MATH 8530 — Midterm Exam

October 21, 2014

NAME: ______________________________

1. Do not open this exam until you are told to begin.
2. This exam has 12 pages including this cover. There are 3 problems.
3. Write your name on the top of EVERY sheet of the exam at the start of the exam!
4. Do not separate the pages of the exam.
5. If you are unsure if you are allowed to use a theorem, ask.
6. Turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	40	
2	30	
3	30	
TOTAL	100	

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1. (10+10 + 5 + 5 + 10 each) Let $T_A \in \mathrm{Hom}_F(F^n, F^m)$ be the linear map given by multiplication by the matrix $A = (a_{i,j}) \in \mathrm{Mat}_{m,n}(F)$.

(a) Prove that a basis for the image of T_A can be found by row reducing A to a matrix B and then taking the columns of A that correspond to the pivot columns of B.

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(b) Find a basis for the kernel and image of the map given by the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{pmatrix}.$$

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(c) Give a matrix representing the dual map $T_A^\vee$. Explain how you arrived at your answer.

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(d) Consider the bases $\mathcal{B} = \{1, x, x^2, x^3\}$ and $\mathcal{C} = \{1, 1+x, (1+x)^2, (1+x)^3\}$ of $V = P_3(F)$ where we recall $P_3(F)$ is the vector space of polynomials of degree less than or equal to 3. Find the change of basis matrix that changes $[v]_{\mathcal{B}}$ into $[v]_{\mathcal{C}}$ for $v \in V$

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(e) Find the eigenvalues and bases for the corresponding eigenspaces of the matrix $A \in \mathrm{Mat}_3(\mathbf{Q})$ given by

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

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2. (5 + 5+ 10 + 10 points) Let $V = P_3(\mathbf{Q})$ be the vector space over $\mathbf{Q}$ of polynomials of degree less than or equal to 3.

(a) Let $\mathcal{B} = \{2, 5x, 3x^2 + x, x^3 + 2x^2 + 3\}$. Show that $\mathcal{B}$ is a basis of V.

(b) Let $T \in \mathrm{Hom}_F(V, V)$ be the map given by differentiation. Find the matrix of this linear map with respect to the basis $\mathcal{B}$.

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(c) Find the rational canonical form of the matrix you found in part (b).

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(d) If the Jordan canonical form of the matrix you found in part (b) exists over **Q**, find it. If it does not exist over **Q**, find it over **C** and explain why it did not exist over **Q**.

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3. (15+5+ 10 points) Let V be a finite dimensional F-vector space. Let $T \in \mathrm{Hom}_F(V, V)$. We say T is *semi-simple* if every T-invariant subspace of V has a T-invariant complement. We say T is *nilpotent* if there is a non-negative integer r so that T^r is the zero map.

(a) Prove that $T \in \mathrm{Hom}_F(V, V)$ is semi-simple if and only if $m_T(x) = p_1(x) \cdots p_k(x)$ with the $p_i(x)$ distinct irreducible polynomials over F.

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(b) Let $\overline{F}$ be the algebraic closure of F. Let $\mathcal{B}$ be a basis of V. Prove that T is semi-simple if and only if $[T]_{\mathcal{B}}$ is similar to a diagonal matrix over $\overline{F}$.

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(c) In this problem we work over $\overline{F}$. Prove that there is a semi-simple linear map $S \in \mathrm{Hom}_{\overline{F}}(V, V)$ and a nilpotent linear map $N \in \mathrm{Hom}_{\overline{F}}(V, V)$ so that

(i) $T = S + N$;

(ii) $SN = NS$.

Furthermore, show that S and N are unique. (Note: One can actually prove this over an arbitrary field, but it is much easier over an algebraically closed field.)