise A.0.35. Let $\phi : R \to S$ be a ring homomorphism. Show $\ker \phi$ is an ideal of R.

Exercise A.0.36. Let F be a field and R a ring. Let $\phi : F \to R$ be a ring homomorphism and assume ϕ is not the zero map, i.e., there exists $x \in F$ so that $\phi(x) \neq 0_R$. Prove that ϕ is an injective map.

Example A.0.37. Let $S \subset R$ be a subring. Then inclusion map $S \hookrightarrow R$ is a ring homomorphism with a trivial kernel.

Example A.0.38. Let R be a ring and $I \subset R$ an ideal. The natural projection map

$$\begin{aligned} R &\to R/I \\ a &\mapsto a + I \end{aligned}$$

is a surjective ring homomorphism with kernel equal to I.

Example A.0.39. Define $\phi : \mathbb{R}[x] \to \mathbb{C}$ by $f(x) \mapsto f(i)$. This is a ring homomorphism with kernel $(x^2 + 1)$.

Exercise A.0.40. Define a map $\phi : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ by $\phi([a]) = [4a]$. What is the kernel of this map? What is the image of this map?

Bibliography

[1] D. Dummit and R. Foote. *Abstract algebra.* John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.

[2] T. Hungerford. *Abstract Algebra: An Introduction*, volume 3. Cengage Learning, 2012.

[3] R. Larson. *Elementary Linear Algebra.* Houghton Mifflin, Boston, 6 edition, 2009.

[4] S. Weintraub. *A guide to advanced linear algebra*, volume 44 of *The Dolciani Mathematical Expositions.* Mathematical Association of America, Washington, DC, 2011. MAA Guides, 6.