MATH 8530 — Final Exam

December 11, 2014

NAME: ____

1. Do not open this exam until you are told to begin.
2. This exam has 9 pages including this cover. There are 4 problems.
3. Write your name on the top of EVERY sheet of the exam at the start of the exam!
4. Do not separate the pages of the exam.
5. If you are unsure if you are allowed to use a theorem, ask.
6. Turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	25	
2	25	
3	30	
4	30	
TOTAL	110	

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1. (10+10 + 5 points) Let V be the vector space of sequences (a_n) of complex numbers, and define the shift operator $S : V \to V$ by $S((a_1, a_2, a_3, ...)) = (a_2, a_3, a_4, ...)$.

(a) Find the eigenvectors of S.

(b) Show that the subspace W consisting of the sequences (a_n) with $a_{n+2} = a_{n+1} + a_n$ is a two-dimensional, S-invariant subspace of V, and exhibit an explicit basis for W.

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(c) Write the Fibonacci sequence using this basis, and use this to find an explicit formula for the nth Fibonacci number, f_n (Recall the Fibonacci sequence is define by $f_1 = 1, f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for all $n \geq 1$.)

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2. (9 + 16 points)

(a) Let $A, B \in \mathrm{Mat}_n(F)$ for a field F.

(i) If $F = \mathbb{Q}$ and A and B are similar over F, are they necessarily similar over $\mathbb{C}$? Be sure to justify your answer.

(ii) If $F = \mathbb{C}$ and A and B are similar over F, are they necessarily similar over $\mathbb{Q}$? Be sure to justify your answer.

(iii) If $F = \mathbb{Q}$ and A and B are similar over F, are they necessarily similar over $\mathbb{F}_3$? Be sure to justify your answer.

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(b) Let $A = \begin{pmatrix} -32 & 9 & 2 & -3 \\ -66 & 20 & 3 & -6 \\ -80 & 21 & 7 & -7 \\ 142 & -36 & -11 & 14 \end{pmatrix}$. Find either the rational or Jordan canonical form of this matrix. It is your choice which you find, but be sure to indicate which you found! Give the minimal and characteristic polynomials.

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3. (5+10+5+10 points)

(a) Let $A \in \mathrm{Mat}_n(\mathbb{C})$. Define what it means for A to be a Hermitian matrix.

(b) Let $A \in \mathrm{Mat}_n(\mathbb{C})$ be a Hermitian matrix. Show that A has n real eigenvalues.

(c) Fill in the following blanks to state the Spectral Theorem for Hermitian matrices:
Given a Hermitian matrix A, there exists a ______________________ matrix P and a diagonal matrix D so that $A =$______________________.

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(d) Let $A = \begin{pmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{pmatrix}$. Find P and D as given in (c) for this particular matrix. Be sure to show your work.

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4. (5+10+5+10 points) Let V and W be finite$F$-vector spaces.

(a) State the universal property for the tensor product $V \otimes_F W$.

(b) Use the universal property to show that $V \otimes_F W \cong W \otimes_F V$.

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(c) Let V be an F-vector space with basis $\mathcal{B} = \{v_1, v_2, v_3\}$. Give a basis for $\Lambda^2 V$.

(d) Define $T : V \to V$ by $T(v_1) = 2v_2 + 3v_3$, $T(v_2) = v_1 + v_3$, and $T(v_3) = 4v_1 + 3v_2$. Calculate $\det(T)$ using the definition of determinant in by using the definition given by $\Lambda^3(T)(v_1 \wedge v_2 \wedge v_3) = (\det T) v_1 \wedge v_2 \wedge v_3$.