Introductory Topology

Jim L. Brown

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Contents

Chapter 1

Introduction

These notes are a work in progress. When finished, they will contain the material covered in a year-long topology sequence taught at Clemson University during the 2009-2010 academic year. I will be working on them throughout the year and posting them as I go so that they can be edited by the students as well as provide details that were omitted in lectures. I will also add a proper introduction at some point.

Things to do yet:

1. More details about holomorphic and anti-holomorphic forms in Chapter 3. Need to add stuff about complex de Rham cohomology, maybe notation for it, etc.

2. Add a chapter on Riemann surfaces that gives the basic theorems necessary for Chapter 5.

3. Beef up Chapter 4. In particular, at least add the duality theorem in this case.

4. Add the proofs of the comparison theorem in Chapter 5.

Chapter 2

Point-Set Topology

This chapter covers the basics of point-set topology that will be needed throughout the rest of these notes. In addition to basic point-set topology, three sections are devoted to developing the notion of topological groups. This topic is normally left to the exercises in a first course in topology, but much more detail is presented here to illustrate how topological notions can be applied effectively to enhance our knowledge of some familiar algebraic structures.

2.1 Basic Definitions and Examples

We begin here with the most fundamental definition, namely that of a topology.

Definition 2.1.1. Let X be a set. A collection $\mathcal{T}$ of subsets of X is called a *topology on* X if they satisfy:

1. $\emptyset$, X both lie in $\mathcal{T}$;
2. Arbitrary unions of elements in $\mathcal{T}$ are in $\mathcal{T}$;
3. Finite intersections of elements in $\mathcal{T}$ are in $\mathcal{T}$.

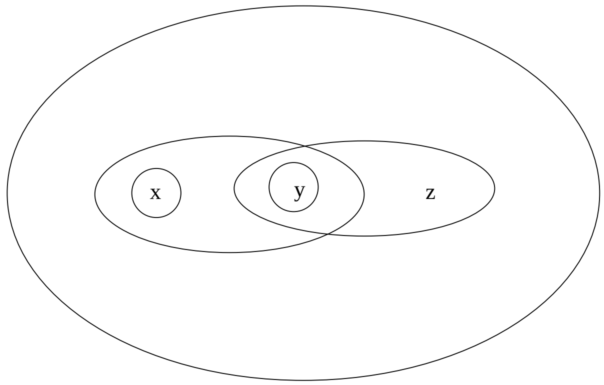
We refer to the elements of $\mathcal{T}$ as *open sets in* X or just *open sets* if X is clear. If $x \in X$ and $U \in \mathcal{T}$ with $x \in U$, we say U is an *open neighborhood* of x.

As this is a fairly abstract definition, before we go any further we give some examples.

Example 2.1.2. For any set X, the collection $\mathcal{T} = \{\emptyset, X\}$ is a topology on X. We refer to this topology as the *trivial topology*.

Example 2.1.3. For any set X, the collection of all subsets of X is a topology. We refer to this topology as the *discrete topology*.

Example 2.1.4. Let $X = \{x, y, z\}$. The collection $\mathcal{T} = \{\emptyset, X, \{x\}, \{y\}, \{x, y\}, \{y, z\}\}$ is a topology on X. It can be pictured as follows where the ovals represent the open sets.



Example 2.1.5. Let $X = \{x, y, z\}$. The collection $\mathcal{T} = \{\emptyset, X, \{x\}, \{y\}\}$ is not a topology because it is not closed under unions.

For a set X, there are generally many different ways to define a topology on X. Sometimes we are able to compare two different topologies defined on X, sometimes we are not.

Definition 2.1.6. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X. If $\mathcal{T}_2 \subset \mathcal{T}_1$ we say that $\mathcal{T}_1$ is *finer* than $\mathcal{T}_2$.

One should note here that given two topologies on a set X, often there will be no containment in one direction or the other so it is not necessarily the case that one topology will be finer than another.

In most cases it is not convenient (or even possible) to give a topology by explicitly listing every open set. It will often be much easier to specify the "important" open sets that can be used to generate the rest of the open sets in the topology.

Definition 2.1.7. Let X be a set. A *basis* for a topology on a set X is a collection $\mathcal{B}$ of subsets of X (called *basis elements*) satisfying:

1. For each $x \in X$ there is a basis element containing x;
2. If there exists $B_1, B_2 \in \mathcal{B}$ so that $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ so that $x \in B_3 \subset B_1 \cap B_2$.

Definition 2.1.8. Let $\mathcal{B}$ be a basis for a topology on X. The topology $\mathcal{T}$ *on* X *generated by* $\mathcal{B}$ is given by declaring that $U \subset X$ is in $\mathcal{T}$ if for each $x \in U$ there exists a $B \in \mathcal{B}$ so that $x \in B \subset U$.

Of course, we are calling such a collection $\mathcal{T}$ given by a basis $\mathcal{B}$ a topology on X, so it is important that we check that $\mathcal{T}$ is actually a topology!

Proposition 2.1.9. *Let X be a set and $\mathcal{B}$ a basis for a topology generating $\mathcal{T}$. The collection $\mathcal{T}$ is a topology on X.*

Proof. It is easy to see that $\emptyset$ and X are both in $\mathcal{T}$.

Let $\{U_i\}_{i\in I}$ be a collection of elements of $\mathcal{T}$. We need to show that $\bigcup_{i\in I} U_i \in \mathcal{T}$. Set $U = \bigcup_{i\in I} U_i$ and let $x \in U$. There exists $j \in I$ so that $x \in U_j$. Since $U_j \in \mathcal{T}$, there is a basis element $B \in \mathcal{B}$ so that $x \in B \subset U_j$. However, $U_j \subset U$ so we have a basis element B so that $x \in B \subset U$. Thus, $U \in \mathcal{T}$.

Finally, we need to show that if $U_1, \ldots, U_n \in \mathcal{T}$, then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$. We show this by induction on n. The base case here is $n = 2$. Let $x \in U_1 \cap U_2$. Since $U_1 \in \mathcal{T}$, there exists $B_1 \in \mathcal{B}$ so that $x \in B_1 \subset U_1$ and similarly there exists a B_2 so that $x \in B_2 \subset U_2$. We now use that $x \in B_1 \cap B_2$ and the fact that $\mathcal{B}$ is a basis to conclude that there exists $B_3 \in \mathcal{B}$ so that $x \in B_3 \subset B_1 \cap B_2$. However, since $B_1 \cap B_2 \subset U_1 \cap U_2$, we have that $x \in\subset B_3 \subset U_1 \cap U_2$ and so $U_1 \cap U_2 \in \mathcal{T}$. Suppose now that the result is true for $n-1$ sets. Consider $U_1 \cap \cdots \cap U_{n-1} \cap U_n = (U_1 \cap \cdots \cap U_{n-1}) \cap U_n$. Our induction hypothesis gives $U_1 \cap \cdots \cap U_{n-1} \in \mathcal{T}$ and $U_n \in \mathcal{T}$ by assumption, so the case $n = 2$ then gives that $(U_1 \cap \cdots \cap U_{n-1}) \cap U_n \in \mathcal{T}$ and so we have the result. □

Lemma 2.1.10. *Let X be a set and $\mathcal{B}$ a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the collection of all unions of elements in $\mathcal{B}$.*

Proof. Clearly one has all the unions of elements in $\mathcal{B}$ are in $\mathcal{T}$ because $\mathcal{T}$ is a topology and $\mathcal{B} \subset \mathcal{T}$. Now let $U \in \mathcal{T}$. For each $x \in U$, there exists $B_x \in \mathcal{B}$ so that $x \in B_x \subset U$. Thus, $U = \bigcup_{x\in U} B_x$ and so the results follows. □

In some cases we will already have a topology defined and we would like to check that a given basis actually gives this topology. Alternatively, we may be interested in whether two bases give the same topology. The following lemmas allow us to check this.

Lemma 2.1.11. *Let X be a set and $\mathcal{B}_1$ and $\mathcal{B}_2$ bases for topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X. One has that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ if and only if for each $x \in X$ and each $B_1 \in \mathcal{B}_1$ with $x \in B_1$, there exists $B_2 \in \mathcal{B}_2$ so that $x \in B_2 \subset B_1$.*

Proof. First, suppose that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$. For $x \in X$, let $B_1 \in \mathcal{B}_1$ be a basis element containing x. Since $\mathcal{T}_2$ is finer than $\mathcal{T}_1$, we must have $B_1 \in \mathcal{T}_2$. However, since $\mathcal{T}_2$ is generated by $\mathcal{B}_2$, there must exist a $B_2 \in \mathcal{B}_2$ so that $x \in B_2 \subset B_1$.

Now suppose that for each $x \in X$ and each $B_1 \in \mathcal{B}_1$ with $x \in B_1$, there exists $B_2 \in \mathcal{B}_2$ so that $x \in B_2 \subset B_1$. Let $U \in \mathcal{T}_1$. Let $x \in U$. Since $\mathcal{B}_1$ generates $\mathcal{T}_1$, there exists $B_1 \in \mathcal{B}_1$ with $x \in B_1 \subset U$. Our hypothesis now gives $B_2 \in \mathcal{B}_2$ so that $x \in B_2 \subset B_1 \subset U$. Thus, we have that $U \in \mathcal{T}_2$ as desired. □

Lemma 2.1.12. *Let X be a set with a topology $\mathcal{T}$. Let $\mathcal{O}$ be a collection of open sets of X such that for each open set $U \subset X$ and each $x \in U$, there exists a $U_x \subset U$ with $x \in U_x$ and $U_x \in \mathcal{O}$. Then $\mathcal{O}$ is a basis for $\mathcal{T}$.*

Proof. There are two steps in this proof: first we must show $\mathcal{O}$ is a basis and second we must show that the topology generated by $\mathcal{O}$ is $\mathcal{T}$.

The first condition of being a basis is clearly satisfied by $\mathcal{O}$. Let $U_1, U_2 \in \mathcal{O}$ so that $x \in U_1 \cap U_2$. Since U_1 and U_2 are open, we have that $U_1 \cap U_2$ is open as well. Thus, there exists $U_3 \in \mathcal{O}$ with $x \in U_3 \subset U_1 \cap U_2$ and so $\mathcal{O}$ is a basis of a topology $\mathcal{T}_{\mathcal{O}}$.

We immediately have from Lemma 2.1.11 that $\mathcal{T}_{\mathcal{O}}$ is finer than $\mathcal{T}$. However, since the elements of $\mathcal{O}$ are in $\mathcal{T}$ and $\mathcal{T}_{\mathcal{O}}$ consists of unions of elements of $\mathcal{O}$ by Lemma 2.1.10, we use that $\mathcal{T}$ is a topology to get that $\mathcal{T}_{\mathcal{O}} \subset \mathcal{T}$ and so we have equality. □

With the notion of bases, we can now give less trivial examples than the ones above.

Example 2.1.13. Consider the set $\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$. Let $\mathcal{B}$ consist of the sets of the form

$$B(x, \epsilon) = \{y \in \mathbb{R}^n : |x - y| < \epsilon\}$$

for $x \in \mathbb{R}^n$ and $\epsilon > 0$. One can easily check that this collection constitutes a basis for a topology on $\mathbb{R}^n$. The topology generated by $\mathcal{B}$ is referred to as the *standard topology* and is the familiar one from classical analysis. Note that the basis elements here are open balls.

Example 2.1.14. Once again we work with $\mathbb{R}^n$, but this time we let $\mathcal{B}'$ consist of sets of the form

$$B(x, \epsilon_1, \dots, \epsilon_n) = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n : |x_i - y_i| < \epsilon_i\}$$

for $x \in \mathbb{R}^n$ and $\epsilon_i > 0$. Once again it is straightforward to check that $\mathcal{B}'$ is a basis for a topology on $\mathbb{R}^n$. The basis elements in this case are open boxes centered at points in $\mathbb{R}^n$.

Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Consider the basis element $B(x, \epsilon) \in \mathcal{B}$. One has that $B(x, \epsilon/2, \dots, \epsilon/2) \subset B(x, \epsilon)$ and $B(x, \epsilon/2, \dots, \epsilon/2) \in \mathcal{B}'$. Thus, the topology generated by $\mathcal{B}'$ is finer than the standard topology on $\mathbb{R}^n$. Now let $B(x, \epsilon_1, \dots, \epsilon_n) \in \mathcal{B}'$. Let $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$. Then we have $B(x, \epsilon) \subset B(x, \epsilon_1, \dots, \epsilon_n)$ and $B(x, \epsilon) \in \mathcal{B}$. Thus, we see that the topology generated by $\mathcal{B}'$ is the standard topology.

Example 2.1.15. Let X and Y be sets with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$ respectively. Consider the set

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

We define a basis for a topology on $X \times Y$ by

$$\mathcal{B} = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}.$$

One should check that $\mathcal{B}$ is in fact a basis. It is also not hard to show that if $\mathcal{B}_X$ is a basis for $\mathcal{T}_X$ and $\mathcal{B}_Y$ is a basis for $\mathcal{T}_Y$, the

$$\mathcal{B}_X \times \mathcal{B}_Y = \{B_X \times B_Y : B_X \in \mathcal{B}_X, B_Y \in \mathcal{B}_Y\}$$

is a basis for the topology given by $\mathcal{B}$. This topology is referred to as the *product topology*. We will study this further in § 2.5.

Note that from what we have shown above the standard topology on $\mathbb{R}^n$ is the same topology as the product topology arising from viewing $\mathbb{R}^n$ as $\mathbb{R} \times \cdots \times \mathbb{R}^n$.

Up to this point the examples given have either been very straightforward or familiar examples from analysis. We now introduce a few less familiar though very important examples. Before we do this, we need to introduce the notion of closed sets. Recall a set $U \subset X$ is an open set if $U \in \mathcal{T}$. We say a set C is *closed* if $C = X - U$ for some $U \in \mathcal{T}$. From this definition it should not be too surprising that one can give a topology on X by specifying the closed sets instead of the open sets. In particular, we have the following result.

Proposition 2.1.16. *Let X be a set and consider a collection $\mathcal{C}$ of subsets of X. Suppose that $\mathcal{C}$ satisfies:*

1. *$\emptyset, X \in \mathcal{C}$;*
2. *$\mathcal{C}$ is closed under finite unions;*
3. *$\mathcal{C}$ is closed under arbitrary intersections.*

Then the set $\mathcal{T} = \{X - C : C \in \mathcal{C}\}$ is a topology on X.

Proof. First, note that $\emptyset = X - X$ and $X = X - \emptyset$, so $\emptyset$ and X are both in $\mathcal{T}$.

Let $U_1 = X - C_1, \ldots, U_n = X - C_n$ be elements of $\mathcal{T}$. We have

$$\begin{aligned} \bigcap_{i=1}^{n} U_i &= \bigcap_{i=1}^{n} (x - C_i) \\ &= X - \left(\bigcup_{i=1}^{n} C_i \right). \end{aligned}$$

Since $\bigcup_{i=1}^{n} C_i \in \mathcal{C}$, we see that $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$ and so $\mathcal{T}$ is closed under finite intersections.

Similarly, if we have an arbitrary collection $\{U_i = X - C_i\}_{i \in I}$ of elements of $\mathcal{T}$, then

$$\begin{aligned} \bigcup_{i \in I} U_i &= \bigcup_{i \in I} (X - C_i) \\ &= X - \left(\bigcap_{i \in I} C_i \right). \end{aligned}$$

Since $\bigcap_{i \in I} C_i \in \mathcal{C}$, we have $\bigcup_{i \in I} U_i \in \mathcal{T}$ and so $\mathcal{T}$ is closed under arbitrary unions. Thus, we have shown that $\mathcal{T}$ is a topology. □

It is often the case that it is easier to specify the closed sets of a set X. In this case one needs to keep in mind that the open sets are the complements of the specified sets!

Example 2.1.17. Let $\mathbb{C}$ be the field of complex numbers. *Affine n-space over* $\mathbb{C}$ is defined to be

$$\mathbb{A}^n_{\mathbb{C}} = \{(a_1, \ldots, a_n) : a_i \in \mathbb{C}\}.$$

One should note that as a set this is just $\mathbb{C}^n$. However, we use the notation $\mathbb{A}^n_{\mathbb{C}}$ to denote the fact that we consider it with a very different topology than the standard topology generated by open balls. Let $A = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over $\mathbb{C}$. We view polynomials $f \in A$ as functions on $\mathbb{A}^n_{\mathbb{C}}$ in the obvious way, namely, for $P = (a_1, \ldots, a_n) \in \mathbb{A}^n_{\mathbb{C}}$ and $f(x_1, \ldots, x_n) \in A$, we have $f(P) = f(a_1, \ldots, a_n) \in \mathbb{C}$. For such an f, set

$$V(f) = \{P \in \mathbb{A}^n_{\mathbb{C}} : f(P) = 0\}.$$

Let $T \subset A$. We define

$$V(T) = \{P \in \mathbb{A}^n_{\mathbb{C}} : f(P) = 0 \text{ for every } f \in T\}.$$

We call a subset $Y \subset \mathbb{A}^n_{\mathbb{C}}$ an *algebraic set* if there exists a $T \subset A$ so that $Y = V(T)$. We declare the algebraic sets to be the closed sets of $\mathbb{A}^n_{\mathbb{C}}$. In order to obtain a topology on $\mathbb{A}^n_{\mathbb{C}}$, we need to show that the collection of algebraic sets satisfies the conditions given in Proposition 2.1.16.

First, note that $V(1) = \emptyset$ and $V(0) = \mathbb{A}^n_{\mathbb{C}}$. Now let $V(T_1), \ldots, V(T_r)$ be algebraic sets. We need to show that $\bigcap_{i=1}^r V(T_i)$ is an algebraic set. We claim that $V(T_1) \cup V(T_2) = V(T_1T_2)$ where T_1T_2 is the set of products of elements in T_1 and T_2. Let $P \in V(T_1) \cup V(T_2)$ so $P \in V(T_1)$ or $P \in V(T_2)$. If $P \in V(T_1)$, then $f(P) = 0$ for every $f \in T_1$. Clearly we then have $fg(P) = f(P)g(P) = 0$ for all $f \in T_1, g \in T_2$. Thus, $P \in V(T_1T_2)$. Similarly if $P \in V(T_2)$ and so $V(T_1) \cup V(T_2) \subset V(T_1T_2)$. Now let $P \in V(T_1T_2)$. Suppose $P \notin V(T_1)$. Then there exists $f \in T_1$ so that $f(P) \neq 0$. If $P \in V(T_2)$ we are done, so suppose $P \notin V(T_2)$, i.e., there exists $g \in T_2$ so that $g(P) \neq 0$. However, this gives $fg(P) \neq 0$, which contradicts $P \in V(T_1T_2)$. Thus, we have equality. One now uses induction to get that $\bigcup_{i=1}^r V(T_i) = V(T_1 \cdots T_r)$ and so $\bigcup_{i=1}^r V(T_i)$ is an algebraic set.

Consider now an arbitrary collection of algebraic sets $\{V(T_i)\}_{i \in I}$. We need to show that $\bigcap_{i \in I} V(T_i)$ is an algebraic set. Note that if $P \in \bigcap_{i \in I} V(T_i)$, then $P \in V(T_i)$ for all $i \in I$. Thus, $f(P) = 0$ for every $f \in \bigcup_{i \in I} T_i$. Hence, we have $\bigcap_{i \in I} V(T_i) \subset V\left(\bigcup_{i \in I} T_i\right)$. It is also easy to see that $V\left(\bigcup_{i \in I} T_i\right) \subset \bigcap_{i \in I} V(T_i)$. Thus, we see that $\bigcap_{i \in I} V(T_i)$ is an algebraic set as well.

The algebraic sets form a topology on $\mathbb{A}^n_{\mathbb{C}}$ called the *Zariski topology*.

Example 2.1.18. We can generalize the previous example. Let R be a commutative ring with identity. Let $\operatorname{Spec} R$ denote the set of prime ideals in R. Recall that an ideal $\mathfrak{p} \subset R$ is a *prime ideal* if $R/\mathfrak{p}$ is an integral domain. Equivalently, $\mathfrak{p}$ is a prime ideal if whenever $xy \in \mathfrak{p}$, either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. We deal here only with proper prime ideals. We define a topology on the space $\operatorname{Spec} R$ as follows. Let I be any subset of R. (It is enough to consider only ideals, but we do not need that here.) We define the closed sets by setting

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec} R : I \subset \mathfrak{p}\}.$$

One must show that these satisfy the criterion given in Proposition 2.1.16. Observe that $V(0) = \operatorname{Spec} R$ since every ideal contains 0. We also have that

$V(1) = \emptyset$ since 1 is not contained in any prime ideal. Much as in the above example, one can show that for $V(I_1), \ldots, V(I_r)$ one has

$$\bigcap_{i=1}^{r} V(I_i) = V\left(\sum_{i=1}^{r} I_i\right)$$

so that $\bigcap_{i=1}^{r} V(I_i)$ is a closed set. Similarly, for an arbitrary collection $\{V(I_i)\}_{i \in I}$, one has

$$\bigcup_{i \in I} V(I_i) = V\left(\prod_{i \in I} I_i\right)$$

so that $\bigcup_{i \in I} V(I_i)$ is also a closed set. Thus, we have a Zariski topology on $\operatorname{Spec} R$ as well.

One should take a moment to think about why Example 2.1.17 is a special case of Example 2.1.18.

2.2 The Subspace Topology

Let X be a set with topology $\mathcal{T}$. Let $Y \subset X$ be a subset. There is a natural way to define a topology on Y using $\mathcal{T}$ called the *subspace topology.* This topology is given by

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}.$$

As an exercise one can show that $\mathcal{T}_Y$ satisfies the definition of a topology on Y.

Lemma 2.2.1. *Let $\mathcal{B}$ be a basis for $\mathcal{T}$ on X. Then*

$$\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$$

is a basis for $\mathcal{T}_Y$.

Proof. We use Lemma 2.1.12. Let $U \cap Y \in \mathcal{T}_Y$. Let $x \in U \cap Y$. In particular, we have $x \in U \in \mathcal{T}$ and so there exists $B \in \mathcal{B}$ so that $x \in B \subset U$ since $\mathcal{B}$ is a basis for $\mathcal{T}$. Thus, we have $x \in B \cap Y \subset U \cap Y$ and so $\mathcal{B}_Y$ is a basis for $\mathcal{T}_Y$. □

Example 2.2.2. Let $X = \mathbb{R}^2$ with the standard topology. Consider $Y = ([0,1] \cup \{2\}) \times \{0\}$ with the subspace topology. The basis we obtain for $\mathcal{T}_Y$ are sets of the form $B(x, \epsilon) \cap Y$ for $\epsilon > 0$ and $x \in \mathbb{R}^2$. For example, $\{(2,0)\} = B((2,0), 1/2) \cap Y$ and so this point is open in Y. Sets of the form $[0, a) \times \{0\}$ as well as $(b, 1] \times \{0\}$ are also open in Y for $0 < a \leq 1$ and $0 \leq b < 1$. This shows there are many sets that are open in Y but not in X.

The best way to get a feel for the subspace topology is to create examples and see what the topologies look like. We see already that we must be careful when specifying a set $U \subset Y$ is open to specify where it is open. It may be the case that U is open in Y but not in X. There is a case when one does not have to be careful, namely when Y is open in X.

Lemma 2.2.3. *Let $Y \subset X$ with $Y \in \mathcal{T}$. Then we have $\mathcal{T}_Y \subset \mathcal{T}$.*

Proof. Let $V \in \mathcal{T}_Y$. Then there exists $U \in \mathcal{T}$ so that $V = Y \cap U$. Since $U \in \mathcal{T}$ and $Y \in \mathcal{T}$, we have $V = Y \cap U \in \mathcal{T}$. □

Consider now two spaces X_1 and X_2 with topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. Let $Y_1 \subset X_1$ and $Y_2 \subset X_2$. We have two natural ways to define a topology on $Y_1 \times Y_2 \subset X_1 \times X_2$. The first way is to consider $Y_1 \times Y_2$ as a subspace of $X_1 \times X_2$ and give it the subspace topology. The second way is to give Y_1 and Y_2 the subspace topologies and then give $Y_1 \times Y_2$ the product topology arising from the subspace topologies. Thankfully, it turns out these are the same topologies.

Theorem 2.2.4. *Let $Y_1 \subset X_1$ and $Y_2 \subset X_2$. The product topology on $Y_1 \times Y_2$ is the same as the subspace topology on $Y_1 \times Y_2$.*

Proof. The basis elements for the product topology on $Y_1 \times Y_2$ are of the form $(U_1 \cap Y_1) \times (U_2 \cap Y_2)$ where $U_i \in \mathcal{T}_i$. Basis elements for the subspace topology on $Y_1 \times Y_2$ are of the form $(U_1 \times U_2) \cap (Y_1 \times Y_2)$. However, by basic set theory we have

$$(U_1 \cap Y_1) \times (U_2 \cap Y_2) = (U_1 \times U_2) \cap (Y_1 \times Y_2).$$

Thus, since the bases are equal the topologies are necessarily equal as well. □

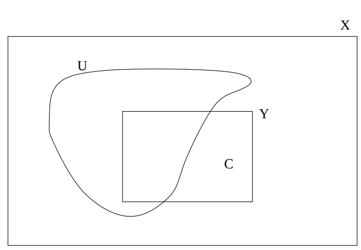
2.3 More Basic Concepts

In this section we introduce some more of the basic concepts that arise in topology such as the closure of a set, limit points, and Hausdorff spaces. First we need some more results on closed sets.

Let $Y \subset X$ be a subset endowed with the subspace topology. (In general we give subsets the subspace topology unless otherwise noted. We refer to them as subspaces.) We say a set $C \subset Y$ is *closed in Y* if it is closed in the subspace topology, i.e., there is an open set $U \in \mathcal{T}_Y$ so that $C = Y - U$.

Lemma 2.3.1. *Let Y be a subspace of X. A set C is closed in Y if and only if it is the intersection of a closed set in X with Y.*

Proof. Let C be closed in Y. Then $Y - C$ is open in Y and so there exists $U \in \mathcal{T}$ such that $Y - C = Y \cap U$. We have that $X - U$ is closed in X. Consider the following figure as motivation.



Based on this figure it is not hard to give a purely set-theoretic proof that $C = (X - U) \cap Y$. Thus, C is the intersection of Y and a closed set in X, as claimed.

Suppose now that $C = Y \cap (X - U)$ for $U \in \mathcal{T}$, i.e., C is the intersection of Y with a closed set in X. Again, using the above picture as motivation one shows that $C = Y - (Y \cap U)$. Since $Y \cap U$ is open in Y, we have that C is closed in Y. □

Proposition 2.3.2. *Let Y be a subspace of X. If C is closed in Y and Y is closed in X, then C is closed in X.*

Proof. The fact that C is closed in Y gives $C = Y \cap E$ for E a closed set in X. However, since Y is also closed in X we have that C is closed in X. □

Definition 2.3.3. Let $A \subset X$. The *interior of* A, denoted $\mathrm{Int}(A)$, is the union of all $U \in \mathcal{T}$ that are contained in A. The *closure of* A, denoted $\mathrm{Cl}(A)$, is the intersection of all closed sets containing A.

Example 2.3.4. Consider $A = [0, 1) \subset \mathbb{R}$. The interior of A is $(0, 1)$ and the closure is $[0, 1]$.

Clearly one always has

$$\mathrm{Int}(A) \subset A \subset \mathrm{Cl}(A).$$

Note that if A is open then $\mathrm{Int}(A) = A$ and if A is closed then $\mathrm{Cl}(A) = A$.

If we have $A \subset Y \subset X$ we must be careful whether we mean the closure of A in X or Y. When dealing with subspaces we write $\mathrm{Cl}_Y(A)$ to denote the closure of A in Y. The following proposition relates $\mathrm{Cl}(A)$ and $\mathrm{Cl}_Y(A)$.

Proposition 2.3.5. *Let $Y \subset X$ be a subspace. We have*

$$\mathrm{Cl}_Y(A) = \mathrm{Cl}(A) \cap Y.$$

Proof. Observe that $\mathrm{Cl}(A) \cap Y$ is a closed set in Y and contains A. Thus, we must have $\mathrm{Cl}_Y(A) \subset \mathrm{Cl}(A) \cap Y$ since $\mathrm{Cl}_Y(A)$ is the intersection of all closed sets in Y containing A.

As for the other direction, observe that since $\mathrm{Cl}_Y(A)$ is closed in Y, there is a closed set C in X so that $\mathrm{Cl}_Y(A) = Y \cap C$ by Lemma 2.3.1. However, we know that necessarily $A \subset C$ and so $\mathrm{Cl}(A) \subset C$. Thus, $\mathrm{Cl}(A) \cap Y \subset Y \cap C \subset \mathrm{Cl}_Y(A)$. □

As we have already seen on several occasions, it is often much easier to work with a basis rather than all of the open sets. The following result allows us to do exactly this when trying to determine the closure of a set.

Theorem 2.3.6. *Let A be a subset of X.*

1. *One has that $x \in \mathrm{Cl}(A)$ if and only if every open set containing x intersects A.*

2. *Let $\mathcal{B}$ be a basis giving the topology $\mathcal{T}$ of X. Then $x \in \mathrm{Cl}(A)$ if and only if every $B \in \mathcal{B}$ such that $x \in B$ satisfies $B \cap A \neq \emptyset$.*

Proof. Note that the first statement is logically equivalent to the statement that $x \notin \mathrm{Cl}(A)$ if and only if there exists $U \in \mathcal{T}$ with $x \in U$ so that $U \cap A = \emptyset$. We prove this statement. Suppose $x \notin \mathrm{Cl}(A)$. Then the U we seek is $X - \mathrm{Cl}(A)$. Conversely, suppose there exists $U \in \mathcal{T}$ with $x \in U$ and $U \cap A = \emptyset$. Then $C = X - U$ is a closed set containing A that does not contain x and so $x \notin \mathrm{Cl}(A)$.

The second results follows from the first. Let $x \in \mathrm{Cl}(A)$. Then every open set containing x intersects A. However, since $\mathcal{B} \subset \mathcal{T}$ we have that every basis element containing x must also intersect A. Conversely, suppose every $B \in \mathcal{B}$ with $x \in B$ satisfies $B \cap A \neq \emptyset$. Let U be an open set containing x. There exists $B \in \mathcal{B}$ so that $x \in B \subset U$. Thus, $U \cap A \neq \emptyset$ and so the first statement gives $x \in \mathrm{Cl}(A)$ as desired. □

Example 2.3.7. Let $A = \{1/n : n \in \mathbb{Z}_{>0}\} \subset \mathbb{R}$ where $\mathbb{R}$ is given the standard topology. Note that there are no nontrivial open sets contained in A so there are no basis elements in A. Thus, $\mathrm{Int}(A) = \emptyset$.

We know that $A \subset \mathrm{Cl}(A)$ automatically. The only point we add when taking the closure of A is 0. If $B = (x, y)$ is a basis element containing 0, then we have $x < 0 < y$ and we can choose M so that $1/M < y$ and so $(x, y) \cap A \neq \emptyset$. Thus, $0 \in \mathrm{Cl}(A)$.

Example 2.3.8. Let $A = \{(x, y) \in \mathbb{R}^2 : x > 0, y \neq 0\}$. This is an open set so $\mathrm{Int}(A) = A$. The closure is given by $\mathrm{Cl}(A) = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$.

Example 2.3.9. Let $A = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q}\}$. Again, we see there are no nontrivial open sets contained in A and so $\mathrm{Int}(A) = \emptyset$.

The fact that the rational numbers are dense in the real numbers gives $\mathrm{Cl}(A) = \mathbb{R}^2$.

Definition 2.3.10. Let $A \subset X$. We say $x \in X$ is a *limit point* of A if every open set containing x intersects A in some point other than x, i.e., x is a limit point of A if it belongs to the closure of $A - \{x\}$.

Theorem 2.3.11. *Let $A \subset X$ and let $\mathrm{LP}(A)$ be the set of limit points of A. Then*

$$\mathrm{Cl}(A) = A \cup \mathrm{LP}(A).$$

Proof. Let $x \in Cl(A)$. If $x \in A$ we are done. Suppose $x \notin A$. Since $x \in \mathrm{Cl}(A)$ we know that every open set containing x intersects A by Theorem 2.3.6. However, since $x \notin A$ it must intersect A in a point other than x and so $x \in \mathrm{LP}(A)$. Thus, $\mathrm{Cl}(A) \subset A \cup \mathrm{LP}(A)$.

Now suppose $x \in A \cup \mathrm{LP}(A)$. If $x \in A$ then clearly $x \in \mathrm{Cl}(A)$. Thus, assume $x \in \mathrm{LP}(A)$ but $x \notin A$. Then every open set containing x must intersect A in a point other than x. Thus, using Theorem 2.3.6 in the other direction we obtain $x \in \mathrm{Cl}(A)$. Hence, $A \cup \mathrm{LP}(A) \subset \mathrm{Cl}(A)$ and so we have equality. □

Corollary 2.3.12. *A set A is closed if and only if it contains all of its limit points.*

Definition 2.3.13. A space X is said to be *Hausdorff* if for every $x, y \in X$ with $x \neq y$ there exists $U, V \in \mathcal{T}$ so that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

A Hausdorff space is one in which one can separate points by open sets. Most familiar spaces that one encounters before studying topology are Hausdorff spaces. It is not difficult to come up with artificial examples of topological spaces that are not Hausdorff.

Example 2.3.14. Let $X = \{x, y, z\}$ with $\mathcal{T} = \{\emptyset, X, \{x, y\}\}$. It is not difficult to check that this is a topology and the points x and y cannot be separated by open sets and so X is not Hausdorff.

It is not uncommon for people to assume all the spaces worth considering are Hausdorff spaces as this eliminates some pathologies that can arise in spaces that are not Hausdorff and will still include most spaces geometers are interested in. However, do not do this as many interesting examples that arise in number theory and algebraic geometry are decidedly not Hausdorff.

Example 2.3.15. Let $X = \operatorname{Spec}\mathbb{Z} = \{0, 2, 3, 5, 7, \dots\}$. Let $p, \ell \in \operatorname{Spec}\mathbb{Z}$ with $p \neq \ell$. Recall the closed sets of $\operatorname{Spec}\mathbb{Z}$ are of the form $V(n) = \{q \in \operatorname{Spec}\mathbb{Z} : q \mid n\}$ along with the empty set and the entire space. Thus, the basic open sets are either the empty set, the entire space or of the form $D(n) = X - V(n) = \{0\} \cup \{q \in \operatorname{Spec}\mathbb{Z} : q \nmid n\}$. From this it is not hard to see that if $p \in U$ and $\ell \in V$, we must have $U \cap V \neq \emptyset$. In fact, $U \cap V$ must contain infinitely many primes. Thus, $\operatorname{Spec}\mathbb{Z}$ is not a Hausdorff space but is an interesting space to arithmetic geometers.

The following theorem is an easy result on Hausdorff spaces. The proof is left as an exercise.

Theorem 2.3.16. *The product of two Hausdorff spaces is Hausdorff. The subspace of a Hausdorff space is Hausdorff.*

Theorem 2.3.17. *Let X be a Hausdorff space. All subsets of X consisting of finitely many points are closed.*

Proof. Note that it is enough to prove the result for a set consisting of a single point as all finite sets can be written as a finite union of one point sets, and finite unions of closed sets are closed. Let $x \in X$. We show that $\operatorname{Cl}(\{x\}) = \{x\}$. Let $y \in X$ be a point with $y \neq x$. Then there exists $U, V \in \mathcal{T}$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Thus, $y \notin \operatorname{Cl}(\{x\})$ since V is an open set that does not intersect $\{x\}$. Since y was any point other than x we must have $\operatorname{Cl}(\{x\}) = \{x\}$ and so $\{x\}$ is closed. □

Corollary 2.3.18. *Let X be Hausdorff and $A \subset X$. One has $x \in \operatorname{LP}(A)$ if and only if every open set containing x intersects A in infinitely many points.*

Proof. Let $x \in \mathrm{LP}(A)$ and suppose there is an open set U with $x \in U$ and $U \cap (A - \{x\}) = \{x_1, \ldots, x_n\}$. We know that $\{x_1, \ldots, x_n\}$ is a closed set in X since X is Hausdorff. Thus, $V = X - \{x_1, \ldots, x_n\}$ is an open set containing x. However, $V \cap (A - \{x\})$ is empty contradicting the fact that $x \in \mathrm{LP}(A)$. Thus, if $x \in U \in \mathcal{T}$ we must have $U \cap A$ contains infinitely many points.

Conversely, if every open set containing x intersects A in infinitely many point, clearly it intersects A in a point other than x and so $x \in \mathrm{LP}(A)$. $\square$

2.4 Continuous Functions

Regardless of the branch of mathematics one is studying, if one wishes to study structure it is important to determine the relevant maps. For instance, in group theory one wishes to look at group homomorphisms and in analysis one works with continuous or differentiable maps. In topology, at least at this point, we are interested in maps that are continuous. (Later we will look at differentiable maps.)

Definition 2.4.1. Let X, Y be sets with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$ respectively. Let $f : X \to Y$ be a map. If for every $V \in \mathcal{T}_Y$ one has $f^{-1}(V) \in \mathcal{T}_X$ we say f is *continuous*.

In general we will omit mention of $\mathcal{T}_X$ and $\mathcal{T}_Y$ and content ourselves with statements of the form "V is open in Y", etc.

Definition 2.4.2. Let $f : X \to Y$. We say f is *continuous at* $x \in X$ if for every open set V in Y with $f(x) \in V$ there is an open set U in X with $f(U) \subset V$.

Proposition 2.4.3. *A function $f : X \to Y$ is continuous if and only if it is continuous at every point in X.*

Proof. It is clear that if f is continuous then it must be continuous at every point in X.

Suppose now that f is continuous at each point $x \in X$. Let V be an open set in Y. Let $x \in f^{-1}(V)$. Then $f(x) \in V$ and so there exists a U_x that is open in X with $x \in U_x$ and $f(U_x) \subset V$ since f is continuous at x. Thus, $U_x \subset f^{-1}(V)$. We can write

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x.$$

Thus, we see that $f^{-1}(V)$ is open since it is the union of open sets. Since V was an arbitrary open set, we see f is continuous. $\square$

The proof of the following proposition is straight-forward and left as an exercise.

Proposition 2.4.4. *Suppose the topology on Y is given by a basis $\mathcal{B}$. Then $f : X \to Y$ is continuous if and only if for every $B \in \mathcal{B}$ one has $f^{-1}(B)$ is open in X.*

Example 2.4.5. Let $f : X \to Y$ be a map and suppose that X has the discrete topology. Then f is continuous as $f^{-1}(V)$ is open for any V, so in particular is open for all V that are open in Y.

Example 2.4.6. Let $X = Y = \mathbb{R}$ with the standard topology. We need to check that our new definition of continuous is equivalent to the $\epsilon - \delta$ definition from elementary analysis. First, suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous in the $\epsilon - \delta$ definition. Let (a, b) be a basis element of the topology. We wish to show that $f^{-1}((a, b))$ is open. Note that if $f(X) \cap (a, b) = \emptyset$ then $f^{-1}((a, b))$ is trivially open. So assume there exists $y \in (a, b)$ so that $y = f(x)$ for some $x \in \mathbb{R}$. Let $\epsilon = \min\{|y - a|, |y - b|\}$. Since f is $\epsilon - \delta$ continuous, there exists a δ_x so that if z satisfies $|x - z| < \delta_x$, then $|f(x) - f(z)| < \epsilon$. Thus, we see that $B(x, \delta_x) \subset f^{-1}((a, b))$ and $x \in B(x, \delta)$. Since we can do this for each $x \in f^{-1}((a, b))$, we can write

$$f^{-1}((a,b)) = \bigcup_{x \in f^{-1}((a,b))} B(x, \delta_x).$$

Thus, $f^{-1}((a, b))$ is open and so by Proposition 2.4.4 we see f is continuous.

Conversely, now assume f is continuous in our new definition. Let $x \in X$ and let $\epsilon > 0$. We have that $B(f(x), \epsilon)$ is an open set, so $f^{-1}(B(f(x), \epsilon))$ is open. We have that $x \in f^{-1}(B(f(x), \epsilon))$ and so there is a basis element $(a, b) \subset f^{-1}(B(f(x), \epsilon))$ so that $x \in (a, b)$. Let $\delta = \min\{|x - a|, |b - x|\}$. Then $B(x, \delta) \subset (a, b) \subset f^{-1}(B(f(x), \epsilon))$. Thus, if $z \in B(x, \delta)$ then $f(z) \in B(f(x), \epsilon)$, i.e., if $|x - z| < \delta$ then $|f(x) - f(z)| < \epsilon$. Since x was arbitrary we have that f is continuous in the $\epsilon - \delta$ definition.

Theorem 2.4.7. *The function $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.*

Proof. First, suppose that f is continuous. Let $C \subset Y$ be closed. Set $A = f^{-1}(C)$. Our goal is to show that $\mathrm{Cl}(A) \subset A$. By definition we have $f(A) \subset C$. Let $x \in \mathrm{Cl}(A)$. If $x \in A$ we are done so assume $x \notin A$. Let V be an open set containing $f(x)$. The fact that f is continuous gives that $f^{-1}(V)$ is open and contains x. Since $x \in \mathrm{Cl}(A)$ we have that $f^{-1}(V) \cap A \neq \emptyset$. Let $y \in f^{-1}(V) \cap A$. Then we have $f(y) \in V \cap f(A)$. Since V was an arbitrary open set containing $f(x)$ and it intersects $f(A)$, we must have $f(x) \in \mathrm{Cl}(f(A))$ and so

$$f(\mathrm{Cl}(A)) \subset \mathrm{Cl}(f(A)). \tag{2.1}$$

Thus,

$$f(x) \in f(\mathrm{Cl}(A)) \subset \mathrm{Cl}(f(A)) \subset \mathrm{Cl}(C) = C$$

and so $\mathrm{Cl}(A) = A$ as claimed.

Now suppose $f^{-1}(C)$ is closed for every closed set C in Y. Let V be an open set in Y. Then $Y - V$ is closed in Y and so $f^{-1}(Y - V)$ is closed in X.

However, basic set theory gives

$$\begin{aligned} f^{-1}(V) &= f^{-1}(Y - C) \\ &= f^{-1}(Y) - f^{-1}(C) \\ &= X - f^{-1}(C) \end{aligned}$$

and so $f^{-1}(V)$ is open. Since V was arbitrary, we have f is continuous. □

Accepting that continuous functions are the "right" functions to study topological spaces, we need to decide under what conditions on $f : X \to Y$ we can reasonably conclude that from a topological point of view that X and Y are the same space. A reasonable guess might be that we want to require f to be continuous and bijective.

Example 2.4.8. Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the unit circle given the subspace topology. Let $X = [0, 1)$ and define $f : X \to S^1$ by $f(x) = (\cos 2\pi x, \sin 2\pi x)$. It is not hard to see using elementary calculus that f is continuous and bijective. However, we would not say that X and S^1 are the same space topologically. For instance, S^1 contains all its limit points where X does not. Pictorially, one has that 0 and 1 are not close together in $\mathbb{R}$, but they map to the same point on S^1 if one considers f defined on all of $\mathbb{R}$. We will see in § 2.8 that S^1 is compact and X is not, so we do not wish to consider them the same space for that reason as well.

The previous example shows that it is not enough to require $f : X \to Y$ to be continuous and bijective.

Definition 2.4.9. We say a continuous bijective map $f : X \to Y$ is a *homeomorphism* if the inverse map $f^{-1} : Y \to X$ is also continuous.

Example 2.4.10. Returning to the example above we see that $g := f^{-1}$ is not continuous. In particular, $[0, 1/4)$ is open in X but $g^{-1}([0, 1/4))$ is not open in S^1.

If there is a homeomorphism between X and Y we say that X and Y are *homeomorphic*. This is the concept of "sameness" that we are looking for. (We will actually give another definition of "sameness" in Chapter 3 that includes homeomorphic spaces.) If whenever a space X satisfies a property, all homeomorphic spaces must also satisfy that property we call the property a *topological property*.

Lemma 2.4.11. *A continuous bijective map $f : X \to Y$ is a homeomorphism if and only if $f(U)$ is open for every open set U in X.*

Proof. Write $g = f^{-1}$ and let U be open in X. Let $V = f(U)$. Since f is bijective, the set $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is precisely U. Thus, $g^{-1}(U) = f(U)$ so g is continuous if and only if $f(U)$ is open for every open set $U \subset X$. □

Example 2.4.12. Let $X = (-1, 1)$ and $Y = \mathbb{R}$. Define $f : X \to Y$ by $f(x) = \frac{x}{1-x^2}$. One can see by graphing this or using calculus that f is bijective and continuous. It is also easy to see that for $(a, b) \subset (-1, 1)$ a basis element of the subspace topology, that $f((a, b)) = \left(\frac{a}{1-a^2}, \frac{b}{1-b^2}\right)$, which is open in $\mathbb{R}$. Thus, f is a homeomorphism.

Let $f : X \to Y$ be continuous and injective. We have that $f : X \to f(X)$ is then continuous and bijective. If $f : X \to f(X)$ is a homeomorphism we say f is an *embedding* and that X *embeds* in the space Y.

Example 2.4.13. Define $f : \mathbb{R} \to \mathbb{R}^2$ by $f(x) = (x, 0)$. Then f is an embedding of $\mathbb{R}$ into $\mathbb{R}^2$.

Example 2.4.14. The map $f : \operatorname{Spec} \mathbb{Z} \to \operatorname{Spec} \mathbb{Z}[x]$ sending (p) to (p) is an embedding where $(p) \subset \mathbb{Z}[x]$ is the extension of the ideal $(p) \subset \mathbb{Z}$ to $\mathbb{Z}[x]$.

In general if X is a subspace of Y with the subspace topology then the identity map $\mathrm{id} : X \to Y$ is an embedding.

We have already made use of some of the following results, but we gather them in one place for convenience. As each item is straight-forward to prove, we leave the proofs to the reader.

Theorem 2.4.15. *Let X, Y and Z be topological spaces.*

1. *The function $f : X \to Y$ defined by $f(x) = y_0$ for a fixed $y_0 \in Y$ is continuous.*
2. *If $A \subset X$ is a subspace, then the inclusion function $f : A \to X$ is continuous.*
3. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*
4. *If $f : X \to Y$ is continuous and $A \subset X$ is a subspace, then $f : A \to Y$ is continuous.*
5. *If $f : X \to Y$ is continuous and $f(X) \subset Z$, then $f : X \to Z$ is continuous assuming the topologies on Z and $f(X)$ agree.*

Theorem 2.4.16. *(The gluing theorem) Let $X = A \cup B$ with A and B closed subsets of X. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions so that $f(x) = g(x)$ for all $x \in A \cap B$. Then the function $h : X \to Y$ given by*

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is continuous.

Proof. First note that h is well-defined because of the assumption that f and g agree on $A \cap B$.

Let $C \subset Y$ be a closed set. We have

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

by set theory. Necessarily we have $f^{-1}(C) \subset A$ and $g^{-1}(C) \subset B$. Since f and g are continuous, $f^{-1}(C)$ is closed in A and $g^{-1}(C)$ is closed in B. However, since A is closed in X we have that $f^{-1}(C)$ is closed in X as well. Similarly we have $g^{-1}(C)$ is closed in X. Thus, $h^{-1}(C)$ is closed in X and so h is continuous as claimed. ☐

2.5 Products of Topological Spaces

Recall that in § 2.1 we defined a topology on $X \times Y$ in terms of the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$. In particular, a basis for the topology on $X \times Y$ was given by

$$\mathcal{B} = \{U \times V : U \in \mathcal{T}_X, V \in T_Y\}.$$

In this section we will generalize this notion to arbitrary products of topological spaces. It turns out there are different ways to generalize the topology given above and these generalizations are not equivalent. We begin with the box topology, which gives the most obvious generalization.

Definition 2.5.1. Let $\{X_i\}_{i \in I}$ be a collection of topological spaces with X_i having topology $\mathcal{T}_i$. The *box topology* on $\prod_{i \in I} X_i$ is the topology generated by the basis

$$\mathcal{B} = \left\{ \prod_{i \in I} U_i : U_i \in \mathcal{T}_i \right\}.$$

One should check that the basis given in the definition of the box topology actually satisfies the requirements to be a basis.

Though the box topology is the obvious generalization of the topology on $X \times Y$, it is actually not the "correct" generalization for most instances as we will see it does not satisfy many of the properties we would expect the topology on $\prod_{i \in I} X_i$ to have. The second way to put a topology on $\prod_{i \in I} X_i$ that generalizes the product topology introduced in § 2.1 is called the product topology.

Definition 2.5.2. Let $\{X_i\}_{i \in I}$ be a collection of topological spaces with X_i having topology $\mathcal{T}_i$. The *product topology* on $\prod_{i \in I} X_i$ is the topology generated by the basis

$$\mathcal{B} = \left\{ \prod_{i \in I} U_i : U_i \in \mathcal{T}_i, U_i = X_i \text{ for all but finitely many } i \right\}.$$

Proposition 2.5.3. *If I is a finite set then the product and box topologies are the same topologies on $\prod_{i \in I} X_i$. For general I the box topology is finer than the product topology.*

We leave the proof of this proposition as an exercise. In many contexts the box topology has too many open sets while the product topology will have the correct number of open sets. In general if we write $\prod_{i\in I} X_i$ without specifying a topology we will mean that it has product topology.

We now state some more easy results. The proofs are left as exercises.

Proposition 2.5.4. *Suppose that for each $i \in I$ that $\mathcal{T}_i$ is given by a basis $\mathcal{B}_i$. Sets of the form $B = \prod_{i\in I} B_i$ with $B_i \in \mathcal{B}_i$ give a basis for the box topology. Sets of the form $\prod_{i\in I} B_i$ with $B_i \in \mathcal{B}_i$ for all i and $B_i = X_i$ for all but finitely many i give a basis for the product topology on $\prod_{i\in I} X_i$.*

Proposition 2.5.5. *If each X_i is Hausdorff, then $\prod_{i\in I} X_i$ is Hausdorff in the box or product topology.*

Proposition 2.5.6. *Let A_i be a subspace of X_i for each $i \in I$. Then $\prod_{i\in I} A_i$ is a subspace of $\prod_{i\in I} X_i$ as long as both are given the product or box topology.*

We close this section with a theorem on continuous functions. This theorem is one that would be expected, but is only true for the product topology. This gives a primary reason that the product topology is the correct topology to put on an infinite product of topological spaces.

Theorem 2.5.7. *Let $\{X_i\}_{i\in I}$ be a collection of topological spaces and let Y be a topological space. For each $i \in I$ let $f : Y \to X_i$ be a function. Define $f : Y \to \prod_{i\in I} X_i$ by $f(y) = (f_i(y))_{i\in I}$. The function f is continuous if and only if f_i is continuous for each $i \in I$.*

Proof. First, suppose that f is continuous. Define $\pi_j : \prod_{i\in I} X_i \to X_j$ by $\pi_j((x_i)) = x_j$, i.e., π_j is the natural projection map onto the j^{th} component. Clearly we have $f_j = \pi_j \circ f$. By assumption we have that f is continuous, and we know the composition of continuous functions is continuous, so it only remains to show that π_j is continuous. Let $U_j \subset X_j$ be open. We have $\pi_j^{-1}(U_j) = \prod_{i\in I} V_i$ where $V_i = X_i$ for all $i \neq j$ and $V_j = U_j$. This is clearly open so π_j is continuous and thus f_j is open as well. Note here that this part is true for the product or box topology.

Now suppose that each f_i is continuous. Let $\prod_{i\in I} U_i$ be a basis element for the product topology. We know that $U_j = X_j$ for all but finitely many j. Let $j_1, \ldots, j_n$ be the indices where $U_j \neq X_j$. We have

$$\prod_{i\in I} U_i = \pi_{j_1}^{-1}(U_{j_1}) \cap \cdots \cap \pi_{j_n}^{-1}(U_{j_n}).$$

Thus, we have that

$$\begin{aligned} f^{-1}\left(\prod_{i\in I} U_i\right) &= f^{-1}\left(\pi_{j_1}^{-1}(U_{j_1}) \cap \cdots \cap \pi_{j_n}^{-1}(U_{j_n})\right) \\ &= f^{-1}(\pi_{j_1}^{-1}(U_{j_1})) \cap \cdots \cap f^{-1}(\pi_{j_n}^{-1}(U_{j_n})) \\ &= f_{j_1}^{-1}(U_{j_1}) \cap \cdots \cap f_{j_n}^{-1}(U_{j_n}). \end{aligned}$$

This last set is open since each f_i is continuous. Thus, $f^{-1}\left(\prod_{i\in I} U_i\right)$ is open and so f is continuous. □

It is important to be sure to understand where the fact that we were using the product topology was used in the second part of the proof of Theorem 2.5.7!

2.6 Metric Spaces

Metric spaces are often covered in an analysis class so it is likely that most already have some experience with metric spaces. In fact, it is generally metric spaces that give most their geometric intuition. This can be a blessing as well as a curse when dealing with more abstract spaces. Even if one has not encountered the definition of a metric space, one is certainly familiar with many of them as we will soon see.

Definition 2.6.1. A *metric* on a set X is a function

$$\rho : X \times X \to \mathbb{R}$$

such that

1. $\rho(x, y) \geq 0$ for every $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;
2. $\rho(x, y) = \rho(y, x)$ for every $x, y \in X$;
3. $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ for every $x, y, z \in X$.

The third condition above is often referred to as the triangle inequality as it generalizes the triangle inequality $|x + y| \leq |x| + |y|$ from $\mathbb{R}$.

Metrics are generalizations of the distance function $\rho(x, y) = |x - y|$ on $\mathbb{R}$, i.e., they measure distance between points of a space. Given a set X and a metric ρ on X, we define the *metric topology* on X to be the topology generated by the basis

$$\mathcal{B} = \{B(x, \epsilon) : x \in X, \epsilon > 0\}$$

where

$$B(x, \epsilon) = \{y \in X : \rho(x, y) < \epsilon\}.$$

We must show that $\mathcal{B}$ satisfies the conditions of being a basis. The first condition is clear as for any $x \in X$, we have $B(x, \epsilon)$ is a basis element containing x for any $\epsilon > 0$. Consider now two basis elements $B(x, \epsilon_1)$, $B(z, \epsilon_2)$. Let $y \in B(x, \epsilon_1) \cap B(z, \epsilon_2)$. Set $\delta = \min(\epsilon_1 - \rho(x_1, y), \epsilon_2 - \rho(x_2, y))$. Then $B(y, \delta) \subset B(x, \epsilon_1) \cap B(z, \epsilon_2)$. Thus, $\mathcal{B}$ is a basis as claimed.

Note that we have shown that U is open in the metric topology if and only if for each $x \in U$ there is an open ball $B(x, \epsilon)$ contained in U. This will be a very useful way to think of open sets in the context of metric spaces.

Example 2.6.2. The space $\mathbb{R}$ with $\rho(x, y) = |x - y|$ is a metric space. The metric topology gives the standard topology on $\mathbb{R}$.

Example 2.6.3. The space $\mathbb{R}$ with $\rho(x,y) = |x-y| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ is a metric space. Note here we write $x = (x_1, \ldots, x_n)$ and similarly for y. Again the metric topology gives the standard topology on $\mathbb{R}^n$.

Example 2.6.4. The space $\mathbb{C}$ with $\rho(z,w) = |z-w|$ is a metric space. The metric topology is the standard topology.

Example 2.6.5. It is often the case that one can define many metrics on a space, each giving a different topology. Consider the space $X = \mathbb{Q}$. The function $\rho(x,y) = |x-y|$ giving the usual metric on $\mathbb{R}$ can be restricted to $\mathbb{Q}$ to give a metric on $\mathbb{Q}$. In fact, in analysis class one learns that $\mathbb{R}$ is constructed from $\mathbb{Q}$ via this metric. In particular, $\mathbb{R}$ is formed from $\mathbb{Q}$ by adjoining the limits of Cauchy sequences where convergence is given in terms of the metric $\rho(x,y) = |x-y|$.

There are other definitions of distance on $\mathbb{Q}$ that are extremely useful and interesting. Let $\frac{a}{b} \in \mathbb{Q}$. For a prime p, we can write $a = p^r c$ and $b = p^s d$ with $r, s \in \mathbb{Z}_{\geq 0}$, $c, d \in \mathbb{Z}$ and $p \nmid cd$. We define a new absolute value on $\mathbb{Q}$ known as the p-adic valuation by setting $\left|\frac{a}{b}\right|_p = p^{s-r}$. For example, one has $\left|\frac{10}{75}\right|_5 = 5^{2-1} = 5$. In this case the absolute value is measuring how divisible by p a number is. The number is small if it is highly divisible by p. Define $\rho_p(x,y) = |x-y|_p$. In this metric two numbers are close together if their difference is divisible by a large power of p. For example, $5^{10}+1$ and 1 are very close together in the metric ρ_5. This metric has many interesting properties that will be given in the following exercises.

One can adjoin to $\mathbb{Q}$ the limits of the Cauchy sequences in the metric ρ_p as was done in forming $\mathbb{R}$. In this case one obtains the field of p-adic numbers, $\mathbb{Q}_p$.

Exercise 2.6.6. Let ρ_p be as in the previous example.

1. Show that $\rho_p(x,y) = |x-y|_p$ defines a metric on $\mathbb{Q}$.

2. Show that this absolute value satisfies

$$|x \pm y|_p \leq \max(|x|_p, |y|_p).$$

3. Let $x \in \mathbb{Q}$ and $\epsilon > 0$. Show that given any $y \in B(x,\epsilon)$ one has $B(x,\epsilon) = B(y,\epsilon)$.

4. As in calculus class, a series $\sum_{n=0}^{\infty} a_n$ with $a_n \in \mathbb{Q}_p$ is said to converge if the sequence of partial sums converge. Show that $\sum_{n=0}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} a_n = 0$.

5. Let $\mathbb{Z}_p \subset \mathbb{Q}_p$ be the set defined by

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

It is a fact that any element $x \in \mathbb{Z}_p$ can be written uniquely in the form

$$x = \sum_{n=0}^{\infty} a_n p^n$$

where $0 \leq a_n \leq p-1$ for all n. Show that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, i.e., if U is an open set in $\mathbb{Z}_p$ then $U \cap \mathbb{Z} \neq \emptyset$.

Definition 2.6.7. Let X be a set with topology $\mathcal{T}$. We say X is *meterizable* if there exists a metric on X so that $\mathcal{T}$ is the topology induced from the metric.

Theorem 2.6.8. *Let X and Y be meterizable with metrics ρ_X and ρ_Y respectively. A function $f : X \to Y$ is continuous if and only if given any $x \in X$ and $\epsilon > 0$ there exists a $\delta > 0$ so that if $\rho_X(x, y) < \delta$ then $\rho_Y(f(x), f(y)) < \epsilon$.*

Proof. First, suppose that f is continuous and let $x \in X$ and $\epsilon > 0$. Consider the open set $B(f(x), \epsilon) \subset Y$. Since f is continuous, we know $f^{-1}(B(f(x), \epsilon))$ is open in X. Thus, there is a $\delta > 0$ so that $B(x, \delta) \subset f^{-1}(B(f(x), \epsilon))$. In other words, if $\rho_X(x, y) < \delta$ then $\rho_Y(f(x), f(y)) < \epsilon$.

Now suppose that given any $x \in X$ and $\epsilon > 0$ there exists a $\delta > 0$ so that if $\rho_X(x, y) < \delta$ then $\rho_Y(f(x), f(y)) < \epsilon$. Let $V \subset Y$ be an open set. If $f^{-1}(V) = \emptyset$, then $f^{-1}(V)$ is clearly open. Assume there is a $x \in f^{-1}(V)$. Since V is an open set, there exists a $\epsilon > 0$ so that $B(f(x), \epsilon) \subset V$. By assumption, there exists a $\delta > 0$ so that if $\rho_X(x, y) < \delta$ then $\rho_Y(f(x), f(y)) < \epsilon$, i.e., $f(B(x, \delta)) \subset B(f(x), \epsilon)$. Thus, we have that $B(x, \delta)$ is an open neighborhood of x contained in $f^{-1}(V)$. Since x was arbitrary, we have that $f^{-1}(V)$ is open and so f is continuous. □

One should compare the proof of this result with that of Proposition 2.4.4 and note the similarity. This shows in many ways a metric space acts much like the familiar Euclidean spaces one is used to.

Recall from calculus the notion of a sequence. A sequence is a function $f : \mathbb{N} \to \mathbb{R}$. We can define a sequence in a topological space X as a function $f : \mathbb{N} \to X$. We again denote the values of f by $x_n := f(n)$. We say the sequence $\{x_n\}$ *converges to a point* $x \in X$ if for every open set U containing x there is a positive integer N so that if $n \geq N$ then $x_n \in U$. We write $x_n \to x$ in this case.

In a general topological space sequences do not behave exactly as one is familiar from calculus. For instance, a sequence can converge to more than one point!

Exercise 2.6.9. Construct a sequence that converges to more than one point.

Proposition 2.6.10. *Let X be Hausdorff and $\{x_n\}$ a sequence in X. If $\{x_n\}$ converges then the limit is unique.*

Proof. Suppose that $\{x_n\}$ converges to x and x' with $x \neq x'$. The fact that X is Hausdorff implies that there exists open sets U and V with $x \in U$, $x' \in V$ and $U \cap V = \emptyset$. However, we know that there exists $N \in \mathbb{N}$ so that if $n \geq N$ then $x_n \in U$ and there exists $M \in \mathbb{N}$ so that if $n \geq M$ then $x_n \in V$. This is a contradiction and so it must be that $x = x'$. □

Proposition 2.6.11. *Let $A \subset X$. If there is a sequence of points of A converging to x then $x \in \mathrm{Cl}(A)$. If X is meterizable and $x \in \mathrm{Cl}(A)$, then there is a sequence $\{x_n\}$ of points in A that converge to x.*

Proof. Suppose there exists $\{x_n\}$ with $x_n \in A$ and $x_n \to x$. Let U be an open set containing x. Since $x_n \to x$, there are infinitely many $x_n \in U \cap A$. If $x \in A$, we are done. If not, every open set containing x intersects $A - \{x\}$ and so $x \in \mathrm{Cl}(A)$.

Now suppose that $x \in \mathrm{Cl}(A)$ and X is meterizable with metric ρ. If $x \in A$ we can set $x_n = x$ for all n and we are done. Assume $x \notin A$. Consider the open set $B(x, 1/n)$ for $n \geq 1$. Since $x \in \mathrm{Cl}(A) - A$, for each n there exists $x_n \in B(x, 1/n) \cap A$. We claim that $x_n \to x$. Let U be an open set containing x. There exists $N > 0$ so that $B(x, 1/N) \subset U$. Thus, for $n > N$, $x_n \in U$ and so we have the claim. □

Example 2.6.12. Let $X = \mathbb{R}^{\mathbb{N}} = \{(x_1, x_2, \dots) : x_i \in \mathbb{R}\}$ and put the box topology on X. Set A to be the set

$$A = \{(x_1, x_2, \dots) : x_i > 0\}.$$

We claim that $0 \in \mathrm{Cl}(A)$. Let $B = (x_1, y_1) \times (x_2, y_2) \times \cdots$ be a basis element of the box topology containing 0. Then clearly we have $B \cap A \neq \emptyset$. For example, the element $(y_1/2, y_2/2, \dots)$ is in the intersection.

Suppose there is a sequence of elements in A converging to 0. Call this sequence $\{x_{i,j}\}$ where

$$x_j = (x_{1,j}, x_{2,j}, \dots) \in A.$$

Consider the basis element

$$B = (-x_{1,1}, x_{1,1}) \times (-x_{2,2}, x_{2,2}) \times \cdots .$$

Then we have $0 \in B$, but $x_j \notin B$ for each j and so the sequence cannot converge to 0. Since the sequence was arbitrary, there can be no sequence in A converging to $0 \in \mathrm{Cl}(A)$.

Note that in light of Proposition 2.6.11 this shows that $\mathbb{R}^{\mathbb{N}}$ with the box topology is not meterizable.

Theorem 2.6.13. *Let $f : X \to Y$ be a function with X meterizable. The function f is continuous if and only if for every sequence $\{x_n\}$ with $x_n \to x$ the sequence $\{f(x_n)\}$ converges to $f(x)$.*

Proof. Let f be a continuous function and $\{x_n\}$ a sequence in X converging to x. Let V be an open set in Y containing $f(x)$. Since f is continuous we have that $f^{-1}(V)$ is open in X and contains x. Thus, there exists $N \in \mathbb{N}$ so that if $n \geq N$ then $x_n \in f^{-1}(V)$. Thus, for $n \geq N$ we have $f(x_n) \in V$. Since V was arbitrary, we have $f(x_n) \to f(x)$. Note that we did not use the fact that X is meterizable for this direction of the proof.

Conversely, suppose that for every sequence $\{x_n\}$ in X with $x_n \to x$ we have $f(x_n) \to f(x)$. Let $A \subset X$ be a subset and let $x \in \mathrm{Cl}(A)$. Since X is meterizable, Proposition 2.6.11 gives a sequence $\{x_n\}$ in A converging to x. Thus, $f(x_n) \to f(x)$. We have $f(x_n) \in f(A)$ for all n and so we must have $f(x) \in \mathrm{Cl}(f(A))$ by Proposition 2.6.11. Let C be a closed subset of Y and set

$A = f^{-1}(C)$. Clearly we have $f(A) \subset C$. Let $x \in \mathrm{Cl}(A)$. However, as we saw in equation (2.1), we have $f(x) \in f(\mathrm{Cl}(A)) \subset \mathrm{Cl}(f(A)) = \mathrm{Cl}(C) = C$ and so $x \in f^{-1}(C) = A$. Thus, $\mathrm{Cl}(A) \subset A$ and so $A = \mathrm{Cl}(A)$ and so $f^{-1}(C)$ is closed. Thus, f is continuous. □

As in analysis, we can consider the notion of a sequence of functions converging uniformly to a function.

Definition 2.6.14. Let $f_n : X \to Y$ be a sequence of functions from a set X to a metric space Y with metric ρ. We say the sequence $\{f_n\}$ *converges uniformly* to the function $f : X \to Y$ if given any $\epsilon > 0$ there exists an integer $N \in \mathbb{N}$ so that

$$\rho(f_n(x), f(x)) < \epsilon$$

for all $n \geq N$ and all $x \in X$.

Theorem 2.6.15. *Let $\{f_n\}$ be a sequence of functions from a topological space X to a metric space Y with metric ρ. If each f_n is continuous and $\{f_n\}$ converges a function f uniformly, then f is continuous.*

Proof. Let V be an open set in Y. We want to show that $f^{-1}(V)$ is open in X, i.e., for each $z \in f^{-1}(V)$ there is an open set U containing z so that $U \subset f^{-1}(V)$, i.e., $f(U) \subset V$.

Let $z \in f^{-1}(V)$ and write $y = f(z)$. Choose $\epsilon > 0$ so that $B(y, \epsilon) \subset V$. Since $\{f_n\}$ converges uniformly to f, there exists $N \in \mathbb{N}$ so that for $n \geq N$ and for all $x \in X$ one has $\rho(f_n(x), z) < \epsilon/4$. Furthermore, since each f_n is continuous, there is an open neighborhood U_1 of z so that $f_N(U_1) \subset B(f_N(z), \epsilon/2)$. Set $U_2 = U \cap U_1$. We claim $f(U_2) \subset B(y, \epsilon)$. Observe that if $x \in U_2$, then

$$\rho(f(x), f_N(x)) < \epsilon/4$$

by our choice of N. Similarly, we have

$$\rho(f_N(z), f_N(x)) < \epsilon/2$$

because f_N is continuous and our choice of U_1. Finally,

$$\rho(f_N(z), f(z)) < \epsilon/4$$

again by the choice of N. Thus, applying the triangle inequality we have

$$\begin{aligned}
\rho(f(x), f(z)) &\leq \rho(f(x), f_N(x)) + \rho(f_N(z), f(z)) \\
&\leq \rho(f(x), f_N(x)) + \rho(f_N(x), f_N(z)) + \rho(f_N(z), f(z)) \\
&< \epsilon/4 + \epsilon/2 + \epsilon/4 \\
&= \epsilon.
\end{aligned}$$

Thus, U_2 is the open set around z we were looking for. □

2.7 Connected Spaces

The notion of a space being connected is a fairly intuitive one. Many of the spaces one would expect to be connected are in fact. However, as with most things, when considering general topological spaces some counterintuitive things can occur.

Definition 2.7.1. Let X be a topological space. We say X is *connected* if whenever $X = U \cup V$ with U, V disjoint open sets, we have $\{U, V\} = \{\emptyset, X\}$. If there exist disjoint open sets U, V with $\{U, V\} \neq \{\emptyset, X\}$ so that $X = U \cup V$, then we call $\{U, V\}$ a separation of X.

Proposition 2.7.2. *A space X is connected if the only sets that are both open and closed are $\emptyset$ and X.*

Proof. Suppose that X is connected. Let U be a subset of X that is open and closed but is not $\emptyset$ or X. Then we have that $X - U$ is closed and open and is not $\emptyset$ or X. Thus, $\{U, X - U\}$ provides a separation of X. This is a contradiction so it must be that there is no such U.

Now suppose that $\emptyset$ and X are the only open and closed sets in X. Let $\{U, V\}$ be a separation of X. Suppose that $\{U, V\}$ gives a separation of X. Then $V = X - U$, so V is open and closed. This is a contradiction so it must be that X is connected. □

Note that we see in the above proof that if $\{U, V\}$ provides a separation of X then U and V are both open and closed in X.

Lemma 2.7.3. *Suppose that $\{U, V\}$ gives a separation of X. If Y is a connected subset of X then Y lies entirely in U or V.*

Proof. This follows immediately from the fact that if not then $\{Y \cap U, Y \cap V\}$ gives a separation of Y. □

Theorem 2.7.4. *The union of a collection of connected sets that have a point in common is connected.*

Proof. Let $\{A_i\}$ be a collection of connected subsets of X with $x \in \bigcap_i A_i$. Set $Y = \bigcup_i A_i$ and suppose that $\{U, V\}$ is a separation of Y. As U and V are disjoint, we must have $x \in U$ or $x \in V$. Without loss of generality we may assume that $x \in U$. Since each A_i is a connected subset of Y, we must have $A_i \subset U$ or $A_i \subset V$ by Lemma 2.7.3. However, since $x \in A_i \cap U$ we must have $A_i \subset U$. Since i was arbitrary, we have $Y \subset U$. Thus, $V = \emptyset$, a contradiction. Thus it must be that Y is connected as claimed. □

Proposition 2.7.5. *Let $A \subset X$ be a connected set. Let B be such that $A \subset B \subset \mathrm{Cl}(A)$. Then B is connected.*

Proof. Suppose that $\{U, V\}$ is a separation of B. We have $(U \cap A) \cup (V \cap A) = A$. Since A is connected we must have $A \subset U$ or $A \subset V$ by Lemma 2.7.3. Without loss of generality we may assume that $A \subset U$. Then we have $\mathrm{Cl}(A) \subset \mathrm{Cl}(U)$ and so $B \subset \mathrm{Cl}(U)$. Observe that we have $\mathrm{Cl}(U) \cap V = \emptyset$ for if $x \in V$, then V is an open set containing x that does not intersect U and so $x \notin \mathrm{Cl}(U)$. Thus, we have $B \cap V = \emptyset$. This is a contradiction so it must be that B is connected. □

The following result shows that the property of being connected is a topological property. This means that we can use this as a way to distinguish different topological spaces. Namely, if X is connected and Y is not connected then X cannot be homeomorphic to Y.

Theorem 2.7.6. *Let $f : X \to Y$ be a continuous map. If $A \subset X$ is connected, then $f(A) \subset Y$ is connected.*

Proof. Let $\{U, V\}$ be a separation of $f(A)$. Since U and V are open and f is continuous, we have $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. They are nonempty because U and V are nonempty and contained in $f(A)$. Similarly, we have $f^{-1}(U) \cup f^{-1}(D) \subset A$ because $U \cup V = f(A)$. Thus we have that $f^{-1}(U) \cup f^{-1}(V) = A$ so it only remains to show the intersection is trivial to obtain a separation of A. Suppose that $x \in f^{-1}(U) \cap f^{-1}(V)$. Then we have $f(x) \in U \cap D$, a contradiction. Thus, the intersection is trivial and we have a separation of A. However, this contradicts the fact that A is connected. Thus, we must have $f(A)$ is connected. □

We now prove the familiar result that $\mathbb{R}$ is connected. We will be able to combine this with other results to obtain many familiar spaces such as $\mathbb{R}^n$ and intervals are connected.

Theorem 2.7.7. *The space $\mathbb{R}$ is connected.*

Proof. Suppose that $\{U, V\}$ is a separation of $\mathbb{R}$. Let $x \in U$ and $y \in V$. Without loss of generality we may assume $x < y$. (If there is no element of U that is less than an element of V, just interchange U and V.

Set $U_0 = [x, y] \cap U$ and $V_0 = [x, y] \cap V$. We have that U_0 is open in $[x, y]$ in the subspace topology since U is open in $\mathbb{R}$ and similarly V_0 is open in $[x, y]$. Let $x_0 = \mathrm{lub}(U_0)$. We split into two cases:

Case 1: Suppose that $x_0 \in U_0$. Then clearly we have $x_0 \neq y$ since $U_0 \cap V_0 = \emptyset$. Now since $x_0 \in U_0$ and U_0 is open, there is an interval of the form $[x_0, z)$ contained in U_0. However, this means we can choose an element $w \in (x_0, z)$ so that $w \in U_0$. This contradicts the fact that $x_0 = \mathrm{lub}(U_0)$. Thus, we cannot have $x_0 \in U_0$.

Case 2: Suppose that $x_0 \in V_0$. Note that is has to be one of these two cases since $y \notin V_0$ so $\mathrm{lub}(U_0) < y$, so lies in $[x, y]$ and $[x, y] = U_0 \cup V_0$ by our assumption that $\{U, V\}$ is a separation of $\mathbb{R}$. Since $x_0 \in V_0$, we have that $x_0 \neq x$. Thus, $x < x_0 \leq y$ and so there is an interval of the form $(z, x_0]$ contained in V_0. If

$x_0 = y$ we have a contradiction because then z is an upper bound of U_0 that is less than $\text{lub}(U_0)$. Suppose that $x_0 < y$. We then have $(x_0, y]$ does not intersect U_0, so

$$(z, y] = (z, x_0] \cup (x_0, y]$$

does not intersect U_0. However, this gives z as a smaller upper bound on U_0 than x_0, a contradiction.

Since we have a contradiction in either case, it must be that $\mathbb{R}$ is connected. $\square$

One can use the same argument to obtain that intervals and rays in $\mathbb{R}$ are connected as well. However, it is often easier to use Theorem 2.7.6. For example, since $\sin : \mathbb{R} \to [-1, 1]$ is continuous, surjective, and $\mathbb{R}$ is connected, we have $[-1, 1]$ is connected. To see that $(-a, a)$ is connected, just observe that $\mathbb{R}$ is homeomorphic to $(-a, a)$ under the map

$$\begin{aligned} f : \mathbb{R} &\longrightarrow (-a, a) \\ f(x) &= \frac{2ax}{1 + \sqrt{1 + 4x^2}}. \end{aligned}$$

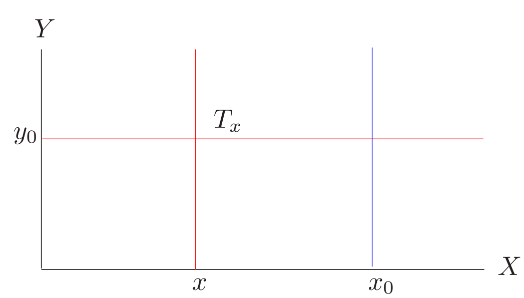
We can use the following theorem to conclude that $\mathbb{R}^n$ is connected for any $n \geq 1$.

Theorem 2.7.8. *The product of connected spaces is connected.*

Proof. We begin by proving the theorem for finite products via induction. Our base case is $n = 2$. Consider two connected spaces X and Y. We wish to show that $X \times Y$ is connected. Pick a point $(x_0, y_0) \in X \times Y$. Observe that $\{x_0\} \times Y$ is homeomorphic to Y so is connected and similarly $X \times \{y_0\}$ is connected. Consider the space

$$T_x = (\{x\} \times Y) \cup (X \times \{y_0\}).$$

Since $\{x\} \times Y$ and $X \times \{y_0\}$ are both connected and both contain the point (x, y_0), Theorem 2.7.4 we have that T_x is connected for every $x \in X$. Given any $x \in X$, we have $(x_0, y_0) \in T_x$. We also have that $\bigcup_{x \in X} T_x = X \times Y$. Since $X \times Y$ is the union of connected sets having a point in common, Theorem 2.7.4 gives that $X \times Y$ is connected. See the following picture where T_x is given in red.



Now suppose the result is true for $n-1$, i.e., if $j \leq n-1$ and $X_1, \ldots, X_j$ are connected, then $X_1 \times \cdots \times X_j$ is connected. Observe that we have $X_1 \times \cdots \times X_n$ is homeomorphic to $(X_1 \times \cdots \times X_{n-1}) \times X_n$. We apply the induction hypothesis to get that $X_1 \times \cdots \times X_{n-1}$ is connected and then apply the base case to obtain $X_1 \times \cdots \times X_n$ is connected as desired.

The more difficult case is when we have an arbitrary product of connected spaces. Let $\{X_i\}_{i \in I}$ be a collection of connected sets. Set $X = \prod_{i \in I} X_i$. Pick a base point $a = (a_i)_{i \in I} \in X$. For any finite set of indices $\{i_1, \ldots, i_m\}$ in I, set

$$X(i_1, \ldots, i_m) \subset X$$

to be the set of points $(x_i)_{i \in I}$ so that $x_i = a_i$ for all $i \notin \{i_1, \ldots, i_m\}$. Let

$$Y = \bigcup X(i_1, \ldots, i_m)$$

where the union is over all finite subsets of I. First, note that Y is not all of X. However, we will show that the closure of Y is all of X. Before we show this, we show why this gives the result.

Observe that we have maps

$$\begin{aligned} \phi : X_{i_1} \times \cdots \times X_{i_m} &\longrightarrow X(i_1, \ldots, i_m) \\ (x_{i_1}, \ldots, x_{i_m}) &\mapsto (y_i)_{i \in I} \end{aligned}$$

where $y_i = a_i$ for $i \notin \{i_1, \ldots, i_m\}$ and $y_{i_j} = x_{i_j}$ for $i_j \in \{i_1, \ldots, i_m\}$. It should also be clear how to define the inverse map as well. Since we are in the product topology, we claim this map is continuous and open. The basis element $B_{i_1} \times \cdots \times B_{i_m}$ maps to $\prod B_j$ where $B_{i_j} = B_{i_j}$ if $i_j \in \{i_1, \ldots, i_m\}$ and otherwise $B_i = \{a_i\} = X(i_1, \ldots, i_m) \cap \prod_{j \neq i} X_j \times B_i$. This gives that the map is open. To see it is continuous, observe a basis element of $X(i_1, \ldots, i_m)$ is of the form $B = \prod B_j$ where $B_j = \{a_j\}$ for $j \notin \{i_1, \ldots, i_m\}$ and B_j is a basis element of X_j for $j \in \{i_1, \ldots, i_m\}$. Thus, $\phi^{-1}(B) = B_{i_1} \times B_{i_m}$. Thus, we have a homeomorphism. The result we have already for finite products gives $X(i_1, \ldots, i_m)$ is connected. Since each $X(i_1, \ldots, i_m)$ contains a we have that the union is connected, i.e., Y is connected. If we can show that $\mathrm{Cl}(Y) = X$, then Proposition 2.7.5 will give that X is connected.

It remains to show that $\mathrm{Cl}(Y) = X$. Let $x = (x_i)_{i \in I} \in X$. Let $B = \prod_{i \in I} B_i$ be a basis element containing x. Each B_i is open in X_i and for all but finitely many indices we have $B_i = X_i$. (Note here it is required we are using the product topology!) Let $\{i_1, \ldots, i_n\}$ be the indices where $B_i \neq X_i$. Define $(y_i)_{i \in I} \in X$ by

$$y_i = \begin{cases} x_i & i \in \{i_1, \ldots, i_m\} \\ a_i & \text{otherwise.} \end{cases}$$

Observe that $y = (y_i)_{i \in I} \in X(i_1, \ldots, i_m) \subset Y$. We also have $y \in B$. Thus, every basis element containing x intersects Y and so $x \in \mathrm{Cl}(Y)$. Thus, $\mathrm{Cl}(Y) = X$ and we are done. $\square$

Definition 2.7.9. A space X is said to be *totally disconnected* if the only subsets of X that are connected are one point sets.

Example 2.7.10. Let $R = \mathbb{R}[x]/(x^2-1)$. Note that since $x^2-1 = (x-1)(x+1)$ we have

$$\mathbb{R}[x]/(x^2-1) \cong \mathbb{R} \oplus \mathbb{R}.$$

The isomorphism is given by the map

$$\begin{aligned} \mathbb{R}[x] &\longrightarrow \mathbb{R} \oplus \mathbb{R} \\ f(x) &\mapsto (f(1), f(-1)). \end{aligned}$$

This map is onto, a homomorphism, and the kernel is (x^2-1). We consider the set of prime ideals $\operatorname{Spec} R$ in R. There are precisely two elements in $\operatorname{Spec} R$, namely, $(x-1)$ and $(x+1)$. These correspond to the two prime ideals in $\mathbb{R} \oplus \mathbb{R}$, namely, $(0) \oplus \mathbb{R}$ and $\mathbb{R} \oplus (0)$. Recall given a general ring R, we defined the Zariski topology on $\operatorname{Spec} R$ in § 2.1. We see that $V((x-1)) = \{(x-1)\}$ and $V((x+1)) = \{(x+1)\}$. Thus, $\{(x+1)\}$ and $\{(x-1)\}$ are both closed in $\operatorname{Spec} R$. Furthermore, since $\{(x+1)\} = \operatorname{Spec} R - V((x-1))$, we have that $\{(x+1)\}$ is open. Similarly, $\{(x-1)\}$ is open as well. We have that $\operatorname{Spec} R = \{(x-1)\} \cup \{(x+1)\}$ so is totally disconnected.

Exercise 2.7.11. Show that $\mathbb{Q}$ with the subspace topology from $\mathbb{R}$ is a totally disconnected space.

There are other notions of how a space can be connected that are useful depending on the situation.

Definition 2.7.12. Let X be a space and $x, y \in X$. A continuous map $f : [a,b] \to X$ with $f(a) = x$, $f(b) = y$ is called a *path from x to y*. If given any $x, y \in X$ there is a path from x to y we say the space X is *path-connected*.

Example 2.7.13. Let $B(a,\epsilon) \subset \mathbb{R}^n$ be an open ball. This is path-connected. To see this, let $x, y \in B(a,\epsilon)$. A path between x and y that lies in $B(a,\epsilon)$ is given by $f : [0,1] \to B(a,\epsilon)$ where $f(t) = (1-t)x + ty$. One should check that the image of f lies in $B(a,\epsilon)$.

Proposition 2.7.14. *Let X be path-connected. Then X is connected.*

Proof. Suppose X is not connected and let $\{U, V\}$ be a separation of X. Let $x \in U, y \in V$. Since X is path-connected, there is a path $f : [a,b] \to X$ with $f(a) = x, f(b) = y$ for some interval $[a,b] \subset \mathbb{R}$. Since $[a,b]$ is connected and f is continuous, $f([a,b])$ is connected. Thus, $f([a,b]) \subset U$ or $f([a,b]) \subset V$. However, this contradicts the fact that $x \in U$, $y \in V$ and f is a path between x and y. Thus, it must be that X is connected. □

It may seem at first glance that a connected space would be path-connected as well. However, this is not the case as the next example demonstrates.

Example 2.7.15. Consider the space $Y = \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) : x \in (0,1)\right\}$. This space is known as the topologist's sine curve. It is connected as it is the image of the connected set $(0,1)$ under the continuous map $y = \sin x$. The topology on Y is the subspace topology it inherits as a subset of $\mathbb{R}^2$.

Our first claim is that $(0,0) \in \mathrm{Cl}(Y)$. Let $U = B(x,\epsilon)$ be a basis element of the topology on $\mathbb{R}^2$ containing $(0,0)$. We can choose $\delta > 0$ so that $B((0,0),\delta) \subset U$. Choose $n \in \mathbb{N}$ so that $\frac{1}{n\pi} < \delta$. Then $\left(\frac{1}{n\pi}, \sin(n\pi)\right) = \left(\frac{1}{n\pi}, 0\right)$ is contained in $B(x,\epsilon) \cap Y$. Since $B(x,\epsilon)$ was arbitrary, this shows $(0,0) \in \mathrm{Cl}(Y)$. One should also note that Y is path-connected by the definition of Y.

Now consider the space $X = Y \cup \{(0,0)\}$. This is a connected set by Proposition 2.7.5 since Y is connected and $(0,0)$ is a limit point of Y. We will now show that X is not path-connected by showing that one cannot connect $(0,0)$ to any other point in Y.

Let $f : [a,b] \to X$ be a path connecting $(0,0)$ to a point $\alpha = \left(x, \sin\left(\frac{1}{x}\right)\right)$ in Y. Since f is continuous and $[a,b]$ is connected, $f([a,b])$ must be connected as well. Consider $f^{-1}(\{(0,0)\})$. Since $\{(0,0)\}$ is closed, we have $f^{-1}(\{(0,0)\})$ is also closed. We now show that $f^{-1}(\{(0,0)\})$ is open as well. This will give that $f^{-1}(\{(0,0)\})$ is a nonempty subset of the connected set $[a,b]$ that is open and closed and so must be the entire interval. This contradicts the fact that f is a path from $(0,0)$ to α. Let $V = B((0,0),1/2)$. Let $t \in f^{-1}(V)$ and $U \subset [a,b]$ a basis element with $t \in U$ and $f(U) \subset B((0,0),1/2)$. We now show that $U \subset f^{-1}(\{(0,0)\})$ and so $f^{-1}(\{(0,0)\})$ is an open set since any point can be surrounded by an open neighborhood that lies in $f^{-1}(\{(0,0)\})$. Since U is a basis element, it is connected and so $f(U)$ is connected as well. We claim that this gives $f(U)$ cannot contain any point other than $(0,0)$. Suppose $\beta = \left(y, \sin\left(\frac{1}{y}\right)\right)$ is in $f(U)$. Choose $n \in \mathbb{N}$ so that $\frac{1}{n\pi} < y$. Then we see that since $f(U) \subset B((0,0),1/2)$, the point $\left(\frac{2}{n\pi}, \sin\left(\frac{n\pi}{2}\right)\right) = \left(\frac{2}{n\pi}, (-1)^{n+1}\right)$ is not in $f(U)$. Consider the disjoint subsets of $\mathbb{R}^2$ given by $\left(-\infty, \frac{2}{n\pi}\right) \times \mathbb{R}$ and $\left(\frac{2}{n\pi}, \infty\right) \times \mathbb{R}$. We see that $f(U)$ must lie completely in the union of these two sets since it does not intersect the line $x = \frac{2}{n\pi}$. However, each set is connected so $f(U)$ must lie entirely in one or the other. Since $(0,0) \in f(U)$, we must have $f(U) \subset \left(-\infty, \frac{2}{n\pi}\right) \times \mathbb{R}$. However, $\beta \in \left(\frac{2}{n\pi}, \infty\right) \times \mathbb{R}$. This is a contradiction. Thus, $f(U) = \{(0,0)\}$ as claimed.

Let X be a topological space. We define an equivalence relation on X by setting $x \sim y$ if there is a connected subset of X that contains x and y. The set of equivalence classes under this relation are called the *connected components* of X. Similarly, we can define another equivalence relation on X by setting $x \sim y$ if there is a path in X that connects x and y. The equivalence classes in this case are called the *path components* of X. One should check that each of these relations is an equivalence relation.

We leave the proof of the following two propositions as exercises.

Proposition 2.7.16. *The connected components of X are connected, disjoint subsets of X whose union is X such that each connected subsets of X intersects only one of the components.*

Proposition 2.7.17. *The path components of X are path-connected disjoint subsets of X whose union is X such that each path-connected subset of X intersects only one of the components.*

Example 2.7.18. Let Y be the topologist's sine curve. We saw above that Y has one connected component and one path component. By adding points in $\{0\} \times [-1, 1]$, we can form a space with one connected component and as many path components as we desire. For instance, $X = \{(0, 0)\} \cup Y$ has two path components. The space $X = \{(0, r) : r \in \mathbb{Q} \cap [-1, 1]\} \cup Y$ has a countably infinite number of path components where $Z = \{(0, r) : r \in (\mathbb{R} - \mathbb{Q}) \cap [-1, 1]\} \cup Y$ has an uncountably infinite number of path components.

We end this section by briefly discussing the notion of a space being locally connected.

Definition 2.7.19. A space X is said to be *locally connected at* $x \in X$ if for every open set U containing x there is a connected open neighborhood of x contained in U. We say X is *locally connected* if it is locally connected at each point.

Similarly one has the notion of locally path-connected, which we leave the reader to define.

Note that being locally connected is equivalent to having a basis of connected sets.

Example 2.7.20. The space $[0, 1) \cup (1, 2]$ is clearly not connected, but it is locally connected. The topologist's sine curve with the points $\{(0, r) : r \in \mathbb{Q} \cap [-1, 1]\}$ added is connected but not locally connected.

Theorem 2.7.21. *A space X is locally connected if and only if for every open set $U \subset X$, each connected component of U is open in X.*

Proof. Let X be locally connected and $U \subset X$ an open set. Let A be a connected component of U. Let $x \in A$. We can choose a connected open neighborhood V of x that is contained in U. However, since A is connected, we must have $V \subset A$ and so A is open in X.

Suppose now that every connected component of every open set in X is open in X. Let $x \in X$ and let U be an open neighborhood of x. There is a connected component A of U that contains x. Since A is open in X by assumption, we have that X is locally connected. □

2.8 Compact Spaces

The notion of a space being compact is not nearly as intuitive as that of being connected. However, the property of compactness is a very powerful property for a space to have so is very important to study.

Definition 2.8.1. Let X be a space and $\{U_i\}$ a collection of open sets. We call $\{U_i\}_{i\in I}$ an *open cover* of X if $X = \bigcup_{i\in I} U_i$. We say the space X is *compact* if every open cover $\{U_i\}_{i\in I}$ of X contains a finite subcover, i.e., there exists $U_1, \ldots, U_n \in \{U_i\}_{i\in I}$ so that $X = U_1 \cup \cdots \cup U_n$.

Example 2.8.2. The space $\mathbb{R}$ is not compact. For instance, if we set $U_i = (-i, i)$, then $\{U_i\}_{i\in\mathbb{N}}$ is an open cover of $\mathbb{R}$ but there is no finite subcover of $\mathbb{R}$.

Example 2.8.3. Let $X = \operatorname{Spec}\mathbb{Z}$. Recall the open sets for the Zariski topology are very large. In fact, the basis elements are given by sets of the form $D(n) = \{(p) \in \operatorname{Spec}\mathbb{Z} : p \nmid n\}$. This should lead us to believe that $\operatorname{Spec}\mathbb{Z}$ is compact, which it is. Let $\{U_i\}$ be an open cover of X. Since each U_i is open, each contains a basis element. Pick any U_i and any basis element contained in U_i, call it $D(n)$. This basis element contains all the elements of $\operatorname{Spec}\mathbb{Z}$ except those primes that divide n. Since there are only finitely many such primes, to cover $\operatorname{Spec}\mathbb{Z}$ we only need to choose the finitely many U_j needed to cover the primes that divide n along with U_i. Thus, we have a finite subcover. Since the open covering was arbitrary, we have that $\operatorname{Spec}\mathbb{Z}$ is compact.

Theorem 2.8.4. *Every closed interval in $\mathbb{R}$ is compact.*

Proof. Let $[a, b]$ be such an interval and let $\mathcal{U} = \{U_i\}$ be an open cover of $[a, b]$. Let $x \in [a, b]$ and let U_i contain x. Since U_i is open, there exists a $y \in [a, b]$ so that $[x, y) \subset U_i$. Choose $z \in [x, y)$. Then $[x, z] \subset U_i$. This can be done for any point $x \in [a, b]$, i.e., for each $x \in [a, b]$ there is a $z \in (a, b)$ so that $[x, z]$ is covered by one element in $\mathcal{U}$.

Let $\mathcal{C}$ be the set of points $c \in (a, b)$ so that $[a, c]$ can be covered by finitely many elements of $\mathcal{U}$. By what we have just shown $\mathcal{C}$ is nonempty. It is clearly bounded above, so there is a least upper bound. Set $c = \operatorname{lub}\mathcal{C}$.

Suppose $c \notin \mathcal{C}$. Choose $U \in \mathcal{U}$ containing c. There exists $d \in [a, c)$ so that $(d, c] \subset U$. Since $c = \operatorname{lub}\mathcal{C}$, there must be an element $z \in \mathcal{C}$ so that $z \in (d, c)$ for otherwise d would be a least upper bound of $\mathcal{C}$. Thus, the interval $[a, z]$ can be covered by finitely many elements in $\mathcal{U}$. However, $[z, c] \subset U$ and so adding this one element to the cover of $[a, z]$ we obtain a finite cover of $[a, c]$, which contradicts the assumption that $c \notin \mathcal{C}$. Thus, $c \in \mathcal{C}$.

If we can show that $c = b$ we will be done. Suppose $c < b$. We can find $x \in (c, b)$ so that $[c, x]$ is covered by only one $U \in \mathcal{U}$. However, we know $[a, c]$ is covered by finitely many intervals, so $[a, x] = [a, c] \cup [c, x]$ is covered by finitely many U_i. This shows $x \in \mathcal{C}$, which contradicts $c = \operatorname{lub}\mathcal{C}$. Thus, $c = b$. □

Proposition 2.8.5. *Every closed subset of a compact space is compact.*

Proof. Let X be compact and let Y be a closed subset of X. Let $\mathcal{U} = \{U_i\}$ be any open covering of Y. Set $\mathcal{U}' = \{U_i\} \cup \{X - Y\}$. We see that $\mathcal{U}'$ is an open covering of X, so has a finite subcover $U_1', \ldots, U_n'$ since X is compact. If $X - Y$ is among the U_i' for $i = 1, \ldots, n$, throw it out. If not, leave the U_i' for $i = 1, \ldots, n$ alone. Either way, we obtain a finite subcover of Y. Thus, Y is compact. □

Proposition 2.8.6. *Every compact subset of a Hausdorff space is closed.*

Proof. Let X be a Hausdorff space and let Y be a compact subset. Let $x_0 \in X - Y$. For each point $y \in Y$, the fact that X is Hausdorff allows us to choose open sets U_y, V_y so that $x_0 \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. Note that $\mathcal{V} = \{V_y\}_{y \in Y}$ forms an open cover of Y. The fact that Y is compact gives a finite subcover $V_{y_1}, \dots, V_{y_n}$ of Y. We have that $\bigcup_{i=1}^n V_{y_i}$ covers Y and is disjoint from the open set $U_{y_1} \cap \cdots \cap U_{y_n}$. Thus, $U = \bigcap_{i=1}^n U_{y_i}$ is an open set in $X - Y$ that contains x_0. Since x_0 was arbitrary, we see that $X - Y$ is open and so Y is closed as claimed. □

It is very important to notice that this last proposition does not say that compact sets are closed in general. This is only true in Hausdorff spaces!

Example 2.8.7. Let $X = \{a, b, c\}$ and set $\mathcal{T} = \{\emptyset, X, \{a\}\}$. Then $\{c\}$ is a compact subset of X but is not closed.

The following result was contained in the proof of Proposition 2.8.6 but we list it as a separate result here as it will be very useful.

Corollary 2.8.8. *Let X be Hausdorff and Y a compact subset of X. If $x \in X - Y$, there are disjoint open sets U, V so that $x \in U$ and $Y \subset V$.*

As with connectedness, the criterion of being compact is a topological property.

Proposition 2.8.9. *The image of a compact space under a continuous map is compact.*

Proof. Let X be a compact space and $f : X \to Y$ a continuous map. Let $\mathcal{V} = \{V_i\}$ be an open cover of $f(X)$. Since f is continuous, $f^{-1}(V_i)$ is continuous for each i and so $\{f^{-1}(V_i)\}$ forms an open cover of X. Since X is compact, there is a finite subcover $f^{-1}(V_1), \dots, f^{-1}(V_n)$. Then $V_1, \dots, v_n$ is a finite cover of $f(X)$. Thus, $f(X)$ is compact. □

In addition to the previous result showing that being compact is a topological property, we can also use it to give an easier criterion one can check to determine if a continuous map from a compact to a Hausdorff space is a homeomorphism.

Theorem 2.8.10. *Let $f : X \to Y$ be a bijective continuous map. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. The bijectivity of f gives the existence of an inverse function $g : Y \to X$. To see that f is a homeomorphism we must show that g is continuous. To see this, it is enough to show that $f(C)$ is closed for every closed set $C \subset X$. The fact that C is closed and X is compact gives that C is compact by Proposition 2.8.5. We now apply Proposition 2.8.9 to see that $f(C)$ is compact. Finally, Proposition 2.8.6 gives that $f(C)$ is closed because Y is assumed to be Hausdorff. □

If we follow the same general outline as when dealing with connected spaces, the next step would be to prove that the product of compact spaces is again compact. This is true in general, though we will only prove it for a finite product. Before we can prove this we need the following lemma.

Lemma 2.8.11. *(Tube Lemma) Let X and Y be topological spaces and assume Y is compact. If N is an open set in $X \times Y$ that contains the slice $\{x_0\} \times Y$ for some $x_0 \in X$, then N contains a tube $U \times Y$ around $\{x_0\} \times Y$ where U is an open set in X containing x_0.*

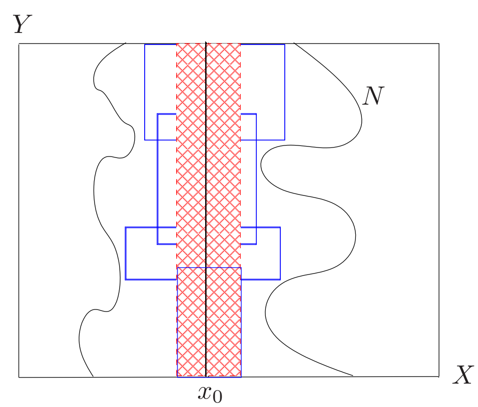
Proof. Note that $\{x_0\} \times Y$ is homeomorphic to Y so it is compact. Let $\{U_i \times V_i\}$ be an open cover of $\{x_0\} \times Y$ with each $U_i \times V_i \subset N$. Since $\{x_0\} \times Y$ is compact, there is a finite subcover $U_1 \times V_1, \ldots, U_n \times V_n$. Set

$$W = U_1 \cap \cdots \cap U_n.$$

This is an open set in X that contains x_0. We claim that $W \times Y \subset N$.

Let $(x, y) \in W \times Y$. There exists j so that $(x_0, y) \in U_j \times V_j$. However, we know that $W \subset U_j$ for all j, so we have $x \in U_j$ as well. Thus, $(x, y) \in U_j \times V_j \subset N$. Hence, we have $W \times Y \subset N$ as claimed.

The following picture illustrates the proof where the red is $W \times Y$ and the blue boxes are the $U_i \times V_i$:



□

It is essential in the tube lemma that Y be compact. If not, the result does not necessarily hold. For example, let $X = Y = \mathbb{R}$. Set

$$N = \left\{(x, y) \in \mathbb{R}^2 : |y| < \frac{1}{|x|}\right\} \cup \{(0, y) : y \in \mathbb{R}\}.$$

This does not contain a tube around Y.

Theorem 2.8.12. *The product of finitely many compact sets is compact.*

Proof. First we note that if we show the result for the product of two compact spaces, the general result for finitely many spaces then follows by induction.

Let X and Y be compact spaces. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of $X \times Y$. Let $x_0 \in X$. The space $\{x_0\} \times Y$ is homeomorphic to Y, so is compact. Thus, there is a finite subcover $U_1, \ldots, U_n$ of $\{x_0\} \times Y$. We have that $U_1 \cup \cdots \cup U_n$ is an open set containing $\{x_0\} \times Y$, so Lemma 2.8.11 gives an open set $W_{x_0} \subset X$ with $x_0 \in W_{x_0}$ and $W_{x_0} \times Y \subset U_1 \cup \cdots \cup U_n$. Thus, the set $W_{x_0} \times Y$ is covered by finitely many open sets $U_1, \ldots, U_n$ in $\mathcal{U}$.

Thus, we see for each $x \in X$, there is an open neighborhood W_x of x so that $W_x \times Y$ can be covered by finitely many elements of $\mathcal{U}$. Observe that $\{W_x\}_{x\in X}$ is an open cover of X and since X is compact, there is a finite subcover $W_{x_1}, \ldots, W_{x_m}$. Thus, $W_{x_1} \times Y, \ldots, W_{x_m} \times Y$ is a covering of $X \times Y$. We know that each $W_{x_j} \times Y$ can be covered by finitely many element of $\mathcal{U}$, and since there are finitely many $W_{x_j} \times Y$ covering $X \times Y$, we obtain a finite cover of $X \times Y$ as desired. □

The more general statement is known as the Tychonoff Theorem. One can find a proof of it in [8].

Theorem 2.8.13. *(Tychonoff Theorem) An arbitrary product of compact spaces is compact in the product topology.*

There are other forms of compactness that are useful as well. We introduce two here.

Definition 2.8.14. A space X is said to be *limit point compact* if every infinite subset of X has a limit point.

Definition 2.8.15. A space X is said to be *sequentially compact* if every sequence in X has a convergent subsequence.

Lemma 2.8.16. *A compact space X is limit point compact.*

Proof. Let A be an infinite subset of X and suppose A has no limit points. Since the closure of A is A along with the limit points of A, we see A must be closed. Since A is closed and X is compact, Proposition 2.8.5 gives that A is necessarily compact as well.

For each $a \in A$ there is an open neighborhood U_a of a so that $U_a \cap A = \emptyset$ since a is not a limit point of A. The collection $\{U_a \cap A\}_{a\in A}$ is an open cover of A with each element containing only one point. Since A is compact, there is a finite subcover. However, this contradicts the fact that A is infinite. Thus, A must have a limit point. □

Lemma 2.8.17. *Let X be a meterizable space. If X is limit point compact then it is sequentially compact.*

Proof. Let ρ be a metric giving the topology on X. Let $\{x_n\}$ be a sequence in X. Set $A = \{x_n : n \in \mathbb{N}\}$. If A is finite there is clearly a convergent subsequence because for some $N \in \mathbb{N}$, if $n > N$ we have $x_{n+j} = x_n$ for all $j \geq 0$. Thus, we have a constant subsequence which obviously converges. Suppose that A is infinite. Since X is limit point compact, A has a limit point x. For each $i \geq 1$, there is an element in $A \cap B(x, 1/i)$ that is not equal to x. Call this element x_{n_i}. It is now clear that the subsequence $\{x_{n_i}\}$ converges to x. Since the sequence $\{x_n\}$ was arbitrary, we see that X is sequentially compact. □

Let X be a metric space with metric ρ. We say a subset A of X is *bounded* if there exists $M \in \mathbb{R}_{>0}$ so that $\rho(a_1, a_2) < M$ for all $a_1, a_2 \in A$. If A is bounded, the *diameter* of A is defined to be

$$\operatorname{diam}(A) = \operatorname{lub}\{\rho(a_1, a_2) : a_1, a_2 \in A\}.$$

Lemma 2.8.18. *(Lebesgue Number Lemma) Let $\mathcal{U}$ be an open cover of a metric space (X, ρ). If X is sequentially compact, there is a $\delta > 0$ (called the Lebesgue number of $\mathcal{U}$) so that for each subset A of X with $\operatorname{diam}(A) < \delta$, there is an element of $\mathcal{U}$ containing A.*

Proof. Let $\mathcal{U}$ be an open cover of X and suppose there is no such δ, i.e., for every $\delta > 0$ there is a subset A_δ of X with $\operatorname{diam}(A_\delta) < \delta$ but A_δ is not contained inside any element of $\mathcal{U}$. In particular, if we set $\delta_i = \frac{1}{i}$, we have sets A_i with $\operatorname{diam}(A_i) < \frac{1}{i}$ so that A_i is not contained in any element of $\mathcal{U}$. For each n, choose a $x_n \in A_n$ and form a sequence $\{x_n\}$.

Suppose that $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ that converges to an element x. Let $U \in \mathcal{U}$ be an element that contains x. Since we are in a metric space, there exists $\epsilon > 0$ so that $B(x, \epsilon) \subset U$. Choose a large j so that $\rho(x, x_{n_j}) < \frac{\epsilon}{2}$ and $\frac{1}{n_j} < \frac{\epsilon}{2}$. Note that $A_{n_j} \subset B\left(x_{n_j}, \frac{1}{n_j}\right)$ since $\operatorname{diam}(A_{n_j}) < \frac{1}{n_j}$, and so $A_{n_j} \subset B(x, \epsilon) \subset U$. This is a contradiction, so the sequence $\{x_n\}$ has no convergent subsequence. This contradicts the fact that X is sequentially compact, so we must have that there exists such a δ. □

In the case that X is meterizable we can now relate the different notions of compactness we have given here.

Corollary 2.8.19. *Let X be meterizable. The following are equivalent:*

1. *X is compact;*
2. *X is limit point compact;*
3. *X is sequentially compact.*

Proof. We have already shown that (1) implies (2) in the general case and that (2) implies (3) in the case of a metric space, so it only remains to show that (3) implies (1).

Let $\epsilon > 0$. We claim that we can cover X by finitely many sets of the form $B(x, \epsilon)$. Suppose not. Let x_1 be any element in X. Note that if $B(x_1, \epsilon)$ is all of X we are done. Otherwise, choose $x_2 \in X - B(x_1, \epsilon)$. We construct a sequence inductively as follows. Suppose we have chosen $x_1, \ldots, x_{n-1}$.

If $B(x_1, \epsilon), \ldots, B(x_{n-1}, \epsilon)$ covers X we are done, if not, choose $x_n \in X - \bigcup_{i=1}^{n-1} B(x_i, \epsilon)$. For each i we have $\rho(x_n, x_i) \geq \epsilon$. Therefore, the sequence can have no convergent subsequence, a contradiction. Thus, the claim follows.

Now let $\mathcal{U}$ be any open cover of X. Since X is sequentially compact there is a Lebesgue number δ associated to $\mathcal{U}$. Choose a finite covering of X by balls of the form $B\left(x, \frac{\delta}{3}\right)$. Each ball has diameter $\frac{2\delta}{3} < \delta$. Thus, for each ball there is an element of $\mathcal{U}$ containing it. Thus, the finitely many elements of $\mathcal{U}$ containing the finitely many $\frac{\delta}{3}$-balls covering X give a finite cover. Since $\mathcal{U}$ was an arbitrary open cover, we see X is compact. □

This allows us to prove the following calculus result generalized to the setting of metric spaces.

Corollary 2.8.20. *Let $f : X \to Y$ be a continuous function with (X, ρ_X) a compact metric space and (Y, ρ_Y) a metric space. Then f is uniformly continuous, i.e., for every $\epsilon > 0$ there exists a $\delta > 0$ so that if $x_1, x_2 \in X$ satisfy $\rho_X(x_1, x_2) < \delta$, then $\rho_Y(f(x_1), f(x_2)) < \epsilon$.*

Proof. Let $\epsilon > 0$. Cover Y by open balls of the form $B\left(y, \frac{\epsilon}{2}\right)$. Let $\mathcal{U}$ be the open cover of X given by the inverse images of the $B\left(y, \frac{\epsilon}{2}\right)$ under f. Let δ be the Lebesgue number of $\mathcal{U}$. If $x_1, x_2 \in X$ satisfy $\rho_X(x_1, x_2) < \delta$, then $x_1, x_2 \in U \in \mathcal{U}$ for some U and so $\rho_Y(f(x_1), f(x_2)) < \epsilon$ since $f(x_1), f(x_2) \in B\left(y, \frac{\epsilon}{2}\right)$ for some $y \in Y$. □

As was the situation when studying connectedness, it is often the case that even though our space is not compact, locally it is compact. Though weaker than being compact, it is still a very useful property to have.

Definition 2.8.21. A space X is said to be *locally compact at a point* $x \in X$ if there is a compact subset C of X that contains an open set containing x. If X is locally compact at each point we say that X is *locally compact*.

Example 2.8.22. The space $\mathbb{R}$ with the standard topology is locally compact. If $x \in \mathbb{R}$, for any $\epsilon > 0$ the set $[x - \epsilon, x + \epsilon]$ is a compact set containing x that contains the open set $(x - \epsilon, x + \epsilon)$.

Lemma 2.8.23. *Let X be Hausdorff. The space X is locally compact at x if and only if for every open neighborhood U of x there is an open neighborhood V of x so that $\mathrm{Cl}(V)$ is compact and $\mathrm{Cl}(V) \subset U$.*

Proof. First suppose that X is locally compact at x. Let C be a compact set containing an open neighborhood of x. Let U be any open neighborhood of x and set $A = C - U$. Since A is closed in C and C is compact, we see A is compact as well. Using Corollary 2.8.8 we can choose disjoint open sets V_1 and V_2 around x and A respectively. Set $W = V_1 \cap \mathrm{Int}(C)$. Note that W is an open neighborhood of x. The fact that X is Hausdorff and C is compact gives that C is closed by Proposition 2.8.6. Thus, $\mathrm{Cl}(W) \subset C$ and so $\mathrm{Cl}(W)$ is compact. Now $W \subset V_1$ so $W \cap A = \emptyset$. Thus, $\mathrm{Cl}(W) \subset C - A$ and so $\mathrm{Cl}(W) \subset U$. This gives the first direction.

The second direction is obvious as one can take $\mathrm{Cl}(V)$ as the desired compact set containing an open neighborhood of the point x. $\square$

Corollary 2.8.24. *Let X be a Hausdorff space that is locally compact. If a subspace Y is open or closed in X then it is locally compact.*

Proof. First we suppose that Y is open in X. Let $y \in Y$. Lemma 2.8.23 allows us to choose an open (in X) neighborhood V of y so that $\mathrm{Cl}(V)$ is compact and $\mathrm{Cl}(V) \subset Y$ where Y is our open neighborhood of y. Then $\mathrm{Cl}(V)$ is a compact set contained in Y containing the open neighborhood V of y. Note that $V \subset Y$ here since $V \subset \mathrm{Cl}(V) \subset Y$. Thus, Y is locally compact.

Now suppose that Y is closed in X. Let $y \in Y$. Let C be a compact set containing an open (in X) neighborhood U of y. Then $C \cap Y$ is closed in Y and hence compact. Thus, we have a compact set $C \cap Y$ containing the open neighborhood $U \cap Y$. Thus, Y is locally compact. $\square$

One should note that in Corollary 2.8.24 the condition that X is Hausdorff was only used in the case that Y is open in X; it was not needed in the case that Y is closed in X.

By now it should be clear that compact spaces are nice ones to work with. Given a locally compact Hausdorff space, there is a way to make the space compact by adding points to it. We will focus on the basic case of a one-point compactification, but one should be aware there are other ways to compactify a space. We will come back to examples and some motivation after definitions.

Definition 2.8.25. Let X be a locally compact Hausdorff space. Let ∞ be an object not in X. We adjoin this to X:

$$Y = X \cup \{\infty\}.$$

We put a topology on Y as follows: $\mathcal{T}_Y$ consists of

1. $\mathcal{T}_X$
2. $Y - C$ where C is a compact set in X.

The space Y is called the *one-point compactification of* X.

Exercise 2.8.26. Check that $\mathcal{T}_Y$ is actually a topology on Y.

We have the following important theorem.

Theorem 2.8.27. *Let X be a locally compact Hausdorff space that is not compact and let Y be the one-point compactification of X. Then Y is a compact Hausdorff space, X is a subspace of Y, the set $Y - X$ consists of a single point, and* $\mathrm{Cl}(X) = Y$.

Proof. First we show X is a subspace of Y. Let $U \in \mathcal{T}_Y$. Then we have either $U \in \mathcal{T}_X$ and so U is open in X or $U = Y - C$ for C a compact subset of X. However, in this second case we have $U \cap X = (Y - C) \cap X = X - C$, which is open in X since C is compact and hence closed because X is Hausdorff. Thus, any $U \in \mathcal{T}_Y$ restricts to an open set in X. Conversely, if $V \in \mathcal{T}_X$, then $V \in \mathcal{T}_Y$ and $V \cap X = V$ so we see that $\mathcal{T}_X$ is the subspace topology induced from Y.

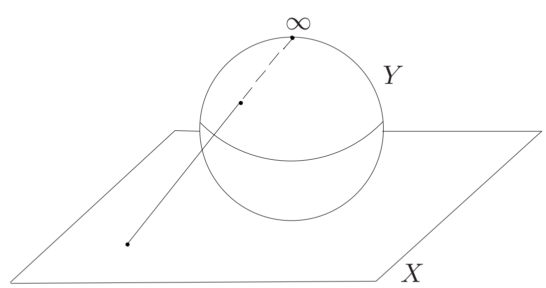
Observe that since X is not compact, $Y - C$ must contain ∞ and intersect X. Thus, ∞ is a limit point of X and so $\text{Cl}(X) = Y$.

Now we show Y is Hausdorff. Let $x, y \in Y$. If both lie in X then there exists $U, V \in \mathcal{T}_X \subset \mathcal{T}_Y$ that are disjoint with $x \in U$, $y \in V$. If $x \in X$ and $y = \infty$, then choose a compact set C containing and open neighborhood U of x. Then U and $Y - C$ are disjoint open neighborhoods of x and ∞ respectively. Thus, Y is Hausdorff.

Finally we show that Y is compact. Let $\mathcal{U}$ be an open cover of Y. There must be at least one open set of the form $Y - C$ in $\mathcal{U}$ in order to cover ∞. Set $U_1 = Y - C$ for one of these sets. Let $\mathcal{U}'$ be the collection of sets $U_i \cap X$ where $U_i \in \mathcal{U} - \{U_1\}$. We see that $\mathcal{U}'$ is an open cover of C and so there is a finite subcover $U_2 \cap X, \ldots, U_n \cap X$. Thus, the finite collection $U_1, U_2, \ldots, U_n$ is a finite cover of Y and so Y is compact. □

Example 2.8.28. Let $X = \mathbb{R}$. The one-point compactification of X is then $Y \cong S^1$.

Example 2.8.29. The one-point compactification of $\mathbb{R}^2$ is the Riemann sphere S^2. One generally encounters this in complex analysis as stereographic projection:



The above picture gives the map from Y to X.

In many cases one does not want to compactify a space by adding a single point.

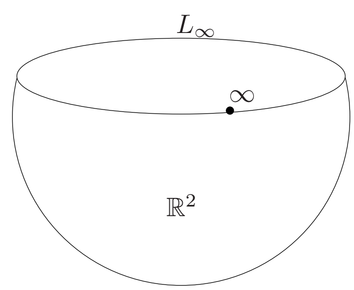
Definition 2.8.30. A *compactification* of a space X is a compact Hausdorff space Y containing X such that X is dense in Y, i.e., $\text{Cl}(X) = Y$. Two compactifications Y_1 and Y_2 of X are said to be *equivalent* if there is a homeomorphism $h : Y_1 \to Y_2$ so that $h(x) = x$ for all $x \in X$.

Not all spaces have compactifications, though we have just seen that all locally compact Hausdorff spaces do. This is enough for many applications. There are particularly useful ways of compactifying Euclidean spaces known as projective spaces. We will deal with these more thoroughly in § 2.9 when dealing with quotient spaces, but we give some motivation here. Consider the space K^n where $K = \mathbb{R}$ or $\mathbb{C}$. We can form a compactification of K^n called projective space and denoted by $K\mathbb{P}^n$. As we will deal with this more in the next section, we restrict to the case of $n = 2$. One has the following very important theorem from algebraic geometry.

Theorem 2.8.31. *(Bezout's Theorem) Let $F(x, y, z)$ and $G(x, y, z)$ be homogeneous polynomials over $\mathbb{C}$ of degree n and m respectively. Suppose F and G have no common factor. Then the curves they define in $\mathbb{CP}^2$ have mn points of intersection counting multiplicity.*

For example, Bezout's Theorem tells us that if we have two lines that are not the same line, they must intersect when considered in $\mathbb{CP}^2$! Thus, in projective space parallel lines also intersect! This type of result makes it much easier to work so that one does not have to constantly split into cases of how many intersections two curves have.

Finally, we give a quick illustration of $\mathbb{RP}^2$. One can think of $\mathbb{RP}^2$ as the plane $\mathbb{R}^2$ along with a "line at infinity" L_∞ with a distinguished point ∞ on this line that compactifies it into a circle:



2.9 Quotient Spaces

The notion of a quotient space is really the first concept we have encountered that is not in some way generalized from classical analysis. The notion of quotient objects is a familiar one from abstract algebra and we will see how the two can be combined when we study topological groups in § 2.10 - 2.12.

Definition 2.9.1. Let $\pi : X \to Y$ be a surjective map between topological spaces. The map π is said to be a *quotient map* provided $U \in \mathcal{T}_Y$ if and only if $\pi^{-1}(U) \in \mathcal{T}_X$.

Recall that $\pi : X \to Y$ is an *open map* if $\pi(U)$ is open for every open set $U \in \mathcal{T}_X$. The map π is said to be a *closed map* if $\pi(C)$ is closed for every closed set C in X. It is straightforward to check that if π is an open or closed map then it is a quotient map.

One is often interested in the case where one has a surjective map $\pi : X \to A$ where X is a topological space and A is a set with no topological structure associated to it. In this case one can put a unique topology on A via the map π. This topology is called the *quotient topology.*

Theorem 2.9.2. *Let $\pi : X \to A$ be a surjective map from a topological space X to a set A. There is a unique topology $\mathcal{T}_A$ on A so that π is a quotient map.*

Proof. Set $\mathcal{T}_A$ to be the set of subsets of A so that $U \in \mathcal{T}_A$ if and only if $\pi^{-1}(U) \in \mathcal{T}_X$. $\square$

Example 2.9.3. Let $X = \mathbb{R}^2$ and $A = \{a, b, c\}$ and define $\pi : X \to A$ by

$$\pi((x,y)) = \begin{cases} a & xy > 0 \\ b & xy = 0 \\ c & xy < 0. \end{cases}$$

The quotient topology induced on A in this case is given by $\mathcal{T}_A = \{\emptyset, A, \{a\}, \{c\}, \{a, c\}\}$.

One of the most common ways one encounters the quotient topology is in the case that A is a partition of X. Recall a partition A of a set X is a collection of subsets $\{U_i\}_{i \in I}$ of X so that $\bigcup_{i \in I} U_i = X$ and $U_i \cap U_j = \emptyset$ for $i \neq j$.

Definition 2.9.4. Let X be a topological space and A a partition of X. Let $\pi : X \to A$ be a surjective map given by $\pi(x) = U_i$ where U_i is the unique element of the partition containing x. If we let $\mathcal{T}_A$ denote the quotient topology on A arising from π, we call A the *quotient space of X with respect to A.* Note that this is often also referred to as the *identification space.*

Note that the quotient space depends upon the partition given. If one gives a different partition, one will construct different spaces in general, as we will see in the following examples and exercises.

Example 2.9.5. Let X be a topological space and Y a subspace. We define the quotient space X/Y by using the partition $A = \{Y\} \cup \bigcup_{x \notin Y} \{x\}$. For example, if we set $D^2 = \{x \in \mathbb{R}^2 : |x| \leq 1\}$, and let $Y = S^1 \subset D^2$, then X/Y is homeomorphic to S^2.

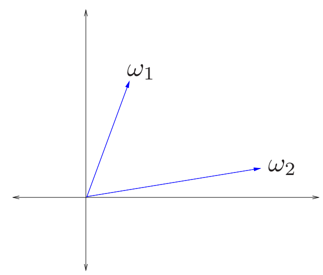
Example 2.9.6. Let ω_1 and ω_2 be complex numbers that are linearly independent over $\mathbb{R}$. The following picture gives an example:

Let

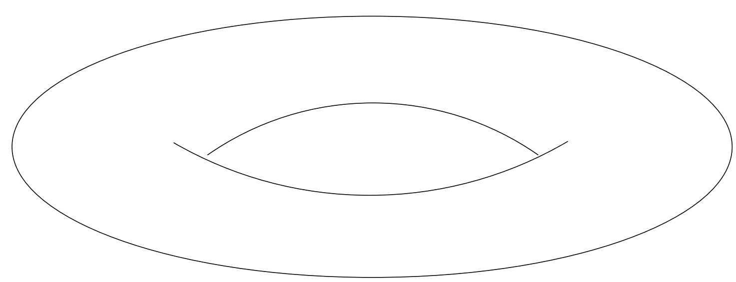
$$X = \{a\omega_1 + b\omega_2 : 0 \leq a, b \leq 1\},$$

i.e., X is the parallelogram spanned by ω_1 and ω_2. Define a partition on X by the sets:

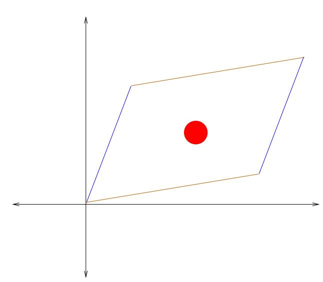
$$\{a\omega_1 + b\omega_2 : 0 < a < 1, 0 < b < 1\}$$
$$\{a\omega_1, a\omega_1 + \omega_2 : 0 < a < 1\}$$
$$\{b\omega_2, \omega_1 + b\omega_2 : 0 < b < 1\}$$
$$\{0, \omega_1, \omega_2, \omega_1 + \omega_2\}.$$

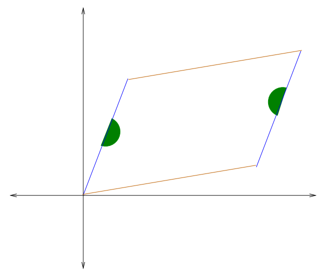


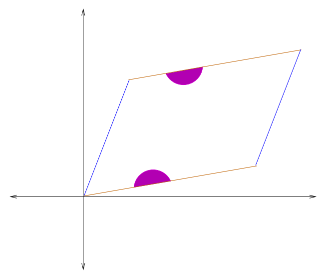
Essentially we are gluing the sides of the parallelogram together to form a tube and then gluing the ends of the tube together to form a doughnut shape. The resulting quotient space is a torus:

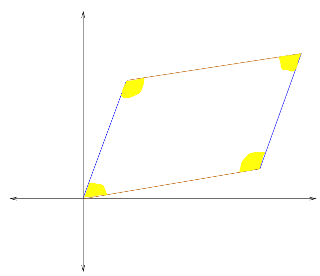


Write T for the torus, i.e., for the quotient space given by the partition given above. Recall a set U is open in T if and only if $\pi^{-1}(U)$ is open in X. The basic open sets in X are given by the subspace topology from $\mathbb{C}$, i.e., they are intersections of open balls with X:

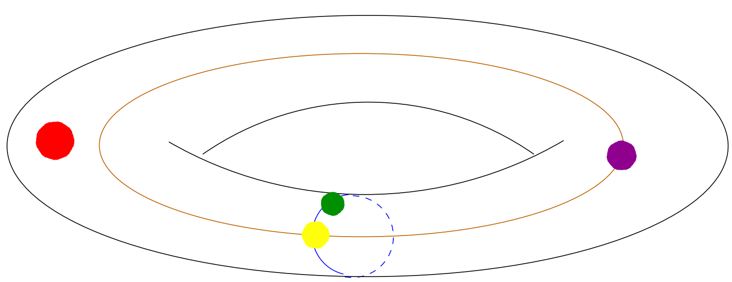








On the torus we can see them easily:

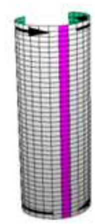


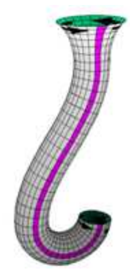
Example 2.9.7. We can construct another example from the same space X used in the previous example by specifying a different partition. The resulting space is referred to as the Klein bottle. Define the partition in this case by the

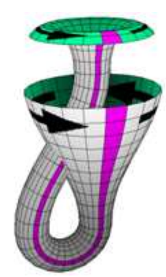
sets:

$$\begin{aligned}
&\{a\omega_1 + b\omega_2 : 0 < a < 1, 0 < b < 1\}\\
&\{a\omega_1, a\omega_1 + \omega_2 : 0 < a < 1\}\\
&\{b\omega_2, \omega_1 + (1-b)\omega_2 : 0 < b < 1\}\\
&\{0, \omega_1, \omega_2, \omega_1 + \omega_2\}.
\end{aligned}$$

Essentially we are identifying sides of the parallelogram to form a tube as in the previous example, but now we twist when we identify the ends of the tube. The following series of pictures shows how the Klein bottle is constructed:

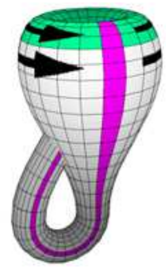






We will return to the Klein bottle in § 3.5 when discussing embedding of manifolds in Euclidean space.

Example 2.9.8. We now can give a precise definition of projective space. Consider the space $\mathbb{R}^{n+1}$. (One could consider $\mathbb{C}^{n+1}$, $\mathbb{F}_p^{n+1}$, etc. instead if one



wanted.) Define an equivalence relation on $\mathbb{R}^{n+1} - \{0\}$ by setting $(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1})$ if there exists $\lambda \in \mathbb{R} - \{0\}$ so that $(x_1, \ldots, x_n) = (\lambda y_1, \ldots, \lambda y_n)$. We write $[x_1 : \cdots : x_{n+1}]$ for the equivalence class containing $(x_1, \ldots, x_{n+1})$. Each equivalence class can be pictured as a line through the origin as all points on the line are equivalent under this equivalence relation. The quotient space obtained from this equivalence relation is denoted $\mathbb{RP}^n$. We will deal specifically with the case $\mathbb{RP}^2$. In this case we have that the "points" of $\mathbb{RP}^2$ are equivalence classes $[x_1 : x_2 : x_3]$ where $(x_1, x_2, x_3) \neq (0, 0, 0)$.

Observe that if $x_3 \neq 0$, then $[x_1 : x_2 : x_3] = \left[\frac{x_1}{x_3} : \frac{x_2}{x_3} : 1\right]$. In this way we have a homeomorphism between the space of equivalence classes with $x_3 \neq 0$ and $\mathbb{R}^2$. Call the space of such equivalence classes U_1. Similarly, one can define $U_2 = \{[x : 1 : z]\}$ and $U_3 = \{[1 : y : z]\}$. Since any $[x : y : z] \in \mathbb{RP}^2$ satisfies x, y, or z is non-zero, we have that $\mathbb{RP}^2$ is covered by U_1, U_2 and U_3. Note that there is a large intersection between the U_i. For example, $U_1 \cap U_2 = \{[x : 1 : 1]\}$, which is homeomorphic to $\mathbb{R}$. Anther way to view this is that $\mathbb{RP}^2 = U_1 \cup \{[x : y : 0]\}$. Note that we can write $\{[x : y : 0]\}$ as $\{[0 : 1 : 0]\} \cup \{[x : 0 : 0]\}$, which is homeomorphic to $\mathbb{RP}^1$. Thus, we have that $\mathbb{RP}^2$ is U_1 along with projective line, often referred to as the "line at infinity. In the notation of § 2.8 we have that $\mathbb{RP}^2 = U_1 \cup L_\infty \cup \{\infty\}$ where $L_\infty = \{[x : 0 : 0]\}$ and $\{\infty\} = \{[0 : 1 : 0]\}$. One can picture this line at infinity as the line one intersects off infinitely far when going out in $\mathbb{R}^2$ away from the origin. The distinguished point at infinity can be viewed as

$$[0 : 1 : 0] = \lim_{y \to \infty} \left[\frac{x}{y} : 1 : 0\right]$$

i.e., you hit this point by traveling vertically in the y-direction.

As was mentioned previously, the notion of forming a quotient space does not directly generalize anything from classical analysis. As such, many of the nice properties we have been studying do not behave nicely when forming a quotient space. We will give several examples of such properties.

Let A be a subspace of a topological space X and let $\pi : X \to Y$ be a quotient map. It is not necessarily true that $\pi|_A : A \to \pi(A)$ is a quotient map.

Example 2.9.9. Let $X = [0, 1] \cup [2, 3]$ and $Y = [0, 2]$. Define $\pi : X \to Y$ by

$$\pi(x) = \begin{cases} x & x \in [0, 1] \\ x - 1 & x \in [2, 3]. \end{cases}$$

One can check that this map is a quotient map.

Set $A = [0,1] \cup (2,3]$. Then $\pi|_A : A \to [0,2]$ is continuous and surjective. However, $\pi|_A^{-1}((1,2]) = (2,3]$. The set $(2,3]$ is closed in A but $(1,2]$ is not closed in A. Thus, π_A is not a quotient map.

Proposition 2.9.10. *Let $\pi : X \to Y$ be a quotient map and A a subspace of X. If π is an open map and A is open in X, then $\pi|_A$ is a quotient map. Similarly, if π is a closed map and A is closed in X then $\pi|_A$ is a quotient map.*

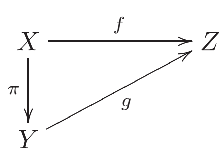
Proof. We already know that in general we have $\pi|_A : A \to \pi(A)$ is continuous and surjective.

Suppose that π is an open map and $A \subset X$ is open. Note that it is enough to show that $\pi|_A(U)$ is open for all U open in A. However, if U is open in A and A is open in X, then $U = V \cap A$ for some $V \in \mathcal{T}_X$ and so $U \in \mathcal{T}_X$. The fact that $\pi|_A$ is an open map gives that $\pi|_A(U) = \pi(U)$ is open in Y. Since $\pi|_A(U) \subset \pi(A)$, we see that $\pi|_A(U) = \pi(U) \cap \pi(A)$ and so is open in $\pi(A)$. Thus, $\pi|_A$ is a quotient map.

A similar argument will give the case that A is closed and π is a closed map. □

Consider the case of a quotient map $\pi : X \to Y$ and a continuous map $f : X \to Z$ for some topological space Z. The natural question is whether f descends to a map from Y to Z. This type of question arises often in abstract algebra. For example, given a group G and a normal subgroup N, one often is interested in determining when a map $\phi : G \to H$ factors through G/N.

Theorem 2.9.11. *Let $\pi : X \to Y$ be a quotient map, Z a topological space, and $f : X \to Z$ a continuous map. Assume that f is constant on $\pi^{-1}(\{y\})$ for each $y \in Y$. Then f descends to a continuous map $g : Y \to Z$ so that the following diagram commutes:*



Proof. Let $y \in Y$. Since f is constant on $\pi^{-1}(\{y\})$, we have that $f(\pi^{-1}(\{y\}))$ is a one point set. Let $g(y)$ be this point. This defines a map $g : Y \to Z$ so that for each $x \in X$, $g(\pi(x)) = f(x)$. It remains to show that g is continuous. Let $V \subset Z$ be open. Since f is continuous we know that $f^{-1}(V)$ is open in X. Note that $f^{-1}(V) = \pi^{-1}(g^{-1}(V))$. The fact that π is a quotient map gives $f^{-1}(V)$ open implies $g^{-1}(V)$ is open in Y. Thus, g is continuous as desired. □

This theorem is very useful in practice. For example, to define a map from $\mathbb{RP}^n$ to a space Z, it is enough to define a continuous map $f : \mathbb{R}^{n+1} \to Z$ so that f is constant on each $\pi^{-1}([x_1 : \cdots : x_{n+1}])$. In other words, we just need f to satisfy

$$f(x_1, \ldots, x_{n+1}) = f(\lambda x_1, \ldots, \lambda x_{n+1})$$

for all $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ and $\lambda \in \mathbb{R}$.

Exercise 2.9.12. Show that the composition of quotient maps is again a quotient map.

Theorem 2.9.13. *Let $f : X \to Z$ be a surjective continuous map. Let*

$$Y = \{f^{-1}(\{x\}) : z \in Z\}.$$

We put the quotient topology on Y.

1. *If Z is Hausdorff, so is Y.*
2. *The map f induces a bijective continuous map $g : Y \to Z$ which is a homeomorphism if and only if f is a quotient map.*

Proof. We apply Theorem 2.9.11 to see that f induces a continuous function $g : Y \to Z$. It is clear that g is a bijective map as well. Suppose that Z is Hausdorff and let $y_1, y_2 \in Y$ be distinct points. The images of these points, $z_1 = g(y_1)$ and $z_2 = g(y_2)$ are distinct points in Z and so there exist disjoint open neighborhoods V_1 and V_2 of these points in Z. However, this gives that $g^{-1}(V_1)$ and $g^{-1}(V_2)$ are disjoint open neighborhoods of y_1 and y_2 in Y. Thus, Y is Hausdorff as well.

Suppose now that g is a homeomorphism. Then we have that g and π are both quotient maps. Thus, $f = g \circ \pi$ is a quotient map as well. Conversely, assume that f is a quotient map. Let $V \subset Y$ be an open set. Then $f^{-1}(g(V)) = \pi^{-1}(V)$, which is open in X because π is continuous. Thus, using that f is a quotient map we have that $g(V)$ is open in Z. Thus, g maps open sets to open sets, hence it is a homeomorphism by Lemma 2.4.11. ☐

In general one does not have that a quotient space of a Hausdorff space is Hausdorff, so this result can be very useful!

Example 2.9.14. Let $A = \{a, b, c\}$ and consider the map $\pi : \mathbb{R} \to A$ defined by

$$\pi(x) = \begin{cases} a & x > 0 \\ b & x < 0 \\ c & x = 0. \end{cases}$$

This is a quotient map inducing the quotient topology $\mathcal{T}_A = \{\emptyset, A, \{a, b\}, \{a\}, \{b\}\}$. Even though $\mathbb{R}$ is Hausdorff, the quotient space A is not Hausdorff with the quotient topology.

Let $\pi_1 : X_1 \to Y_1$ and $\pi_2 : X_2 \to Y_2$ be quotient maps. A natural question is whether we can conclude that the product map

$$\begin{aligned} \pi_1 \times \pi_2 &: X_1 \times X_2 \to Y_1 \times Y_2 \\ (x_1, x_2) &\mapsto (\pi_1(x_1), \pi_2(x_2)) \end{aligned}$$

is a quotient map. In general, this will not be a quotient map. One can see [8] for a counterexample. Fortunately, if we add some conditions on the spaces we can conclude that the product map is a quotient map.

Definition 2.9.15. Given a map $f : X \to Y$ we say a subset $A \subset X$ is *saturated* if $f^{-1}(f(A)) = A$. Given an arbitrary subset $B \subset X$, we define the *saturation of B* to be

$$\mathrm{Sat}(B) = f^{-1}(f(B)).$$

We leave the proof of the following lemma as an exercise.

Lemma 2.9.16. *A surjective continuous map $\pi : X \to Y$ is a quotient map if and only if it takes saturated open sets to open sets.*

Theorem 2.9.17. *Let $\pi : X \to Y$ be a quotient map and assume Z is a locally compact Hausdorff space. Let $\mathrm{id} : Z \to Z$ be the identity map. Then the product map*

$$h := \pi \times \mathrm{id} : X \times Z \to Y \times Z$$
$$(x, z) \mapsto (\pi(x), z)$$

is a quotient map.

Proof. The fact that π is a quotient map gives that h is continuous and surjective (id is clearly continuous and surjective as well.) It only remains to show that h takes saturated open sets to open sets.

Let $U \subset X \times Z$ be a saturated open set and let $(x_0, z_0) \in U$. Suppose we can find a saturated open neighborhood $V \times W$ of (x_0, z_0) contained in U. Then $h(V \times W) = \pi(V) \times W$ contains $(\pi(x_0), z_0)$ and is contained in $h(U)$. Since π is a quotient map and V is necessarily saturated and open, we have $\pi(V)$, and hence $h(V \times W)$, is open. Thus, $h(U)$ is open. It remains to show that we can find such a $V \times W$.

Let $V^1 \times W^1$ be a basis element of $X \times Z$ containing (x_0, z_0) that is contained in U. As we have seen before, since Z is locally compact and Hausdorff there is an open set W_0 so that $\mathrm{Cl}(W_0)$ is compact and $\mathrm{Cl}(W_0) \subset W^1$. Thus,

$$(x_0, z_0) \in V^1 \times \mathrm{Cl}(W_0) \subset V^1 \times W^1 \subset U.$$

Since U is assumed to be saturated one can check that $\mathrm{Sat}(V^1) \times \mathrm{Cl}(W_0) \subset U$. The definition of h gives that $\mathrm{Sat}(V^1) \times W_0$ is a saturated subset of $X \times Z$ contained in U that contains (x_0, z_0). (Note that h is the identity on W_0 so saturation is automatic there.) It remains to show it is open.

We now show that there exists an open $V^2 \subset X$ with $\mathrm{Sat}(V^1) \subset V^2$ so that $V^2 \times \mathrm{Cl}(W_0) \subset U$. Fix $x \in \mathrm{Sat}(V^1)$. For any $z \in \mathrm{Cl}(W_0)$, there is a basis element $V_x \times W_z \subset U$ in $X \times Z$. We can cover the compact set $\{x\} \times \mathrm{Cl}(W_0)$ with these open sets and so obtain a finite subcover

$$V_{x_1} \times W_{z_1}, \ldots, V_{x_n} \times W_{z_n}.$$

Set $V_x = \cap_{i=1}^n V_{x_i}$. We have that V_x is an open neighborhood of x with $V_x \times \mathrm{Cl}(W_0) \subset U$. Set $V^2 = \cup_x V_x$ where the union is over $x \in \mathrm{Sat}(V^1)$. Repeating this construction we form a sequence

$$V^1 \subset \mathrm{Sat}(V^1) \subset V^2 \subset \mathrm{Sat}(V^2) \subset V^3 \subset \cdots$$

with $V^i \times \mathrm{Cl}(W_0) \subset U$. Let $V = \cup V^i$. We have that V is open as it is the union of open sets and $V \times W_0 \subset U$. We also have that $V \times W_0$ is saturated. If $(x, z) \in V \times W_0$, then x is in some V^i and if x' is in the same factor as x, then $x' \in V^{i+1}$ and so $(x', z) \in V \times W_0$ as well. Thus, $V \times W_0$ is the required saturated open neighborhood of (x_0, z_0). □

Corollary 2.9.18. *Let $\pi_1 : X_1 \to Y_1$ and $\pi_2 : X_2 \to Y_2$ be quotient maps. If X_2 and Y_1 are locally compact Hausdorff spaces, then $\pi_1 \times \pi_2$ is a quotient map.*

Proof. Consider the map

$$\pi_1 \times \mathrm{id}_{X_2} : X_1 \times X_2 \to Y_1 \times X_2.$$

Theorem 2.9.17 gives that this is a quotient map since X_2 is locally compact and Hausdorff. Similarly, we have

$$\mathrm{id}_{Y_1} \times \pi_2 : Y_1 \times X_2 \to Y_1 \times Y_2$$

is a quotient map because Y_1 is locally compact and Hausdorff. We now can use that the composition of quotient maps is again a quotient map to finish the proof. □

2.10 Topological Groups: Definitions and Basic Properties

In this section we introduce the notion of a topological group. It is the first instance of topology and algebra mixing together. In the subsequent two sections we will see how topologies can be put on familiar algebraic structures to enhance our understanding of them.

Definition 2.10.1. A *topological group* is a group G that has a topology so that the following conditions hold:

1. If we endow $G \times G$ with the product topology, then the group operation

$$G \times G \longrightarrow G$$
$$(g, h) \mapsto gh$$

   is continuous.

2. The inversion map

$$G \longrightarrow G$$
$$g \mapsto g^{-1}$$

is continuous.

For general groups we will denote the identity element as e and the group operation as multiplication. It is also convention that unless otherwise noted, finite groups are all given the discrete topology.

Example 2.10.2. Let G be any group. We can form a topological group by putting the discrete topology on G. This may seem to be a trivial example, but we will see that we can use finite groups with the discrete topology to build very interesting topological groups.

Example 2.10.3. The spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ with the usual topology are topological groups with the operation being addition and the identity being $(0, \ldots, 0)$.

Example 2.10.4. The spaces $\mathbb{R}^\times$ and $\mathbb{C}^\times$ with the subspace topology are topological groups under the operation of multiplication. Note that even though they are subspaces of $\mathbb{R}$ and $\mathbb{C}$ respectively, they are not subgroups!

Example 2.10.5. Let V be a finite dimensional vector space over k where we take k to be $\mathbb{R}$ or $\mathbb{C}$. We can think of V as just a group by focusing on the addition and ignoring scalar multiplication. (One can actually define a topological vector space, but we omit that.) As a vector space, we have $V \cong k^n$ if $n = \dim_k V$. We can use this to define a topology on V by declaring that $U \subset V$ is open if and only if $T(U)$ is open in k^n where T is the linear map giving the isomorphism $V \cong k^n$. A priori this depends on the choice of T, but it turns out that one gets homeomorphic spaces for different choices of T.

Example 2.10.6. Let k be $\mathbb{R}$ or $\mathbb{C}$ again and consider

$$\mathrm{GL}_n(k) = \{g \in \mathrm{M}_n(k) : \det(g) \neq 0\}.$$

This is a finite dimensional vector space with operation given by matrix multiplication. As such, it is a topological group.

Let

$$\mathrm{SL}_n(k) = \{g \in \mathrm{GL}_n(k) : \det(g) = 1\}.$$

This is a subgroup of $\mathrm{GL}_n(k)$ that one gives the subspace topology. In fact, this is a closed subgroup of $\mathrm{GL}_n(k)$ as $g \in \mathrm{GL}_n(k)$ is in $\mathrm{SL}_n(k)$ if and only if g satisfies the polynomial equation

$$\det(g) - 1 = 0.$$

Let G be a topological group. Given any $g \in G$, we can define left and right translation maps:

$$L_g : G \longrightarrow G$$
$$h \mapsto gh$$

$$R_g : G \longrightarrow G$$
$$h \mapsto hg.$$

It is clear from the definitions that L_g and R_g are homeomorphisms for any $g \in G$. Thus, we see that $U \subset G$ is open if and only if gU is open if and only if Ug is open. Similarly, we use that inversion is a homeomorphism to see U is open if and only if $U^{-1} = \{g^{-1} : g \in U\}$ is open.

Proposition 2.10.7. *Let G be a topological group. It is enough to give a basis of open neighborhoods around the identity in order to give a basis for the entire space.*

Proof. This follows from the fact that L_g is a homeomorphism for each g. Thus, U is an open neighborhood of g if and only if $g^{-1}U$ is an open neighborhood of e. □

Definition 2.10.8. Let S be a subset of a group G. We say S is *symmetric* if $S = S^{-1}$.

Proposition 2.10.9. *Let G be a topological group.*

1. *Every neighborhood U of the identity contains a neighborhood V of the identity so that $V \cdot V \subset U$.*

2. *Every neighborhood U of the identity contains a symmetric neighborhood V of the identity.*

3. *If H is a subgroup of G, $\mathrm{Cl}(H)$ is also a subgroup.*

4. *Every open subgroup of G is also closed.*

5. *If K_1 and K_2 are compact subsets of G, so is $K_1 \cdot K_2$.*

Proof.

1. Note that since every neighborhood of the identity contains an open neighborhood of the identity, we can assume without loss of generality that U is open. Let $m : U \times U \to G$ denote the continuous map arising from the group operation. Since U is open, $m^{-1}(U)$ is open in $U \times U$ and contains the point (e, e). The fact that $U \times U$ has the product topology gives that there are open sets $V_1, V_2 \subset U$ so that $(e, e) \in V_1 \times V_2$. Set $V = V_1 \cap V_2$. Then $e \in V$, $V \subset U$, and by construction we have $V \cdot V \subset U$.

2. Note that $g \in U \cap U^{-1}$ if and only if $g, g^{-1} \in U$. Thus, set $V = U \cap U^{-1}$. This is clearly a symmetric neighborhood of the identity contained in U.

3. Recall that the map $f : G \times G \to G$ defined by $f(x, y) = xy^{-1}$ is a continuous map as it is the composition of two continuous maps. Also recall that under a continuous map $f : G \times G \to G$ we have $f(\mathrm{Cl}(H) \times \mathrm{Cl}(H)) \subset \mathrm{Cl}(f(H \times H))$ by equation (2.1). Let $h_1, h_2 \in \mathrm{Cl}(H)$. Then we have $f(h_1, h_2) = h_1 h_2^{-1} \in f(\mathrm{Cl}(H) \times \mathrm{Cl}(H)) \subset \mathrm{Cl}(f(H \times H)) = \mathrm{Cl}(H)$ since H is a subgroup. Thus, we have $\mathrm{Cl}(H)$ is a subgroup.

4. Let H be a subgroup of G. Then $G = \coprod Hg_i$ for some set of coset representatives $\{g_i\}$. If H is open, we have Hg_i is open for every g_i and so $\coprod Hg_i$ is open. In particular, if we consider $\{g_i\} - \{e\}$, then we get

$$H = G - \coprod_{g_i \neq e} Hg_i,$$

which shows H is a closed set.

5. The set $K_1 \cdot K_2$ is the continuous image of the compact set $K_1 \times K_2$ in $G \times G$ under the multiplication map. Thus, it is compact.

□

Note that by combining the first two parts of the previous proposition we see that every neighborhood of the identity contains a symmetric neighborhood of the identity so that $V \cdot V \subset U$. Also note that if G is connected, the only open subgroups of G are the empty set and G. This can be used to show G is not connected in many instances.

Proposition 2.10.10. *Let G be a topological group. The following are equivalent:*

1. *G is Hausdorff.*
2. *The set $\{e\}$ is closed.*
3. *Every point of G is closed.*

Proof. Suppose that G is Hausdorff. Then we know every one point set in a Hausdorff space is closed, so we get the first statement implies the second and third.

Suppose now that $\{e\}$ is closed. Let $g \in G$ and consider the map $L_{g^{-1}}$. This is a continuous map and $L_{g^{-1}}^{-1}(\{e\}) = \{g\}$. Since $\{e\}$ is closed, we must have $\{g\}$ is closed as well.

Finally, assume that every point of G is closed. Let $g, h \in G$ be distinct points. The set $\{gh^{-1}\}$ is a closed set and so there is an open neighborhood U of e that does not contain gh^{-1}. Choose V to be an open symmetric neighborhood of e contained in U as given by Proposition 2.10.9. Then Vg and Vh are open neighborhoods of g and h respectively. Note that they must be disjoint as well for if not, we'd have $xg = yh$ for some $x, y \in V$, i.e., $gh^{-1} = x^{-1}y \in V$, a contradiction. Thus, G is Hausdorff. □

As was mentioned in § 2.9, we can also study quotient groups in the setting of topological groups. Let $H \subset G$ be a subgroup and consider the set G/H of left cosets of G. Recall this partitions G into disjoint sets so we can put the quotient topology on this space via the usual projection map

$$\pi : G \to G/H,$$

i.e., $U \subset G/H$ is open if and only if $\pi^{-1}(U)$ is open in G. If H is normal in G, then G/H is also a topological group as we will see in the next proposition.

Proposition 2.10.11. *Let G be a topological group and H a subgroup.*

1. *The quotient space G/H is homogeneous under G, i.e., left translation is a homeomorphism.*
2. *The projection $\pi : G \to G/H$ is an open map.*
3. *The quotient space G/H is discrete if and only if H is open. Moreover, if G is compact, then H is open if and only if G/H is finite.*
4. *If H is normal in G, then G/H is a topological group with respect to multiplication of cosets and the quotient topology.*
5. *If H is normal in G, the topological group G/H is Hausdorff if and only if H is closed.*
6. *Set $H = Cl(\{e\})$. Then H is normal in G and the quotient group G/H is Hausdorff with respect to the quotient topology.*

Proof.

1. Let $x \in G$. We need to show the map

$$\begin{aligned} L_x : G/H &\to G/H \\ gH &\mapsto xgH \end{aligned}$$

is a homeomorphism. Note that the inverse map is given by $L_{x^{-1}}$,so it is enough to show that L_x is an open map. Let $\overline{U}$ be an open set in G/H and set $U = \pi^{-1}(U)$. We have that U is an open set by the definition of the quotient topology. Given any $x \in G$, it is easy to see that $\pi^{-1}(x\overline{U}) = xU$. Since U is open, xU is open as well. Now we use the definition of the quotient topology again to see that $x\overline{U}$ is open. Thus, L_x is an open map.

2. Let $U \subset G$ be an open set. We need to show that $\pi(U) \subset G/H$ is an open set. We know that $\pi(U)$ is open if and only if $\pi^{-1}(\pi(U))$ is open. Group theory gives that $\pi^{-1}(\pi(U)) = U \cdot H$. Let $g \in U \cdot H$. Then there exists $u \in U$ and $h \in H$ so that $g = uh$. Since U is open, there is an open neighborhood V_u of u contained in U. Thus, $V_u h$ is an open neighborhood of g contained in $U \cdot H$. Thus, $U \cdot H$ is open as desired.

3. Let H be an open subgroup of G. By what we have just shown, $\pi(H)$ is open in G/H, i.e., H is an open point in G/H. However, we know that L_x is a homeomorphism on G/H for every $x \in G$, so xH is open for every $x \in G$. Thus, all the points in G/H are open and so G/H has the discrete topology.

   Conversely, if G/H has the discrete topology the point H in G/H is open and so $\pi^{-1}(H) = H$ is open in G.

   Suppose now that G is compact. We have that H is open if and only if G/H has the discrete topology. However, since G/H is the continuous image of the compact set G, it is compact as well. But this means it must be finite since it is compact and discrete.

4. Let H be a normal subgroup so that G/H is a group with multiplication of cosets being the operation. We must show multiplication and inversion in G/H are continuous in the quotient topology. For multiplication, observe we have a commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ {\scriptstyle \pi\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G/H \times G/H & \xrightarrow{\overline{m}} & G/H. \end{array}$$

Let $\overline{U} \subset G/H$ be open. Then we must show that $\overline{m}^{-1}(\overline{U})$ is open in $G/H \times G/H$. We have by the commutativity of the diagram that

$$\overline{m}^{-1}(\overline{U}) = (\pi \times \pi)(m^{-1}(\pi^{-1}(\overline{U}))).$$

Since π is continuous and m is continuous, we have that $m^{-1}(\pi^{-1}(\overline{U}))$ is open in $G \times G$. We have shown above that $\pi : G \to G/H$ is an open map, so $\pi \times \pi$ is also an open map. Thus, $\overline{m}^{-1}(\overline{U})$ is open in $G/H \times G/H$. A similar argument shows inversion is a continuous map on G/H and so G/H is a topological group with respect to the quotient topology.

5. Let H be a normal subgroup of G. Proposition 2.10.10 applied to the group G/H gives that G/H is Hausdorff if and only if H is closed as a point in G/H. If H is closed as a point in G/H, we have $\pi^{-1}(H) = H$ is closed in G. Thus, if G/H is Hausdorff then H is closed in G. Suppose now that H is closed in G. Then $G - H$ is open. Note that $\pi(G - H)$ is open in G/H since π is an open map. Furthermore, $\pi(G - H) = (\bigcup gH) - H$, i.e., all the cosets except H. Thus, $H = (G/H) - \pi(G - H)$ and so is closed as a point in G/H. Now we use Proposition 2.10.10 again to conclude that G/H is Hausdorff.

6. We know that $\{e\}$ is a subgroup of G and so $H = \mathrm{Cl}(\{e\})$ is also a subgroup by Proposition 2.10.9 Observe that for any $g \in G$, the map

$$\begin{aligned} f : G &\to G \\ h &\mapsto ghg^{-1} \end{aligned}$$

is a continuous map on G. Thus, we know that $f(\mathrm{Cl}(\{e\})) \subset \mathrm{Cl}(f(\{e\}))$, i.e., $gHg^{-1} \subset H$ for all $g \in G$. On the other hand, since H is the smallest closed subgroup containing e, we have $H \subset gHg^{-1}$ for otherwise $H \cap gHg^{-1}$ would be a smaller closed subgroup containing e. Thus, $H = gHg^{-1}$ for every $g \in G$ and so H is normal. The rest now follows from the previous part.

□

Example 2.10.12. Consider the topological group $\mathbb{C}$. Let $\omega_1, \omega_2 \in \mathbb{C}$ with ω_1 and ω_2 linearly independent over $\mathbb{R}$. Consider the subgroup of $\mathbb{C}$ given by

$$\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.$$

This is clearly a group under addition. We refer to such a group in $\mathbb{C}$ as a *lattice*. The topology is the subspace topology, which one can easily check is actually the discrete topology in this case. Since $\mathbb{C}$ is an abelian group, all subgroups are normal so we can form the topological group $\mathbb{C}/\Lambda$. This is precisely the torus we studied in § 2.9. Note that since Λ is easily seen to be closed in $\mathbb{C}$, we recover the (also easy) fact that the torus is Hausdorff.

Definition 2.10.13. A topological group G that is locally compact and Hausdorff is called a *locally compact group*.

Note that we require Hausdorff in this definition. The reason is that under such an assumption one has the existence of a Haar measure on G. Though we will not prove such a measure exists, we will briefly review some definitions from measure theory and then precisely state the theorem. First, we prove the following result.

Proposition 2.10.14. *Let G be a Hausdorff topological group. A subgroup H of G that is locally compact in the subspace topology is closed. In particular, every discrete subgroup of G is closed.*

Proof. Let K be a compact neighborhood of e sitting in H. Since H is Hausdorff, we have that K is closed in H. Thus, there is a closed set C in G so that $K = H \cap C$. Now $H \cap C$ is compact and sits inside the Hausdorff space G, so it is closed in G as well. Proposition 2.10.9 gives a neighborhood V of e in G so that $V \cdot V \subset C$. Our goal is to show that if $g \in \mathrm{Cl}(H)$, then $g \in H$.

The fact that H is a subgroup gives that $\mathrm{Cl}(H)$ is a subgroup as well. Let $g \in \mathrm{Cl}(H)$. This means that $g^{-1} \in \mathrm{Cl}(H)$ as well. If $g \in H$, we are done so assume $g \notin H$. We know that $g^{-1} \notin H$ as well then. Since $g^{-1} \in \mathrm{Cl}(H)$, every neighborhood of g^{-1} must intersect H. In particular, there exists a $y \in Vg^{-1} \cap H$. We claim that $yg \in C \cap H$. Since $C \cap H$ is closed, showing that $yg \in \mathrm{Cl}(C \cap H)$ is the same as showing $yg \in C \cap H$. Thus, to show the claim it is enough to show every neighborhood of yg meets $C \cap H$. Let W be a neighborhood of yg. Then we have that $y^{-1}W$ is a neighborhood of g. Observe that $y^{-1}W \cap gV$ is also a neighborhood of g. Since $g \in \mathrm{Cl}(H)$, there is an element $z \in y^{-1}W \cap gV \cap H$. We then have $yz \in W \cap H$, $y \in Vg^{-1}$, and $z \in gV$. Thus, $yz \in Vg^{-1} \cdot gV = V \cdot V \subset C$. Thus, the intersection $W \cap C \cap H \neq \emptyset$. Thus, $yg \in \mathrm{Cl}(C \cap H)$ and so the claim is true.

We can now use the claim to see that $yg \in H$ and $y \in H$, so $g \in H$ as desired. □

We now begin our review of the relevant definitions from measure theory.

Definition 2.10.15. A collection $\mathfrak{M}$ of subsets of a set X is called a σ-algebra if it satisfies:

1. $X \in \mathfrak{M}$;
2. If $A \in \mathfrak{M}$, then $X - A \in \mathfrak{M}$;
3. If $A_n \in \mathfrak{M}$ for all $n \geq 1$, then $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{M}$, i.e., $\mathfrak{M}$ is closed under countable unions.

One should note that the definition implies that $\emptyset \in \mathfrak{M}$ and $\mathfrak{M}$ is closed under finite and countably infinite intersections.

Definition 2.10.16. A set X along with a σ-algebra $\mathfrak{M}$ is called a *measurable space*. If X is a topological space with topology $\mathcal{T}$, the smallest σ-algebra containing $\mathcal{T}$ is denoted by $\mathfrak{B}$. The elements of $\mathfrak{B}$ are called the *Borel subsets* of X.

Definition 2.10.17. Given a measurable space $(X, \mathfrak{M})$, a function

$$\mu : \mathfrak{M} \to \mathbb{R}_{>0} \cup \{\infty\}$$

is called a *positive measure* if given any family $\{A_n\}$ of disjoint sets in $\mathfrak{M}$, one has that μ is countably additive, i.e.,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Moreover, if X is a locally compact Hausdorff space and μ is a positive measure defined on the Borel sets, μ is called a *Borel measure*.

Definition 2.10.18. Let μ be a Borel measure on X and let E be a Borel subset of X. We say μ is *outer regular* on E if

$$\mu(E) = \inf\{\mu(U) : U \supset E, U \in \mathcal{T}\}.$$

We say μ is *inner regular* on E if

$$\mu(E) = \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}.$$

A *Radon measure* on X is a Borel measure that is finite on compact sets, outer regular on all Borel sets, and inner regular on all open sets.

Consider now a group G. A Borel measure μ on G is said to be *left translation invariant* if for all Borel subsets E of G one has

$$\mu(gE) = \mu(E)$$

for all $g \in G$. We say μ is *right translation invariant* if

$$\mu(Eg) = \mu(E)$$

for all $g \in G$ and all Borel subsets E of G.

Definition 2.10.19. Let G be a locally compact topological group. A left (resp. right) *Haar measure* on G is a nonzero Radon measure μ on G that is left (resp. right) translation invariant. A *bi-invariant Haar measure* is a nonzero Radon measure that is left and right invariant.

Theorem 2.10.20. *Let G be a locally compact group. Then G admits a left invariant Haar measure. Moreover, this measure is unique up to a scalar multiple.*

The proof of this theorem is not terribly difficult, but it would take us too far afield. The interested reader can consult [9] for a proof of this result. One has the same result for a right invariant Haar measure. As the measure is only unique up to scalar multiple, in practice one must choose a Haar measure. One generally does this by picking a set that one specifies to have measure 1 which fixes the scalar. Having a Haar measure allows one to define integration on general locally compact groups.

Example 2.10.21. The topological group $\mathbb{R}$ is locally compact and hence has a Haar measure. The Haar measure in this case is the familiar Lebesgue measure.

Example 2.10.22. The topological group $\mathbb{Q}_p$ is a locally compact groups and so has a Haar measure. One generally normalizes it so that the measure of $\mathbb{Z}_p$ is 1.

Example 2.10.23. If G is locally compact, then so is $\mathrm{GL}_n(G)$ and so $\mathrm{GL}_n(G)$ has a Haar measure as well.

2.11 Profinite Groups

Profinite groups given an example of how one can build an interesting topology out of discrete topologies. Profinite groups arise in many situations. For example, infinite Galois groups are profinite. Profinite groups are also very prevalent in number theory. Before we define profinite groups, we give a quick review of inverse limits.

Let I be a nonempty set. We say I is *preordered* with respect to a relation $\leq$ if $i \leq i$ for every $i \in I$ and if $i \leq j$ and $j \leq k$, then $i \leq k$. Note that we do not require that if $i \leq j$ and $j \leq i$ then $i = j$.

Example 2.11.1. The set $\mathbb{R}$ is preordered with respect to the usual inequality $\leq$.

Example 2.11.2. The set $\mathbb{Z}$ is preordered with respect to the operation of division.

Let I be a preordered set and let $\{G_i\}$ be a family of sets indexed by I. We assume that for every $i, j \in I$ with $i \leq j$ we have a mapping $\varphi_{ij} : G_j \to G_i$ so that

1. $\varphi_{ii} = \mathrm{id}_{G_i}$ for every $i \in I$;

2. $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ for every $i, j, k \in I$ with $i \leq j \leq k$.

The collection (G_i, φ_{ij}) is called an *inverse system* or *projective system*.

Definition 2.11.3. Let (G_i, φ_{ij}) be a projective system of sets. The *projective limit* or *inverse limit* of the system is defined by

$$\varprojlim_i G_i = \left\{ (g_i) \in \prod_{i \in I} G_i : \text{ if } i \leq j, \text{ then } \varphi_{ij}(g_j) = g_i \right\}.$$

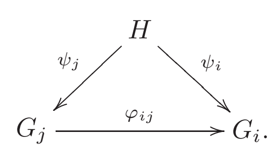
It is clear that $\varprojlim G_i$ is a subset of the product $\prod_{i \in I} G_i$. As such, one obtains for each $i \in I$ natural projection maps

$$\pi_j : \varprojlim_i G_i \to G_j.$$

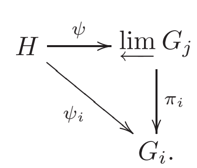
For our interests we will require the G_i to be topological groups and the maps φ_{ij} to be continuous maps. In this case we see that $\varprojlim G_i$ is a topological group with componentwise multiplication and the subspace topology coming from the product topology on $\prod_i G_i$.

We recall the following universal property of inverse limits. One can find a proof of this result in any elementary commutative algebra or graduate level abstract algebra book. One should adapt the proofs given there to our setting of topological groups.

Theorem 2.11.4. *(Universal Property of Inverse Limits) Let H be a topological group and let there be a system of continuous maps $\psi_j : H \to G_j$ for all $j \in I$ that is compatible with the given projective system (G_i, φ_{ij}) in the sense that for each $i, j \in I$ with $i \leq j$, the following diagram commutes:*



Then there exists a unique continuous map $\psi : H \to \varprojlim G_j$ such that for each $i \in I$ the following diagram commutes



The main case of interest is when each G_i is a finite group with the discrete topology. The projective limit $\varprojlim G_i$ acquires the subspace topology from the product topology as mentioned above. One should immediately note that even though each G_i has the discrete topology, $\prod_{i \in I} G_i$ does not have the discrete topology and neither does $\varprojlim G_i$. The topology on $\varprojlim G_i$ is called the *profinite topology.*

Definition 2.11.5. A topological group G is said to be a *profinite group* if it is isomorphic (as a topological group) to the projective limit of a projective system of finite groups with the profinite topology.

One should note here that as is always the case, when we use the term "isomorphic" we mean that the relevant structures agree. So in the the case of topological groups isomorphic means they are isomorphic as groups and homeomorphic as topological spaces.

Proposition 2.11.6. *Let G be a profinite group with $G \cong \varprojlim G_i$. Then we have*

1. *G is Hausdorff in the profinite topology;*
2. *G is a closed subset of $\prod_i G_i$;*
3. *G is compact.*

Proof.

1. We know that each G_i is Hausdorff because it has discrete topology. The product of Hausdorff spaces is Hausdorff, and the subspace of a Hausdorff space is Hausdorff.
2. Note that we have
$$\prod_i G_i - G = \bigcup_i \bigcup_{j \geq i} \left\{ (g_k) \in \prod_k G_k : \varphi_{ij}(g_j) \neq g_i \right\}.$$
This is an open set as it is the union of open sets, thus G is closed in $\prod_i G_i$.
3. Since each G_i is compact, $\prod_i G_i$ is compact. Thus, we have a closed subset of a compact space, hence it is compact.

□

This proposition can be very useful in showing spaces are compact. It can be easier to realize a space as a profinite than to show it is compact directly. We now give a few examples before studying the topology of profinite groups further.

Example 2.11.7. Let p be a prime number and consider $G_i = \mathbb{Z}/p^i\mathbb{Z}$. We have natural maps
$$\varphi_{ij} : \mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$$

for $j \geq i$ given by reduction modulo p^i. The projective limit is denoted by $\mathbb{Z}_p$, the ring of p-adic integers:

$$\mathbb{Z}_p = \varprojlim_i \mathbb{Z}/p^i\mathbb{Z}.$$

We can view $\mathbb{Z}_p$ in other ways as well. The most common way is as a ring of power series in p. Let $x \in \mathbb{Z}_p$. Then $x = (x_j)$ with each $x_j \in \mathbb{Z}/p^j\mathbb{Z}$ so that if $j \geq i$ then $x_j \equiv x_i \pmod{p^i}$. We can define a power series inductively as follows. Set $a_0 = x_1$. For $x_2 \in \mathbb{Z}/p^2\mathbb{Z}$, write $x_2 = a_0 + a_1 p$. For $x_3 \in \mathbb{Z}/p^3\mathbb{Z}$, since $\varphi_{23}(x_3) = x_2$, we can write $x_3 = a_0 + a_1 p + a_2 p^2$ for some a_2. We continue in this pattern to get the power series in p.

Another way that $\mathbb{Z}_p$ arises is as the valuation ring of $\mathbb{Q}_p$, i.e.,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

It is clear from the above description of $\mathbb{Z}_p$ as a power series that these two definitions coincide algebraically. One must still check that the metric topology arising from $|\cdot|_p$ is the same as the profinite topology. It is easy to see the only values $|\cdot|_p$ takes on $\mathbb{Z}_p$ are $1, \frac{1}{p}, \frac{1}{p^2}, \ldots$. Thus, given an $\epsilon > 0$, we choose n_0 so that $\frac{1}{p^{n_0+1}} < \epsilon \leq \frac{1}{p^{n_0}}$. In this case, if we look at $B(x, \epsilon)$ with $x = a_0 + a_1 p + a_2 p^2 + \cdots$, we see $y = \sum_{i=0}^{\infty} b_i p^i \in B(x, \epsilon)$ if and only if $a_i = b_i$ for $0 \leq i \leq n_0$. Thus, we have that $B(x, \epsilon)$ is contained in the open set

$$\{x_1\} \times \{x_2\} \times \cdots \times \{x_{n_0}\} \times \mathbb{Z}/p^{n_0+1}\mathbb{Z} \times \cdots$$

where $x_1 = a_0$, $x_2 = a_0 + a_1 p$, etc. Now we just need to show we can find an ϵ so that given an open set U around x in $\mathbb{Z}_p$ in the profinite topology, that $U \subset B(x, \epsilon)$. We will then have that the topologies match up as well. In general, a basic open set in $\mathbb{Z}_p$ in the profinite topology will be of the form

$$\{x_1\} \times \{x_2\} \times \cdots \times \{x_n\} \times \mathbb{Z}/p^n\mathbb{Z} \times \cdots$$

This follows from the fact that we must have $x_j \equiv x_i \pmod{p^i}$ for $x = (x_n) \in \mathbb{Z}_p$. It now follows that if we choose $\epsilon = \frac{1}{p^n}$, then $U \subset B(x, \epsilon)$ as claimed.

Example 2.11.8. We briefly note here that one can do the same procedure as the previous example with the groups $\mathbb{Z}/N\mathbb{Z}$. In this case the ordering is divisibility. Thus, if $N \mid M$ we write $N \leq M$ and have the natural projection

$$\varphi_{NM} : \mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}.$$

If we take the projective limit in this case we obtain the group $\widehat{\mathbb{Z}}$. We will see later that in fact one has

$$\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p.$$

Example 2.11.9. Consider the collection of finite Galois extensions of $\mathbb{Q}$ in some fixed algebraic closure $\overline{\mathbb{Q}}$. This collection forms a directed set with respect to inclusion. We have a direct system of finite groups where if L and K are finite Galois extensions of $\mathbb{Q}$ with $\mathbb{Q} \subset K \subset L$, then

$$\varphi_{KL} : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Gal}(K/\mathbb{Q})$$

via restriction. We can consider the inverse limit $\varprojlim_L \mathrm{Gal}(L/\mathbb{Q})$. It is not hard to see that

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\simeq} \varprojlim_{L} \mathrm{Gal}(L/\mathbb{Q})$$
$$\sigma \mapsto (\sigma|_L).$$

Recall that a topological space is said to be totally disconnected if every point is its own connected component. For a topological group G, the fact that G is homogeneous gives that G is totally disconnected if and only if the connected component of the identity is $\{e\}$. We write the connected component of the identity in G as G^0. Thus, G is totally disconnected if and only if $G^0 = \{e\}$.

Lemma 2.11.10. *The set G^0 is a normal subgroup of G. The quotient space G/G^0 is totally disconnected and so $\left(G/G^0\right)^0$ is the trivial subgroup of G/G^0.*

Proof. Let $g \in G^0$ be any element. The fact that G is homogeneous gives $g^{-1}G^0$ is connected. Now since $g \in G^0$, we have $e \in g^{-1}G^0$. Thus, $g^{-1}G^0$ is a connected set containing e and so $g^{-1}G^0 \subset G^0$. Thus, G^0 is closed under inverses. The same type of argument gives $gG^0 \subset G^0$ and so G^0 is a subgroup of G. Let $g \in G$. Then by homogeneity we have gG^0g^{-1} is connected and contains e, so $gG^0g^{-1} \subset G^0$. Thus, G^0 is normal in G.

The homogeneity of G gives that the connected components of G are precisely the elements of G/G^0. Thus, G/G^0 is totally disconnected. ☐

Lemma 2.11.11. *Let G be a profinite topological group. Then G is compact and totally disconnected.*

Proof. We have already seen that G is necessarily compact, so it only remains to show that G is totally disconnected, i.e., that $G^0 = \{e\}$.

Let $U \subset G$ be an open subgroup. Since $e \in U$ necessarily, we have $U \cap G^0$ is a nonempty open subgroup of G^0. Set

$$V = \coprod_{x \in G^0 - U} x(U \cap G^0).$$

We have that V is open in G^0. Suppose that there exists $y \in U \cap V$. Then there exists $x \in G^0 - U$ so that $y \in x(U \cap G^0)$, i.e., there exists $u \in U \cap G^0$ so that $y = xu$. But this gives that $x = yu^{-1} \in U$, a contradiction. Thus, $U \cap V = \emptyset$. Furthermore, one has $G^0 = (G^0 \cap U) \coprod V$. However, since G^0 is connected we must have $U \cap G^0 = \emptyset$ or $V = \emptyset$. We know that $e \in U \cap G^0$, so it must be that $V = \emptyset$. Thus, we have $G^0 \subset U$. Since U was arbitrary, we have

$$G^0 \subset \bigcap U$$

where the intersection is over open subgroups of G. Note that up to this point we have not used that G is profinite.

Write $G = \varprojlim G_i$ where each G_i is finite with discrete topology. Let $g \in G$ with $g \neq e$. Then there exists a j so that $g_j \neq e$. Let $\pi_i : G \to G_i$ be the natural projection map. Since G_i has discrete topology, $\{e\}$ is open in G_i. The fact that $\{e\}$ is open and π_i is continuous gives that $U = \pi_i^{-1}(\{e\})$ is an open set in G which by construction does not contain g. Since g was any element not equal to e, we see

$$\bigcap U = \{e\}$$

where the intersection is over the open subgroups of G. Thus, $G^0 = \{e\}$ as claimed. $\square$

The converse to this theorem is also true, namely, if G is compact and totally disconnected then G is profinite. The proof of this result is more involved so we break it into pieces. First we define the profinite group that we will eventually show G is isomorphic to.

Let $\mathcal{N}$ be the collection of open normal subgroups of G. This is a directed set where we say $M \leq N$ if $N \subset M$. If we assume that G is compact and totally disconnected, we see that Proposition 2.10.11 gives that G/N is finite with the discrete topology for each $N \in \mathcal{N}$. Letting $M, N \in \mathcal{N}$ with $M \leq N$, we have that the natural projection map $G \to G/M$ must have N in the kernel, so it descends to

$$\begin{aligned} \varphi_{MN} : G/N &\longrightarrow G/M \\ gN &\mapsto gM. \end{aligned}$$

Thus, if $N_1 \leq N_2 \leq N_3$ with $N_i \in \mathcal{N}$, we see that

$$\varphi_{N_1N_2} \circ \varphi_{N_2N_3} = \varphi_{N_1N_3}$$

so $\{G/N\}_{N\in\mathcal{N}}$ gives a projective system. One goal will be to show that $G \cong \varprojlim G/N$.

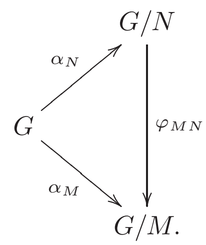
Lemma 2.11.12. *Set $G' = \varprojlim G/N$. There exists a surjective, continuous homomorphism $\alpha : G \to G'$.*

Proof. Let $N \in \mathcal{N}$ and let $\alpha_N : G \to G/N$ be the canonical projection map. Recall that G/N is homogeneous because G is. The map α_N is continuous because G/N is homogeneous and $\alpha_N^{-1}(e_{G/N}) = N$, an open set. If $M \leq N$ we have that the following diagram commutes:

Thus, applying the universal property of projective limits we have a continuous map

$$\alpha : G \to G' = \varprojlim_{N\in\mathcal{N}} G/N$$

so that $\pi_N \circ \alpha = \alpha_N$ for all $N \in \mathcal{N}$.



We still must show that α is surjective. First, we show that α has dense image in G', i.e., there is no open set in G' disjoint from $\alpha(G)$. The topology of G' is generated by open sets of the form $\pi_N^{-1}(U_N)$ where U_N is any subset of G/N since G/N has the discrete topology. Thus, given any open set in G', we can express it as a union of finite intersections of sets of the form $\pi_N^{-1}(U_N)$. Let U be such a set. Then U consists of elements of the form $(x_N)_{N\in\mathcal{N}} \in \prod_{N\in\mathcal{N}} G/N$ where $x_N \in G/N$ with finitely many of the x_N required to lie in a subset of G/N. Let $N_1, \ldots, N_r$ be the groups where the coordinates of (x_N) are constrained to lie in a subset. Set

$$M = \bigcap_{j=1}^{r} N_j.$$

Then for $(x_N) \in G'$, the coordinates of x_{N_j} are all determined by the projection of x_M under the map $\varphi_{N_jM} : G/M \to G/N_j$. Since α_M is surjective there exists a $g \in G$ so that $\alpha_M(g) = x_M$ and so $\alpha_{N_j}(g) = x_{N_j}$ for $j = 1, \ldots, r$. Now if $(x_N) \in U$, then $\alpha(g) \in U$ as well since $\alpha(g)$ agrees with (x_N) on the constrained coordinates. Thus, $U \cap \alpha(G) \neq \emptyset$ and so $\alpha(G)$ is dense in G'.

We can now show α is in fact surjective. Recall that G is compact and G' is Hausdorff and thus $\alpha(G)$ is compact and Hausdorff, hence closed. However, since it is also dense we must have $\alpha(G) = G'$. $\square$

Lemma 2.11.13. *Let X be a compact Hausdorff space. For any fixed $x_0 \in X$, set*

$$\mathcal{U}_{x_0} = \{K : K \text{ compact and open, } x_0 \in K\}.$$

Set

$$Y = \bigcap_{K\in\mathcal{U}_{x_0}} K.$$

The set Y is connected.

Proof. First, observe that $\mathcal{U}_{x_0}$ is nonempty because $X \in \mathcal{U}_{x_0}$. Suppose that there exist nonempty disjoint closed sets Y_1, Y_2 so that $Y = Y_1 \cup Y_2$. Recall that since X is compact and Hausdorff and Y_1 and Y_2 are closed, there exist disjoint open sets U_1 and U_2 so that $Y_i \subset U_i$ for $i = 1, 2$. Set $Z = X - (U_1 \cup U_2)$ so that Z is closed and hence compact. By construction we have Z and Y are disjoint, i.e., $Z \subset X - Y$. Thus, the sets $\{X - K\}$ form an open cover of Z. There is a finite subcover $X - K_1, \ldots, X - K_r$. Thus, there exist $K_1, \ldots, K_r \in \mathcal{U}_{x_0}$

so that $Z \cap \left(\bigcap_{j=1}^{r} K_j\right) = \emptyset$. Set $W = \bigcap_{j=1}^{r} K_j$. Then W is a compact open neighborhood of x_0 and so $W \in \mathcal{U}_{x_0}$. However, since W is disjoint from Z we have

$$W = (W \cap U_1) \coprod (W \cap U_2).$$

Now $W \cap U_1$ and $W \cap U_2$ are compact open subsets of X and x_0 must lie in $W \cap U_1$ or $W \cap U_2$, but not both. Suppose $x_0 \in W \cap U_1$. Then $W \cap U_1 \in \mathcal{U}_{x_0}$ and so $Y \subset W \cap U_1$. Since $Y_2 \subset Y$ and Y_2 is disjoint from U_1, we must have $Y_2 = \emptyset$. This is a contradiction. Thus, Y is connected as claimed. ☐

Finally, we come to the last lemma we need before we can prove the converse to Lemma 2.11.11.

Lemma 2.11.14. *Let G be a compact, totally disconnected topological group. Then every neighborhood of the identity contains an open normal subgroup.*

Proof. The proof this lemma consists of three steps. First, we show every open neighborhood of e contains a compact open neighborhood W of e. The second step consists of showing that W contains an open symmetric neighborhood V of e so that $W \cdot V \subset W$. Finally, using V we construct an open subgroup and then an open normal subgroup of G contained in U.

Let $\mathcal{U}$ be the set of compact open neighborhoods of e. Then, as in the previous lemma using e for x_0, we see that $Y = \bigcap_{K \in \mathcal{U}} K$ is a connected set containing e. However, since we are assuming G is totally disconnected we must have $Y = \{e\}$. Let U be any open neighborhood of e. Then $G - U$ is closed and hence compact. The fact that e is the only element in every $K \in \mathcal{U}$ shows that we can cover $G - U$ with $\{X - K\}_{K \in \mathcal{U}}$. Since $G - U$ is compact, there finitely many $K \in \mathcal{U}$ so that $G - U$ is covered by $X - K_1, \ldots, X - K_r$. Thus, $W = \bigcap_{j=1}^{r} K_j$ must be a subset of U. It is also a compact open neighborhood of e. Thus, $W \in \mathcal{U}$ as desired. This gives the first step.

We consider the continuous map $m : W \times W \to G$ given by restricting the group operation to W. Note that

1. For every $w \in W$, $(w, e) \in m^{-1}(W)$.
2. Since W is open, $m^{-1}(W)$ is open in $W \times W$.
3. The first two imply that for every $w \in W$, there exist open neighborhoods U_w of w and V_w of e so that $U_w \times V_w \subset m^{-1}(W)$. Moreover, we can assume V_w is symmetric by Proposition 2.10.9.
4. The collection $\{U_w\}$ is an open cover of W. Since W is compact, we can choose a finite subcover $U_1, \ldots, U_r$.

Let $V_1, \ldots, V_r$ be the sets corresponding to $U_1, \ldots, U_r$ so that $U_i \times V_i \subset m^{-1}(W)$. Define $V \subset W$ by

$$V = \bigcap_{j=1}^{r} V_j.$$

By construction we have $W \cdot V \subset W$. By induction we have $W \cdot V^n \subset W$ for all $n \geq 0$. Thus, $V^n \subset W$ for all $n \geq 0$. This gives the second step.

Set $\mathcal{O} = \bigcup_{n=1}^{\infty} V^n$. Note that $\mathcal{O}$ is open in G and contained in W. Moreover, since V is symmetric $\mathcal{O}$ is closed under inversion and so is an open subgroup contained in W. The space $G/\mathcal{O}$ is compact and discrete and hence finite. Thus, we can find a finite set of cost representatives $x_1, \ldots, x_m$ for $\mathcal{O}$ in G. Similarly, $x_j \mathcal{O} x_j^{-1}$ for $j = 1, \ldots, m$ are the finitely many conjugates of $\mathcal{O}$ in G. Thus, $N = \bigcap_{j=1}^{m} x_j \mathcal{O} x_j^{-1}$ is an open normal subgroup of G. Moreover, since necessarily $x_j \mathcal{O} x_j^{-1} = \mathcal{O}$ for one of the j's, we see that $N \subset \mathcal{O} \subset W$ as desired. □

Theorem 2.11.15. *Let G be a topological group. Then G is profinite if and only if G is compact and totally disconnected.*

Proof. We have already seen that if G is profinite then it is compact and totally disconnected. Suppose now that G is compact and totally disconnected. We have seen in Lemma 2.11.12 that there is a surjective homomorphism $\alpha : G \to G'$ where $G' = \varprojlim G/N$. It is enough to show that α is injective by Theorem 2.8.10.

It is easy to see that $\ker(\alpha) = \bigcap_{N \in \mathcal{N}} N$. Lemma 2.11.14 gives that every open neighborhood of e contains an open normal subgroup, which is necessarily in $\mathcal{N}$. Thus, $\ker(\alpha)$ is contained in every neighborhood of e, and hence is in the intersection of all such neighborhoods. However, since G is Hausdorff, the intersection of all neighborhoods of e is $\{e\}$ and so α is injective. □

Theorem 2.11.15 is important because it may not immediately be clear a group is profinite. Moreover, not only does it tell us that a compact totally disconnected topological group G is profinite, but it gives us exactly how the profinite group is realized:

$$G \cong \varprojlim_{N \in \mathcal{N}} G/N$$

where $\mathcal{N}$ consists of the open normal subgroups of G. We will see a particularly interesting example of this in the next section.

We close this section with the following theorem on the closed subgroups of a profinite group.

Theorem 2.11.16. *Let G be a profinite group and H a subgroup of G. The subgroup H is open if and only if G/H is finite. Moreover, the following three statements are equivalent:*

1. *H is closed.*
2. *H is profinite.*
3. *H is the intersection of a family of open subgroups.*

Finally, if the above conditions are satisfied, then G/H is compact and totally disconnected.

Proof. The fact that H is open if and only if G/H is finite follows from Proposition 2.10.11.

Suppose that H is closed. Since H is a closed subset of a compact space, H is compact. We also have that $G^0 = \{e\}$, so $H^0 = \{e\}$. Thus, H is compact and totally disconnected and so profinite.

Suppose now that H is profinite. Then H is a compact subset of a Hausdorff space, so it must be closed. This shows the first two statements are equivalent.

Now suppose that H is the intersection of a family of open subgroups of G. Recall that every open subgroup is also closed by Proposition 2.10.9. Thus, we have that H is the intersection of a family of closed subgroups, and so must be closed.

Let H be closed. Once again, let $\mathcal{N}$ denote the family of all open normal subgroups of G. For each $N \in \mathcal{N}$, NH is a subgroup of G because N is normal. We also see by the first statement that $[G : N]$ is finite because N is open. Thus, $[G : NH]$ is also finite and hence NH is open. We also have

$$H \subset \bigcap_{N \in \mathcal{N}} NH.$$

Let $x \in \bigcap_{N \in \mathcal{N}} NH$. Let U be any neighborhood of x. Then Ux^{-1} is a neighborhood of e and so contains a $N_x \in \mathcal{N}$ by Lemma 2.11.14 where we have used that G is profinite since we have already shown the equivalence of the first two statements in the theorem. So $x \in N_x H$. Since $e \in N_x$, $x \in N_x x$ as well. Thus, $N_x x = N_x h$ for some $h \in H$ and so $h \in N_x x \subset U$. Thus, every neighborhood of x intersects H and so $x \in \mathrm{Cl}(H)$. However, we assumed H is closed and so $\mathrm{Cl}(H) = H$. Thus, $\bigcap_{N \in \mathcal{N}} NH = H$ as desired. This shows that the first and third condition are equivalent, which when combined with what we have already shown gives the equivalence of all three conditions.

Finally, we must show if one of the three equivalent conditions is satisfied then G/H is compact and totally disconnected. Since G is compact and G/H is the continuous image of G, we have that G/H is compact as well. Let $\pi : G \to G/H$ be the canonical projection map. Suppose that G/H is not totally disconnected. Let $\pi(X)$ be a connected subset of G/H that properly contains $\pi(H)$. Note that we may assume $H \neq \{e\}$ for otherwise the result is trivial. We also clearly see that H is a proper subset of X and so $X - H$ contains at least one point. Suppose $X - H = \{x\}$. Then we have $\{x\}$ is closed because G, and hence X is Hausdorff. Thus, H is open in X. Now observe that $\pi(\{x\})$ is nonempty, $\pi(\{x\}) \neq \pi(X)$ since $\pi(H)$ is a proper subset of $\pi(X)$, and $\pi(\{x\})$ is open because projection is an open map, and closed because once again H is closed and we are in a Hausdorff space and $\pi(\{x\})$ is a point. This contradicts $\pi(X)$ being connected and so it must be that $\pi(H)$ is its own connected component.

Suppose now that $X - H$ has at least two points. In this case we set $Y = X - H$. Then Y cannot be connected because G is totally disconnected so there exist disjoint nonempty open (and hence closed) sets F_1 and F_2 in X so that $Y = F_1 \cup F_2$. Then $X = (F_1 \cup H) \cup F_2$. Now repeat the argument above replacing $\{x\}$ by F_2. ∎

One should note that the last part of the previous theorem could also be shown by directly showing

$$G/H \cong \varprojlim_{N \in \mathcal{N}} G/NH.$$

2.12 Examples of profinite groups

In this section we work out a couple of important examples of profinite groups based on the material covered in the previous sections. We also introduce the notion of pro-p groups and examine what they say for these examples.

It is customary in a beginning abstract algebra class to cover Galois theory for finite extensions of fields. However, one is often interested in studying infinite field extensions. For example, if we have a field K, the separable closure K^{sep} is a very interesting infinite field extension. Often by studying properties of the larger field extension one can recover information about the finite extensions. The theory of profinite groups is very important in this study. We quickly recall some notions from abstract algebra.

Definition 2.12.1. Let F be a field and K a field that contains F. We say $\alpha \in K$ is *algebraic* over F if there is a monic polynomial $f(x) \in F[x]$ so that $f(\alpha) = 0$. We say that α is *separable* if $f(x)$ is irreducible and has no repeated roots. The field K is said to be a *separable extension* of F if it is generated over F by separable elements.

Definition 2.12.2. Let $\overline{F}$ be an algebraic closure of F and K a field so that $F \subset K \subset \overline{F}$. The field K is said to be a *normal extension of* F if every embedding $\sigma : K \hookrightarrow \overline{F}$ so that $\sigma|_F = \text{id}$ is an automorphism of K.

Definition 2.12.3. A field extension K/F is said to be *Galois* if it is separable and normal. The set of all automorphisms of K that fix F form a group under composition denoted $\text{Gal}(K/F)$.

One should note that there is no assumption the field extensions were finite extensions in the above definitions!

One of the deepest and most beautiful theorems one learns as an undergraduate or graduate student is the Fundamental Theorem of Galois Theory for finite extensions:

Theorem 2.12.4. *Let K/F be a finite Galois extension. Set $G = \text{Gal}(K/F)$. There is a bijection*

$$\{\textit{subfields } E \textit{ of } K \textit{ containing } F\} \leftrightarrow \{\textit{subgroups } H \textit{ of } G\}$$

given by

$$E \mapsto \text{Gal}(K/E)$$

and

$$H \mapsto K^H$$

which are inverse to each other where

$$K^H = \{x \in K : \sigma x = x \text{ for every } \sigma \in H\}.$$

Furthermore, we have that E is Galois over F if and only if $\mathrm{Gal}(K/E)$ *is normal in* $\mathrm{Gal}(K/F)$.

One can see [3] for a more detailed discussion of this result as well as a proof. Out interest is in the corresponding statement for infinite extensions.

Let K/F be a Galois extension of fields, not necessarily finite. Let $\mathcal{N}$ be the set of normal subgroups of $G = \mathrm{Gal}(K/F)$ of finite index. If $N_1, N_2 \in \mathcal{N}$ with $N_1 \subset N_2$, we have a natural projection map

$$\varphi_{N_2,N_1} : G/N_1 \longrightarrow G/N_2.$$

Thus, we obtain a projective system $\{G/N\}_{N\in\mathcal{N}}$. It is easy to see the projections φ_{N_2,N_1} are compatible with the natural projection

$$\varphi_{N_2} : G \to G/N_2.$$

Note that φ_{N_2} is the restriction map $\mathrm{Gal}(K/F) \to \mathrm{Gal}(K^{N_2}/F)$. Thus, as before we have a canonical induced homomorphism

$$\varphi : G \longrightarrow \varprojlim_{N\in\mathcal{N}} G/N.$$

Proposition 2.12.5. *The canonical map*

$$\varphi : \mathrm{Gal}(K/F) \longrightarrow \varprojlim_{N\in\mathcal{N}} \mathrm{Gal}(K/F)/N$$

is an isomorphism of groups and so G is a profinite group in the topology induced by φ.

Proof. It is clear that

$$\ker\varphi = \bigcap_{N\in\mathcal{N}} N.$$

We wish to show that $\ker\varphi$ is trivial and φ is onto. Let $\sigma \in \ker\varphi$ and let $x \in K$. Then there exists a finite Galois extension E of F where $F \subset E \subset K$ with $x \in E$. The restriction map $\mathrm{Gal}(K/F) \to \mathrm{Gal}(E/F)$ has kernel $\mathrm{Gal}(K/E)$. Thus, $\mathrm{Gal}(K/E)$ is normal and has finite index so $\mathrm{Gal}(K/E) \in \mathcal{N}$. Thus, $\sigma \in \mathrm{Gal}(K/E)$, which gives $\sigma(x) = x$ since $x \in E$. Since x was arbitrary, we must have $\sigma(x) = x$ for every $x \in K$, i.e., $\sigma = e$ and so $\ker\varphi$ must be trivial as desired.

Now let $(\sigma_N)_{N\in\mathcal{N}} \in \varprojlim G/N$. We need to show there is a $\sigma \in G$ so that $\varphi_N(\sigma) = \sigma_N$ for every $N \in \mathcal{N}$. Let $x \in K$. There exists a finite Galois extension E of F contained in K that contains x with $N = \mathrm{Gal}(K/E)$ normal and of finite index in G. Note $G/N \cong \mathrm{Gal}(E/F)$. Define $\sigma \in \mathrm{Gal}(K/F)$ by $\sigma(x) = \sigma_N(x)$. The definition of projective limit shows that this is well-defined, i.e., if we choose another field E' satisfying the same conditions as E, the definition of σ on x does not change. Doing this for each $x \in K$ defines σ and it is clear from construction that $\varphi_N(\sigma) = \sigma_N$. □

From this proposition we obtain that $\mathrm{Gal}(K/F)$ is compact and totally disconnected. We have that a subgroup $H \subset \mathrm{Gal}(K/F)$is open if and only if G/H is finite.

Theorem 2.12.6. *Let K/F be any Galois extension of fields and set $G = \mathrm{Gal}(K/F)$ endowed with the profinite topology. The maps*

$$E \mapsto \mathrm{Gal}(K/E)$$

and

$$H \mapsto E = K^H$$

give a bijection between the intermediate fields $F \subset E \subset K$ and the closed subgroups of G. The maps are inverse to each other and are order reversing. Furthermore, $E/F$is Galois if and only if $\mathrm{Gal}(K/E)$ is normal in $\mathrm{Gal}(K/F)$.

One should note that if G is finite, the topology on G is the discrete topology and this theorem reduces to Theorem 2.12.4. We will assume the that Theorem 2.12.4 is known in the process of proving Theorem 2.12.6.

Proof. Let f be the map that sends E to $\mathrm{Gal}(K/E)$ and g the map that sends H to K^H. It is clear that for any subset $H \subset G$, the set K^H is a field between F and K so g is well-defined. We must show that f is well-defined, i.e., for $F \subset E \subset K$, the group $\mathrm{Gal}(K/E)$ is a closed subgroup of G. Consider the collection $\{L \text{ a field} : L/F \text{ is finite and Galois}, L \subset E\}$. For each such L, we have $\mathrm{Gal}(K/L)$ is a subgroup of $\mathrm{Gal}(K/F)$. It is open because $\mathrm{Gal}(K/F)/\mathrm{Gal}(K/L) \cong \mathrm{Gal}(L/F)$, a finite group, so we can apply Theorem 2.11.16 to conclude it is open. Now recall from Proposition 2.10.9 that every open subgroup of a topological group is also closed, so $\mathrm{Gal}(K/L)$ is closed for each such L as well. It is easy to see that $\mathrm{Gal}(K/E) \subset \bigcap_L \mathrm{Gal}(K/L)$ since if $\sigma \in \mathrm{Gal}(K/E)$, then certainly $\sigma \in \mathrm{Gal}(K/L)$ for every $L \subset E$. Conversely, let $\sigma \in \mathrm{Gal}(K/L)$ for every such L. Suppose there is an $\alpha \in E$ so that $\sigma(\alpha) \neq \alpha$. This is a contradiction because by adjoining α and the rest of the roots of the minimal polynomial of α to F we obtain a finite Galois extension of F, and so σ must fix this extension. Thus, $\mathrm{Gal}(K/E) = \bigcap_L \mathrm{Gal}(K/L)$, and so being the intersection of closed sets, $\mathrm{Gal}(K/E)$ is itself closed.

Our next step is to show that $g \circ f$ is the identity map. Let $F \subset E \subset K$ so $f(E) = \mathrm{Gal}(K/E)$, which clearly fixes E. Thus, we must have $(g \circ f)(E) \supset E$. Let $x \in (g \circ f)(E)$. There exists a finite Galois extension of E containing x that is contained in K. Call this extension L. Let $\sigma \in \mathrm{Gal}(L/E)$. There exists a $\tau \in \mathrm{Gal}(K/E)$ so that $\tau|_L = \sigma$. This follows exactly as in the finite case only requiring an application of Zorn's Lemma in this instance. By definition, we have $x \in (g \circ f)(E) = K^{\mathrm{Gal}(K/E)}$, and so $\tau(x) = x$. However, since $\tau|_L = \sigma$ and $x \in L$, we see that $\sigma(x) = x$ as well. We now use finite Galois theory on $\mathrm{Gal}(L/E)$ to conclude that $x \in E$ and so $g \circ f$ is the identity map.

We now show that $f \circ g$ is the identity map. Let H be a subgroup of G. We have

$$(f \circ g)(H) = f(K^H) = \mathrm{Gal}(K/K^H)$$

and so clearly we have H is a subgroup of $(f \circ g)(H)$. Now let H be a closed subgroup. Theorem 2.11.16 gives that H is the intersection of a family of open subgroups, call the family $\mathcal{U} = \{U_i\}$. We have

$$\begin{aligned} g(H) &= g(\cap U_i) \\ &= K^{\cap U_i} \\ &\supset \bigcup K^{U_i} \\ &= \bigcup g(U_i). \end{aligned}$$

If we apply f we obtain

$$\begin{aligned} (f \circ g)(H) &= \mathrm{Gal}(K/g(H)) \\ &\subset \mathrm{Gal}(K/\cup g(U_i)) \\ &\subset \bigcap \mathrm{Gal}(K/g(U_i)) \\ &= \bigcap U_i \\ &= H \end{aligned}$$

where we have used that since the U_i are open, they have finite index and so we can use the finite case to get $\bigcap \mathrm{Gal}(K/g(U_i)) = \bigcap U_i$. Thus, we have that $f \circ g$ is the identity.

It only remains to show that given an intermediate field E, E/F is Galois if and only if $H = \mathrm{Gal}(K/E)$ is normal in $G = \mathrm{Gal}(K/F)$. Let $\sigma \in G$. We claim that

$$\sigma H \sigma^{-1} = \mathrm{Gal}(K/\sigma(E)).$$

Let $\tau \in H$. If $x \in \sigma(E)$, we have $\sigma^{-1}(x) \in E$ and so $\tau\sigma^{-1}(x) \in E$ and thus $\sigma\tau\sigma^{-1}(x) \in \sigma(E)$. Hence, $\sigma\tau\sigma^{-1} \in \mathrm{Gal}(K/\sigma(E))$ and so $\sigma H \sigma^{-1} \subset \mathrm{Gal}(K/\sigma(E))$. Now suppose that $\chi \in \mathrm{Gal}(K/\sigma(E))$. Let $x \in E$. Then $\sigma(x) \in \sigma(E)$ so $\chi(\sigma(x)) = \sigma(x)$, and so $\sigma^{-1}\chi\sigma(x) = x$. Thus, $\sigma^{-1}\chi\sigma \in \mathrm{Gal}(K/E)$. From this we see that $\chi = \sigma(\sigma^{-1}\chi\sigma)\sigma^{-1} \in \sigma H \sigma^{-1}$. Thus, $\mathrm{Gal}(K/\sigma(E)) = \sigma H \sigma^{-1}$. From what we have shown above, we have $\sigma(E) = E$ for every $\sigma \in G$ if and only if $\sigma H \sigma^{-1} = H$ for every $\sigma \in G$, i.e., E is Galois over F if and only if H is normal in G. □

Before our next examples we need some definitions and theorems.

Definition 2.12.7. Let p be a prime number. We say a profinite group is a *pro-p group* if it is the projective limit of finite p-groups.

Example 2.12.8. We constructed $\mathbb{Z}_p$ as

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}.$$

Thus, $\mathbb{Z}_p$ is a pro-p group.

Definition 2.12.9. A *supernatural number* is a formal product

$$\prod_p p^{n_p}$$

where p runs over the set of primes and $n_p \in \mathbb{N} \cup \{\infty\}$.

Definition 2.12.10. Let G be a profinite group and let $\mathcal{N}$ denote the set of open normal subgroups in G. Let H be a closed subgroup of G. The *index of H in G*, denoted $[G : H]$, is defined by

$$[G : H] = \mathrm{lcm}_{N \in \mathcal{N}} \left[G/N : HN/N\right].$$

The *order of G* is defined by

$$|G| = [G : \{e\}].$$

Note that since $N \in \mathcal{N}$ is an open subgroup in G, $|G/N| < \infty$ so $[G/N : HN/N] < \infty$ for each N. However, $[G : H]$ is a supernatural number.

Example 2.12.11. We claim that $|\mathbb{Z}_p| = p^\infty$. To see this, observe that the open subgroups of $\mathbb{Z}_p$ are $p^n\mathbb{Z}_p$ for $n \geq 0$. Thus,

$$\begin{aligned} |\mathbb{Z}_p| &= \mathrm{lcm}_{n \geq 0}[\mathbb{Z}_p : p^n\mathbb{Z}_p] \\ &= \mathrm{lcm}_{n \geq 0}\, p^n \\ &= p^\infty. \end{aligned}$$

It is possible to prove many of the standard results of group theory in this setting. We state one of them here but omit the proof.

Proposition 2.12.12. *Let G be a profinite group, H and K closed subgroups of G with $H \subset K$. Then*

$$[G : K] = [G : H][K : H].$$

Definition 2.12.13. Let G be a profinite group. A maximal pro-p subgroup of G is called a *pro-p Sylow subgroup of G*, or simply a *p-Sylow subgroup of G*.

Theorem 2.12.14. *Let G be a profinite group and p a prime number. One has*

1. *p-Sylow subgroups of G exist (they may be trivial.)*
2. *Any pair of p-Sylow subgroups of G are conjugate.*
3. *If H is a p-Sylow subgroup of G, then $[G : H]$ is relatively prime to p.*
4. *Each p-Sylow subgroup of G is nontrivial if and only if p divides the order of G.*

Proof. We begin with a basic observation. If we let $\mathcal{N}$ denote the open normal subgroups of G, then we have

$$G \xrightarrow{\simeq} \varprojlim_{N \in \mathcal{N}} G/N$$
$$x \mapsto (x_N)_{N \in \mathcal{N}}.$$

Thus, given $g, h \in G$, $g = h$ if and only if $g_N = h_N$ for all $N \in \mathcal{N}$.

1. Let $N \in \mathcal{N}$ and write $\mathcal{P}(N)$ for the (possibly empty) set of p-Sylow subgroups of the finite group G/N. Let $M, N \in \mathcal{P}(N)$ with $N \subset M$. There is a natural surjection

$$\varphi_{M,N} : G/N \longrightarrow G/M.$$

For $P_N \in \mathcal{P}(N)$ we have a natural induced surjection

$$(G/N)/P_N \longrightarrow (G/M)/(\varphi_{M,N}(P_N)).$$

The kernel of this map is $P_N \ker \varphi_{M,N}/P_N$. In particular, we have

$$[G/N : P_N] = [G/M : \varphi_{M,N}(P_N)][P_N \ker \varphi_{M,N} : P_N].$$

It is clear that $\varphi_{M,N}(P_N)$ is a p-group and this equality shows it is a p-Sylow subgroup of G/M since $p \nmid [G/N : P_N]$ implies $p \nmid [G/M : \varphi_{M,N}(P_N)]$. Thus, we have a map

$$\varphi_{M,N} : \mathcal{P}(N) \longrightarrow \mathcal{P}(M)$$

induced from the original $\varphi_{M,N}$. Thus, the collection $(\{\mathcal{P}(N)\}, \varphi_{M,N})$ forms a projective system of finite nonempty sets. This projective limit is nonempty, so we can take an element of it. An element in this projective limit is a collection of p-Sylow subgroups $P_N \subset G/N$, one for each $N \in \mathcal{N}$. However, these are finite groups themselves so form a projective system themselves:

$$P = \varprojlim_{N} P_N.$$

We have that P is then a subgroup of G and by construction we have that P is a pro-p group. We must show that P is maximal. Let Q be any p-Sylow subgroup of G containing P. Then for each $N \in \mathcal{N}$ we have

$$QN/N \supset PN/N = P_N,$$

and is a p-group. Since P_N is a p-Sylow subgroup of G/N, the finite Sylow theorem gives $P_N = QN/N$ for every $N \in \mathcal{N}$. Thus, by the comment at the beginning of the proof we have $Q = P$.

2. Let P and Q be p-Sylow subgroups of G. Set $P_N = PN/N$, $Q_N = QN/N$ and
$$A_N = \{g_N \in G/N : g_N P_N g_N^{-1} = Q_N\}.$$
By the finite Sylow theorems we have that $A_N \neq \emptyset$ for all $N \in \mathcal{N}$. The sets A_N form a projective system. Set
$$A = \varprojlim_N A_N,$$
which is a subset of G. Let $a \in A$. Then we have that aPa^{-1} and Q have the same projection in G/N for each $N \in \mathcal{N}$, so they must be equal.

3. This is a homework exercise.

4. This follows immediately from the previous parts.

□

Corollary 2.12.15. *Let G be an abelian profinite group. Then:*

1. *For every prime p, G has a unique p-Sylow subgroup.*

2. *Let p and q be distinct primes and P a p-Sylow subgroup and Q a q-Sylow subgroup. Then $P \cap Q = \{e\}$.*

3. *The group G is isomorphic to a direct product of its Sylow subgroups.*

Proof. The first two results follow immediately from the previous theorem. The third statement follows from the isomorphism
$$G/N \cong \prod PN/N$$
where the product is over the p-Sylow subgroups of G and the fact that $P \cap N$ forms a cofinal system among the open subgroups of G. □

We close this section and chapter by returning to the example of $\widehat{\mathbb{Z}}$ formed by looking at the projective system formed by $\mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/M\mathbb{Z}$. We begin by observing that $|\widehat{\mathbb{Z}}| = \prod_p p^\infty$. To see this, let $n \geq 1$. We have a surjection $\widehat{\mathbb{Z}} \to \mathbb{Z}/n\mathbb{Z}$ with a kernel, call it H_n. Thus, every integer n divides $|\widehat{\mathbb{Z}}|$.

Let p be a prime and let P be the p-Sylow subgroup of $\widehat{\mathbb{Z}}$. Let P_n be the unique p-Sylow subgroup of $\mathbb{Z}/n\mathbb{Z}$. Then
$$\begin{aligned} P &= \varprojlim_n P_n \\ &= \varprojlim_n \mathbb{Z}/p^{v_p(n)}\mathbb{Z} \\ &= \varprojlim_m \mathbb{Z}/p^n\mathbb{Z} \\ &= \mathbb{Z}_p \end{aligned}$$

where $\upsilon_p(n)$ is the power of p exactly dividing n. Thus,

$$\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p.$$

Chapter 3

Differential Topology and de Rham cohomology

In the last few sections of Chapter 2 we saw how topology can be useful in giving more information to algebraic settings such as Galois theory. In this chapter we begin to see how algebra can be used to study topological notions. This chapter focuses on the differential theory, which one can think of as generalized calculus. As such, we draw our motivation from calculus and develop the theory over $\mathbb{R}^n$ before moving on to the more abstract theory over differentiable manifolds. One should keep in mind that this theory can be developed over $\mathbb{C}^n$ as well. In this case one needs to use the theory of several complex variables. We will return to this later and develop what is needed for covering some of the desired examples from algebraic geometry.

3.1 Motivation

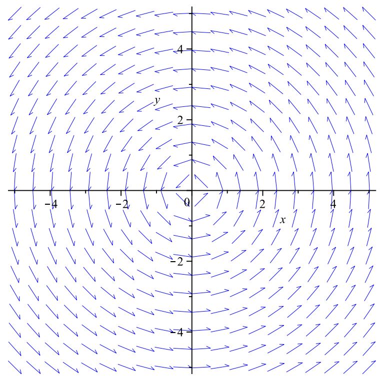
In this section we give the basic motivation for deRham cohomology by studying a familiar problem from multivariable calculus and rephrasing it in a more algebraic language.

Definition 3.1.1. Let $D \subset \mathbb{R}^2$. A *vector field* on D is a function that assigns a vector $\mathbf{F}(x, y)$ to each point $(x, y) \in D$.

Example 3.1.2. Define a vector field on $\mathbb{R}^2 - \{(0,0)\}$ by setting

$$\mathbf{F}(x, y) = \left(-\frac{y}{\sqrt{x^2 + y^2}}\right)\mathbf{i} + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)\mathbf{j}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the standard unit vectors in the x and y directions respectively. This vector field can be pictured as follows:



Recall that given a function $z = f(x,y)$, the *gradient of* f is defined by

$$\nabla f(x,y) = \frac{\partial f}{\partial x}(x,y)\mathbf{i} + \frac{\partial f}{\partial y}(x,y)\mathbf{j}.$$

The gradient of a function is a vector field where it is defined.

One knows from elementary physics class that work done by a constant force $\mathbf{F}$ over a straight line displacement $\mathbf{D}$ is given by

$$W = \mathbf{F} \cdot \mathbf{D}.$$

More generally, we can consider the work done by a variable force $\mathbf{F}(x,y)$ in moving a particle over a path C. Suppose the path is traced out as the tips of the vectors $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ for $a \le t \le b$. Let $t_0 = a < t_1 < \cdots < t_n = b$ be a partition of the interval and let s_j be the arc-length of the curve from $\mathbf{r}(t_j)$ to $\mathbf{r}(t_{j+1})$ for $0 \le j \le n-1$. Let $\mathbf{T}(t)$ be the unit tangent vector to C at t, i.e., $\mathbf{T}(t) = \frac{\mathbf{r}(t)}{|\mathbf{r}(t)|}$. As long as s_j is small for each j, we can approximate the force over this interval by $\mathbf{F}(\mathbf{r}(t_j))$ and the displacement by $s_j\mathbf{T}(t_j)$. Thus, one has that the work done over the interval t_j to t_{j+1} is approximately equal to $\mathbf{F}(\mathbf{r}(t_j)) \cdot \mathbf{T}(t_j)(s_j)$. If we add up all of these contributions we end up with a Riemann sum approximating the value of the total work:

$$\sum_{j=0}^{n-1} (\mathbf{F}(\mathbf{r}(t_j)) \cdot \mathbf{T}(t_j))s_j.$$

Now, let the width of the partition, i.e., $\min_{1 \le j \le n}(t_j - t_{j-1})$, go to 0. This gives that the work over this curve is given by

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot \mathbf{T}\, ds \\ &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{T}(t)\, dt. \end{aligned}$$

Of course, we recall that this is how a line integral is defined even when one does not have the physical interpretation of $\mathbf{F}$ as a force moving something.

Suppose now that there is a function $z = f(x, y)$ so that $\nabla f(x, y) = \mathbf{F}(x, y)$. In this case we see from the fundamental theorem of line integrals (which follows immediately from the standard fundamental theorem of calculus) that

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

In other words, if there is such a f then the line integral along C from the point $\mathbf{r}(a)$ to the point $\mathbf{r}(b)$ is independent of the path C! We call such a f a *potential function* and such a vector field $\mathbf{F}$ a *conservative vector field*. The question is, given a vector field $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$, is $\mathbf{F}$ conservative, i.e., when does there exist a function $z = f(x, y)$ so that $F_1 = \frac{\partial f}{\partial x}$ and $F_2 = \frac{\partial f}{\partial y}$? Using the equality of mixed partial derivatives we see immediately that it is necessary for

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{\partial^2 f}{\partial y \partial x} \\ &= \frac{\partial^2 f}{\partial x \partial y} \\ &= \frac{\partial F_2}{\partial x}. \end{aligned}$$

However, it is not immediately clear if this is also a sufficient condition.

Example 3.1.3. Consider the vector field $\mathbf{F}$ given in Example 3.1.2 and let C be the unit circle parameterized by $x(t) = \cos t$, $y(t) = \sin t$ for $0 \leq t \leq 2\pi$. Then we have

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{T}\, ds &= \int_0^{2\pi} ((-\sin t\mathbf{i} + \cos t\mathbf{j}) \cdot (-\sin t\mathbf{i} + \cos t\mathbf{j}))\, dt \\ &= 2\pi. \end{aligned}$$

Thus, we have that $\mathbf{F}$ is not conservative because C is a closed curve and the line integral of a conservative vector field over a closed curve must be 0. However, it is easy to check that $\mathbf{F}$ satisfies the necessary condition given above. Thus, this condition cannot be a sufficient condition as well!

It turns out that the topology of the region D the vector field is defined on is fundamental to whether or not the condition is also sufficient.

Definition 3.1.4. A region $U \subset \mathbb{R}^2$ is said to be *star-shaped with respect to the point* (x_0, y_0) if the line segment $\{t(x_0, y_0) + (1 - t)(x, y) : 0 \leq t \leq 1\}$ lies in U for all $(x, y) \in U$.

Theorem 3.1.5. *Let $U \subset \mathbb{R}^2$ be open and star-shaped. Let $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$ be a vector field defined on U with F_1 and F_2 having continuous partial derivatives on U. Furthermore, assume $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ on U. Then $\mathbf{F}$ is conservative.*

Proof. Without loss of generality we may assume that U is star-shaped with respect to $(0,0)$. Define $f : U \to \mathbb{R}$ by

$$f(x,y) = \int_0^1 xF_1(tx,ty) + yF_2(tx,ty)\,dt.$$

Note that this is well-defined because U is star-shaped with respect to $(0,0)$ and $\mathbf{F}$ is defined on U. Observe that we have

$$\frac{\partial f}{\partial x}(x,y) = \int_0^1 \left(F_1(tx,ty) + tx\frac{\partial F_1}{\partial x}(tx,ty) + ty\frac{\partial F_2}{\partial x}(tx,ty) \right) dt.$$

An application of the chain rule gives

$$\frac{d}{dt}\left(tF_1(tx,ty)\right) = F_1(tx,ty) + tx\frac{\partial F_1}{\partial x}(tx,ty) + ty\frac{\partial F_1}{\partial y}(tx,ty).$$

Thus, we see that

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= \int_0^1 \left(\frac{d}{dt}(tF_1(tx,ty)) + ty\left(\frac{\partial F_2}{\partial x}(tx,ty) - \frac{\partial F_1}{\partial y}(tx,ty) \right) \right) dt \\
&= \int_0^1 \frac{d}{dt}(tF_1(tx,ty))\,dt \\
&= F_1(x,y)
\end{aligned}$$

where we have used that $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$. Similarly, we obtain $\frac{\partial f}{\partial y}(x,y) = F_2(x,y)$ and so $\mathbf{F}$ is conservative. □

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$. We write $C^\infty(U,V)$ for the set of functions $f : U \to V$ that have continuous partial derivatives of all orders. We refer to this as the set of *smooth functions*. For most instances continuous partial derivatives to the second order, but we don't lose much by focusing on the smooth functions and it simplifies the exposition.

Let $U \subset \mathbb{R}^2$. Define

$$\text{curl} : C^\infty(U,\mathbb{R}^2) \to C^\infty(U,\mathbb{R})$$

by setting

$$\text{curl}(F_1,F_2) = \frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}$$

and

$$\text{grad} : C^\infty(U,\mathbb{R}) \to C^\infty(U,\mathbb{R}^2)$$

by setting

$$\text{grad}(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right).$$

Observe that we have

$$\text{curl}(\text{grad}(f)) = 0$$

for all $f \in C^\infty(U, \mathbb{R})$. Thus, we see that the image of ∇ lies in the kernel of curl. The problem we are trying to solve is to determine when the image of grad is precisely the kernel of curl. To this end, define

$$\mathrm{H}^1_{\mathrm{dR}}(U) = \ker(\mathrm{curl})/\operatorname{im}(\mathrm{grad})$$

and

$$\mathrm{H}^0_{\mathrm{dR}}(U) = \ker(\mathrm{grad}).$$

Observe that Theorem 3.1.5 can be restated as follows.

Theorem 3.1.6. *If U is star-shaped then $\mathrm{H}^1_{\mathrm{dR}}(U) = 0$.*

It is clear that $C^\infty(U, \mathbb{R}^2)$ has the structure of a $\mathbb{R}$-vector space. Furthermore, ker(grad) and im(curl) are both $\mathbb{R}$-subspaces and so $\mathrm{H}^0_{\mathrm{dR}}(U)$ and $\mathrm{H}^1_{\mathrm{dR}}(U)$ both have natural $\mathbb{R}$-vector space structures. This structure will be essential in the forthcoming sections to establish a nice algebraic theory.

At this point we do not have enough machinery to calculate $\mathrm{H}^1_{\mathrm{dR}}(U)$ for any nontrivial spaces U. However, we can calculate $\mathrm{H}^0_{\mathrm{dR}}(U)$.

Theorem 3.1.7. *The dimension of $\mathrm{H}^0_{\mathrm{dR}}(U)$ as a $\mathbb{R}$-vector space is precisely the number of connected components of U.*

Proof. Let U have k connected components $U_1, \ldots, U_k$. Define functions $f_1, \ldots, f_k$ by

$$f_i(x, y) = \begin{cases} 1 & (x, y) \in U_i \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate that each f_i satisfies $\mathrm{grad}(f_i) = 0$ and so $f_1, \ldots, f_k \in \mathrm{H}^0_{\mathrm{dR}}(U)$. It is also clear that $f_1, \ldots, f_k$ are linearly independent, so it only remains to show they span $\mathrm{H}^0_{\mathrm{dR}}(U)$.

Let $f \in \mathrm{H}^0_{\mathrm{dR}}(U)$. Since $\mathrm{grad}(f) = 0$ we have that f is locally constant, i.e., for each $(x_0, y_0) \in U$ there is an open set $V_{(x_0,y_0)}$ containing (x_0, y_0) so that $f(x, y) = f(x_0, y_0)$ for all $(x, y) \in V_{(x_0,y_0)}$. Let $(x_i, y_i) \in U_i$ be fixed points and set $f(x_i, y_i) = z_i$ for $1 \leq i \leq k$. We claim that $f(x, y) = z_i$ for al $(x, y) \in U_i$. To see this, consider the set

$$f^{-1}(f(x_i, y_i)) \cap U_i = \{(x, y) \in U_i : f(x, y) = z_i\}.$$

This set is closed in U_i since it is the preimage of the closed set $\{z_i\}$ under the continuous map f and it is open because f is locally constant. Thus, it must be that it is all of U_i as claimed. Thus, we see that we can write

$$f(x, y) = \sum_{i=1}^{k} z_i f_i(x, y)$$

and so $f_1, \ldots, f_k$ span $\mathrm{H}^0_{\mathrm{dR}}(U)$ as claimed. $\square$

3.2 Some General Homological Algebra

Before we proceed further with de Rham cohomology we give some basic homological algebra results. One could prove all the results in this section for de Rham cohomology in particular, but as we will encounter other types of cohomology theories it is best to have some of the machinery set up generally. We also briefly review some basic algebra facts that we will use often. For the reader that desires to see a concrete example before seeing such general theory, the following section can safely be read before this one. Section 3.3 is spent constructing an example of a chain complex that satisfies the definitions laid out in this section and whose cohomology groups generalize those constructed in § 3.1.

Let V^1, V^2, and V^3 be vector spaces and $T^i : V^i \to V^{i+1}$ be linear maps. Note we use superscripts here as we are going to be dealing with cohomology, so superscripts are standard. One can also keep in mind that spaces $\Omega^i(U)$ are the vector spaces that will take the place of these when we specialize to the case of de Rham cohomology. We say the sequence

$$V^1 \xrightarrow{T^1} V^2 \xrightarrow{T^2} V_3$$

is *exact* when $\ker(T^2) = \operatorname{im}(T^1)$.

Let $\{V^i\}$ be a collection of vector spaces and $\{d^i : V^i \to V^{i+1}\}$ a collection of linear maps. We call the sequence

$$\cdots \longrightarrow V^{i-1} \xrightarrow{d^{i-1}} V^i \xrightarrow{d^i} V^{i+1} \xrightarrow{d^{i+1}} V^{i+2} \longrightarrow \cdots$$

a *chain complex* if $d^{i+1} \circ d^i = 0$ for all i. We say the chain complex is *exact* if $\ker(d^i) = \operatorname{im}(d^{i-1})$ for all i. If the chain complex is exact, we obtain a short exact sequence

$$0 \longrightarrow \operatorname{im}(d^{i-1}) \longrightarrow V^i \longrightarrow \operatorname{im}(d^i) \longrightarrow 0.$$

Let $V^* = \{V^i, d^i\}$ be a chain complex. Note that the maps d^i are understood from context. The cohomology groups of this chain complex are defined by

$$\mathrm{H}^m(V^*) = \frac{\ker(d^m : V^m \to V^{m+1})}{\operatorname{im}(d^{m-1} : V^{m-1} \to V^m)}.$$

Note that this is a natural vector space to consider as it measures how far the complex V^* is from being exact. In general, we refer to the elements in $\ker(d^m)$ as the *m-cocycles* and the elements in $\operatorname{im}(d^{m-1})$ as the *m-coboundaries* (or simply the cocyles and coboundaries if m is clear from context.) Note in the de Rham setting we will refer to these as closed and exact instead.

Let V^* and W^* be chain complexes. A *chain map* $T^* : V^* \to W^*$ is a collection of linear maps $T^i : V^i \to W^i$ satisfying $d^i_W \circ T^i = T^{i+1} d^i_V$ for all i, i.e., the following diagram commutes:

Lemma 3.2.1. *A chain map $T^* : V^* \to W^*$ induces a linear map*

$$\mathrm{H}^m(T^*) : \mathrm{H}^m(V^*) \to \mathrm{H}^m(W^*)$$

for all m.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & V^{i-1} & \xrightarrow{d_V^{i-1}} & V_i & \xrightarrow{d_V^{i}} & V^{i+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle T^{i-1}} & & \downarrow{\scriptstyle T^{i}} & & \downarrow{\scriptstyle T^{i+1}} & & \\
\cdots & \longrightarrow & W^{i-1} & \xrightarrow{d_W^{i-1}} & W_i & \xrightarrow{d_W^{i}} & W^{i+1} & \longrightarrow & \cdots
\end{array}$$

Proof. Let $v \in V^m$ be a cocycle and $[v] = v + \mathrm{im}(d^{m-1})$ be the corresponding cohomology class. Define

$$\mathrm{H}^m(T^*)([v]) = [T^m(v)].$$

We must show this map is well-defined. The first step of this is to show that $T^m(v)$ is a cocycle in $\mathrm{H}^m(W^*)$, i.e., that $d_W^m(T^m(v)) = 0$. Observe that

$$\begin{aligned} d_W^m(T^m(v)) &= T^{m+1}(d_V^m(v)) \\ &= T^{m+1}(0) \\ &= 0 \end{aligned}$$

since v is a cocycle.

It remains to show that $\mathrm{H}^m(T^*)([v])$ is independent of the representative of $[v]$ we choose. Let v_1 and v_2 be representatives of $[v]$. Note that this means there is a $x \in V^{m-1}$ so that $v_1 - v_2 = d_V^{m-1}(x)$. So we have

$$\begin{aligned} T^m(v_1 - v_2) &= T^m(d_V^{m-1}(x)) \\ &= d_W^m(T^{m-1}(x)) \end{aligned}$$

i.e., we have $T^m(v_1) - T^m(v_2) = d_W^m(T^{m-1}(x))$. Thus,

$$[T^m(v_1)] = [T^m(v_2)]$$

and so $\mathrm{H}^m(T^*)([v_1]) = \mathrm{H}^m(T^*)([v_2])$. Hence, the map is well-defined. $\square$

We say a sequence of chain complexes

$$0 \longrightarrow U^* \xrightarrow{S^*} V^* \xrightarrow{T^*} W^* \longrightarrow 0$$

is a *short exact sequence of chain complexes* provided that the sequence

$$0 \longrightarrow U^m \xrightarrow{S^m} V^m \xrightarrow{T^m} W^m \longrightarrow 0$$

is exact for every m.

From this we get the following commutative diagram for any m:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle d_U^{m-1}} & & \downarrow{\scriptstyle d_V^{m-1}} & & \downarrow{\scriptstyle d_W^{m-1}} & & \\
0 & \longrightarrow & U^m & \xrightarrow{S^m} & V^m & \xrightarrow{T^m} & W^m & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_U^{m}} & & \downarrow{\scriptstyle d_V^{m}} & & \downarrow{\scriptstyle d_W^{m}} & & \\
0 & \longrightarrow & U^{m+1} & \xrightarrow{S^{m+1}} & V^{m+1} & \xrightarrow{T^{m+1}} & W^{m+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_U^{m+1}} & & \downarrow{\scriptstyle d_V^{m+1}} & & \downarrow{\scriptstyle d_W^{m+1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Lemma 3.2.2. *Given a short exact sequence of chain complexes as above, one as that the sequence*

$$\mathrm{H}^m(U^*) \stackrel{\mathrm{H}^m(S^*)}{\longrightarrow} \mathrm{H}^m(V^*) \stackrel{\mathrm{H}^m(T^*)}{\longrightarrow} \mathrm{H}^m(W^*)$$

is exact for every m.

Proof. Let $[u] \in \mathrm{H}^m(U^*)$. We have

$$\begin{aligned}
\mathrm{H}^m(T^*) \circ \mathrm{H}^m(S^*)([u]) &= \mathrm{H}^m(T^*)([S^m(u)]) \\
&= [T^m \circ S^m(u)] \\
&= 0
\end{aligned}$$

since $T^m \circ S^m = 0$ for all m because of the fact that we have an exact sequence of chain complexes. Thus, we have $\mathrm{im}(\mathrm{H}^m(S^*)) \subset \ker(\mathrm{H}^m(T^*))$.

Suppose now that $[v] \in \ker(\mathrm{H}^m(T^*))$. Thus we have $T^m(v) \in \mathrm{im}(d_W^{m-1})$ so there exists $w \in W^{m-1}$ so that $T^m(v) = d_W^{m-1}(w)$. Since the sequence is exact at the level of chain complexes we know that T^{m-1} is surjective. Thus, there is a $v_1 \in V^{m-1}$ so that $T^{m-1}(v_1) = w$. Hence, we have

$$T^m(v - d_V^{m-1}(v_1)) = 0.$$

Thus, using exactness there is a $u \in U^m$ with $S^m(u) = v - d_V^{m-1}(v_1)$. We need to show that u is a cocycle. Observe that we have

$$\begin{aligned}
S^{m+1}(d_U^m(u)) &= d_V^m(S^m(u)) \\
&= d_V^m(v - d_V^{m-1}(v_1)) \\
&= 0
\end{aligned}$$

since v is a cocycle by definition and $d_V^m \circ d_V^{m-1} = 0$. The fact that S^{m+1} is injective gives that $d_U^m(u) = 0$ and so u is a cocycle. Thus, given $[v] \in \ker(\mathrm{H}^m(T^*))$ we have found $[u] \in \mathrm{H}^m(U^*)$ so that

$$\begin{aligned}
\mathrm{H}^m(S^*)([u]) &= [S^m(u)] \\
&= [v - d_V^{m-1}(v_1)] \\
&= [v].
\end{aligned}$$

Thus, $\ker(\mathrm{H}^m(T^*)) \subset \mathrm{im}(\mathrm{H}^m(S^*))$ and so we have equality as desired. $\square$

One should note that given an exact sequence of complexes

$$0 \longrightarrow U^* \xrightarrow{S^*} V^* \xrightarrow{T^*} W^* \longrightarrow 0$$

in general one does not obtain a short exact sequence of cohomology groups. The issue arises in the fact that even though $T^m : V^m \to W^m$ is surjective for all m, it can be the case that $(T^m)^{-1}(w)$ does not contain any m-cocycles even if w is a m-cocycle. In fact, this is measured via a map

$$\partial^m : \mathrm{H}^m(W^*) \to \mathrm{H}^{m+1}(U^*).$$

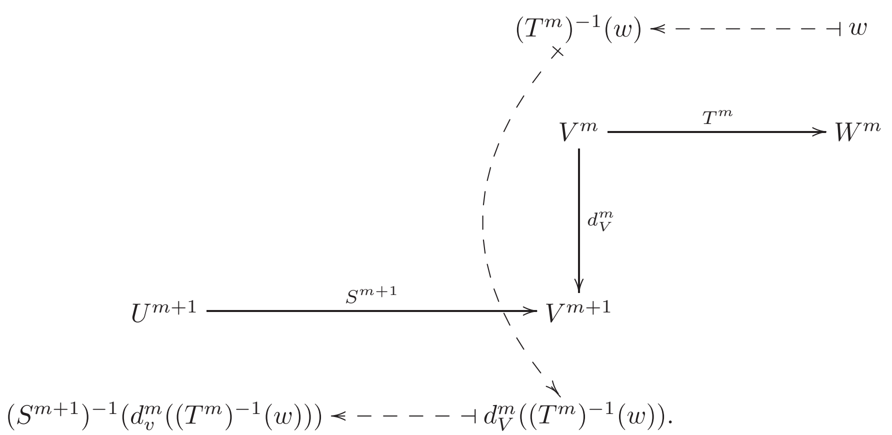
Our goal will be to show that we can define such a map so that we obtain a long exact sequence of cohomology groups

$$\cdots \xrightarrow{\partial^{m-1}} \mathrm{H}^m(U^*) \xrightarrow{\mathrm{H}^m(S^*)} \mathrm{H}^m(V^*) \xrightarrow{\mathrm{H}^m(T^*)} \mathrm{H}^m(W^*) \xrightarrow{\partial^m} \mathrm{H}^{m+1}(U^*) \xrightarrow{\mathrm{H}^{m+1}(S^*)} \mathrm{H}^{m+1}(V^*) \longrightarrow \cdots .$$

Define

$$\partial^m([w]) = [(S^{m+1})^{-1}\left(d_V^m((T^m)^{-1}(w))\right)].$$

Graphically, we have



We must show that this map is well-defined. In order to do this, we must show

1. If $T^m(v) = w$ and $d_W^m(w) = 0$, then $d_V^m(v) \in \mathrm{im}(S^{m+1})$.
2. If $S^{m+1}(u) = d_V^m(v)$, then $d_U^{m+1}(u) = 0$, i.e., the elements that map to $d_V^m(v)$ are cocycles.
3. If $T^m(v_1) = T^m(v_2) = w$ and $S^{m+1}(u_i) = d_V^m(v_i)$, then $[u_1] = [u_2]$ in $\mathrm{H}^{m+1}((^*)U)$.

Note that the following diagram commutes:

$$\begin{CD} V^m @>T^m>> W_m \\ @VV{d_V^m}V @VV{d_W^m}V \\ V^{m+1} @>T^{m+1}>> W^{m+1} \end{CD}$$

Thus, we have that $T^{m+1}(d_V^m(v)) = 0$, i.e., $d_V^m(v) \in \ker(T^{m+1})$. The fact that the sequence

$$0 \longrightarrow U^{m+1} \xrightarrow{S^{m+1}} V^{m+1} \xrightarrow{T^{m+1}} W^{m+1} \longrightarrow 0$$

is exact implies that there must be a $u \in U^{m+1}$ so that $S^{m+1}(u) = d_V^m(v)$. This gives the first part.

For the second part, we use the following diagram:

$$\begin{CD} @. @. V^m \\ @. @. @VV{d_V^m}V \\ 0 @>>> U^{m+1} @>S^{m+1}>> V^{m+1} \\ @. @VV{d_U^{m+1}}V @VV{d_V^{m+1}}V \\ 0 @>>> U^{m+2} @>S^{m+2}>> V^{m+2}. \end{CD}$$

If $S^{m+1}(u) = d_V^m(v)$, then we have $d_V^{m+1}(S^{m+1}(u)) = d_V^{m+1}(d_V^m(v)) = 0$, since we have a chain complex. Using that the diagram commutes we have

$$S^{m+2}(d_U^{m+1}(u)) = 0.$$

However, the fact that S^{m+2} is injective gives that $d_U^{m+1}(u) = 0$. Thus, we have the second part.

Finally, we show the third statement and thus conclude that ∂^m is well-defined. Note that $T^m(v_1) = T^m(v_2)$ implies that $v_1 - v_2 \in \ker(T^m) = \operatorname{im}(S^m)$. Thus, there is a $u \in U^m$ so that $S^m(u) = v_1 - v_2$. We use that $d_V^m \circ S^m = S^{m+1} \circ d_U^m$ to conclude that we have

$$\begin{aligned} d_V^m(v_1 - v_2) &= d_V^m(S^m(u)) \\ &= S^{m+1}(d_U^m(u)). \end{aligned}$$

Thus,

$$(S^{m+1})^{-1}(d_V^m(v_1)) = (S^{m+1})^{-1}(d_V^m(v_2)) + d_U^m(u),$$

i.e., $[u_1] = [u_2]$ as desired.

We can now show that we obtain the long exact sequence of cohomology as stated above.

Theorem 3.2.3. *Let*

$$0 \longrightarrow U^* \xrightarrow{S^*} V^* \xrightarrow{T^*} W^* \longrightarrow 0$$

be a short exact sequence of chain complexes. Then the sequence

$$\cdots \xrightarrow{\partial^{m-1}} \mathrm{H}^m(U^*) \xrightarrow{\mathrm{H}^m(S^*)} \mathrm{H}^m(V^*) \xrightarrow{\mathrm{H}^m(T^*)} \mathrm{H}^m(W^*) \xrightarrow{\partial^m} \mathrm{H}^{m+1}(U^*) \xrightarrow{\mathrm{H}^{m+1}(S^*)} \mathrm{H}^{m+1}(V^*) \longrightarrow \cdots$$

is exact.

Proof. There are two things to check, namely, that given any m, the sequences

$$\mathrm{H}^m(V^*) \xrightarrow{\mathrm{H}^m(T^*)} \mathrm{H}^m(W^*) \xrightarrow{\partial^m} \mathrm{H}^{m+1}(U^*)$$

and

$$\mathrm{H}^m(W^*) \xrightarrow{\partial^m} \mathrm{H}^{m+1}(U^*) \xrightarrow{\mathrm{H}^{m+1}(S^*)} \mathrm{H}^{m+1}(V^*)$$

are exact. We begin with the first sequence.

Let $[v] \in \mathrm{H}^m(V^*)$. We have

$$\begin{aligned} \partial^m(\mathrm{H}^m(T^*)([v])) &= \partial^m([T^m(v)]) \\ &= [(S^{m+1})^{-1}(d_V^m(T^m)^{-1}(T^m(v)))] \\ &= [(S^{m+1})^{-1}(d_V^m(v))] \\ &= 0 \end{aligned}$$

since v is a cocycle. Thus, $\mathrm{im}(\mathrm{H}^m(T^*)) \subset \ker(\partial^m)$.

Now let $[w] \in \ker(\partial^m)$. Let $v \in V^m$ so that $T^m(v) = w$. (Recall the sequence is exact at the level of chain complexes!) Observe that since w is a cocycle, we have that $d_W^m(w) = 0$. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U^m & \xrightarrow{S^m} & V^m & \xrightarrow{T^m} & W^m & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_U^m} & & \downarrow{\scriptstyle d_V^m} & & \downarrow{\scriptstyle d_W^m} & & \\ 0 & \longrightarrow & U^{m+1} & \xrightarrow{S^{m+1}} & V^{m+1} & \xrightarrow{T^{m+1}} & W^{m+1} & \longrightarrow & 0. \end{array}$$

Since this diagram commutes and we have $d_W^m(w) = 0$, we have that $d_V^m(v) \in \ker(T^{m+1})$. Thus, using the exactness we see there is a $u \in U^{m+1}$ so that

$$S^{m+1}(u) = d_V^m(v).$$

Using the definition of ∂^m we have

$$\begin{aligned} \partial^m([w]) &= [(S^{m+1})^{-1}(d_V^m((T^m)^{-1}(w)))] \\ &= [(S^{m+1})^{-1}(d_V^m(v))] \\ &= [u]. \end{aligned}$$

However, we know that $\partial^m([w]) = 0$ so we must have $u' \in U^m$ with $u = d_U^m(u')$. Now observe that we have

$$\begin{aligned} T^m(v - S^m(u')) &= T^m(v) - T^m(S^m(u')) \\ &= T^m(v) \\ &= w. \end{aligned}$$

Moreover, we have

$$\begin{aligned} d_V^m(v - S^m(u')) &= d_V^m(v) - d_V^m(S^m(u')) \\ &= d_V^m(v) - S^{m+1}(d_U^m(u')) \\ &= d_V^m(v) - S^{m+1}(u) \\ &= d_V^m(v) - d_V^m(v) \\ &= 0 \end{aligned}$$

Thus, we have that $v - S^m(u')$ is a cocycle that maps to w and so

$$\mathrm{H}^m(T^*)([v - S^m(u')]) = [w].$$

Thus, we have exactness of the first sequence.

Let $[w] \in \mathrm{H}^m(W^*)$. We have

$$\begin{aligned} \mathrm{H}^{m+1}(S^*)(\partial^m([w])) &= [S^{m+1}((S^{m+1})^{-1} d_V^m (T^m)^{-1}(w))] \\ &= [d_V^m(v)] \\ &= 0 \end{aligned}$$

where $v \in V^m$ such that $T^m(v) = w$. Thus, $\mathrm{im}(\partial^m) \subset \ker(\mathrm{H}^{m+1}(S^*))$.

Now let $[u] \in \ker(\mathrm{H}^{m+1}(S^*))$, i.e., $S^{m+1}(u) = d_V^m(v)$ for some $v \in V^m$. We have

$$\begin{aligned} d_W^m(T^m(v)) &= T^{m+1}(d_V^m(v)) \\ &= T^{m+1}(S^{m+1}(u)) \\ &= 0. \end{aligned}$$

Thus, we have that $T^m(v)$ is a cocycle. We also see that

$$\begin{aligned} \partial^m([T^m(v)]) &= [(S^{m+1})^{-1}(d_V^m(T^m)^{-1}(T^m(v)))] \\ &= [(S^{m+1})^{-1}(d_V^m(v))] \\ &= [u]. \end{aligned}$$

This shows that $\ker(\mathrm{H}^{m+1}(S^*)) \subset \mathrm{im}(\partial^m)$ and so the second sequence is exact as well. □

Exercise 3.2.4. Show that ∂^m is a linear map.

Definition 3.2.5. Let $S^*, T^* : V^* \to W^*$ be chain maps. We say S^* and T^* are *chain-homotopic* if there are linear maps $\Psi^m : V^m \to W^{m-1}$ satisfying

$$d_W^{m-1}\Psi^m + \Psi^{m+1}d_V^m = S^m - T^m : V^m \to W^m$$

for all m.

Chain homotopies will be very important for calculating cohomology groups in § 3.4. We will also construct a chain homotopy in Theorem **??** of § 3.3 to calculate the cohomology groups of a star-shaped region in $\mathbb{R}^n$.

Proposition 3.2.6. *Let $S^*, T^* : V^* \to W^*$ be chain homotopic maps. Then for all m we have*

$$\mathrm{H}^m(S^*) = \mathrm{H}^m(T^*) : \mathrm{H}^m(V^*) \to \mathrm{H}^m(W^*).$$

Proof. Let $[v] \in \mathrm{H}^m(V^*)$. We have

$$\begin{aligned}(\mathrm{H}^m(T^*) - \mathrm{H}^m(S^*))([v]) &= [(T^m - S^m)(v)] \\ &= [d_W^{m-1}(\Psi^m(v)) + \Psi^{m+1}(d_V^m(v))] \\ &= [d_W^{m-1}(\Psi^m(v)) + \Psi^{m+1}(0)] \\ &= [d_W^{m-1}(\Psi^m(v))] \\ &= 0.\end{aligned}$$

□

Exercise 3.2.7. Let V^* and W^* be chain complexes. Show that

$$\mathrm{H}^m(V^* \oplus W^*) = \mathrm{H}^m(V^*) \oplus \mathrm{H}^m(W^*)$$

where $V^* \oplus W^*$ has the obvious definition. Extend this to a finite number of chain complexes.

3.3 de Rham Cohomology on Open Subsets of $\mathbb{R}^n$

We set up the basic definitions of de Rham cohomology groups for open subsets of $\mathbb{R}^n$ as well as prove the basic properties in this section. The main purpose of this section is to show that we can generalize the results of § 3.1 and define a chain complex of "differential forms" with the resulting cohomology groups as defined in § 3.2 recovering the ones defined in § 3.1.

Let K be a field of characteristic 0. We will be interested in the case when $K = \mathbb{R}$ or $K = \mathbb{C}$, so one can specialize to those cases immediately if one likes. Let V be a K-vector space. We write V^k for the product $V \times V \times \cdots \times V$ where there are k terms.

Definition 3.3.1. Let F be a field. A map $f : V^k \to F$ is said to be *k-linear* if it is linear in each of the k variables.

In general we will take $F = K$ in the above definition, but we will also be interested in the case that the maps are on a $\mathbb{R}$-vector space but map into $\mathbb{C}$ as in Example 3.3.4 below.

Example 3.3.2. The natural projection maps

$$x_i : \mathbb{R}^k \to \mathbb{R}$$

given by

$$x_i(a_1, \ldots, a_k) = a_i$$

are k-linear for $1 \leq i \leq k$.

Example 3.3.3. The projection maps on $\mathbb{C}^k$ are k-linear as well. We denote them by z_i in this case.

Example 3.3.4. We can consider $\mathbb{C}^n$ as a $2n$-dimensional $\mathbb{R}$-vector space. If we do this, then we have that the maps

$$\overline{z}_i : \mathbb{C}^k \to \mathbb{C}$$

defined by

$$\overline{z}_i(a_1, \ldots, a_k) = \overline{a}_i$$

are k-linear for $1 \leq i \leq k$.

Definition 3.3.5. A k-linear map f is said to be *alternating* if $f(v_1, v_2, \ldots, v_k) = 0$ whenever $v_i = v_j$ for some $i \neq j$. We denote the set of alternating k-linear maps by $\mathrm{Alt}^k(V, K)$. We set $\mathrm{Alt}^0(V, K) = K$.

It is easy to see that for a K-vector space V, the set $\mathrm{Alt}^k(V, K)$ is a K-vector space as well. One should also note that $\mathrm{Alt}^1(V, K)$ is the dual space to V, i.e., $\mathrm{Alt}^1(V, K) = \mathrm{Hom}_K(V, K)$.

Exercise 3.3.6. Show that $\mathrm{Alt}^k(V, K) = 0$ for all $k > \dim_K V$.

Let S_k denote the symmetric group on k letters. As is standard, we write a transposition interchanging i and j by (i, j). Recall that any permutation $\sigma \in S_k$ can be written as a product of transpositions. Moreover, there is a well-defined homomorphism

$$\mathrm{sgn} : S_k \to \{\pm 1\}$$

where $\mathrm{sgn}(\tau) = -1$ for any transposition τ.

Lemma 3.3.7. *Let $f \in \mathrm{Alt}^k(V, K)$ and $\sigma \in S_k$. Then we have*

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \mathrm{sgn}(\sigma) f(v_1, \ldots, v_k).$$

Proof. We prove the result for a transposition $\sigma = (i, j)$. The general result then follows by induction and the fact that any permutation can be written as a product of transpositions. Set $f_{i,j}(v_i, v_j) = f(v_1, \dots, v_i, \dots, v_j, \dots, v_k)$ where we view the v_r's with $r \notin \{i, j\}$ as arbitrary but fixed vectors. We see immediately that $f_{i,j} \in \mathrm{Alt}^2(V, K)$ and so

$$f_{i,j}(v_i + v_j, v_i + v_j) = 0.$$

We use the linearity and the fact that $f_{i,j}$ is alternating to obtain

$$f_{i,j}(v_i, v_j) + f_{i,j}(v_j, v_i) = 0,$$

i.e.,

$$f_{i,j}(v_i, v_j) = -f_{i,j}(v_j, v_i).$$

This gives the result for σ and so finishes the proof. □

Exercise 3.3.8. Let $V = K^2$. Let $v = (v_1, v_2)$ and $w = (w_1, w_2)$ be vectors in V. Show that the map

$$f(v, w) = \det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$$

is alternating. More generally, prove the corresponding statement for $V = K^k$.

Definition 3.3.9. Let m, n be positive integers. A (m, n)-*shuffle* σ is a permutation of $\{1, \dots, m+n\}$ that satisfies

$$\sigma(1) < \cdots < \sigma(m)$$

and

$$\sigma(m+1) < \cdots < \sigma(m+n).$$

The set of all such permutations is denoted $S(m, n)$.

Exercise 3.3.10. Show $\#S(m, n) = \binom{m+n}{m}$.

Definition 3.3.11. Let $f \in \mathrm{Alt}^m(V, K)$ and $g \in \mathrm{Alt}^n(V, K)$. Define

$$(f \wedge g)(v_1, \dots, v_{m+n}) = \sum_{\sigma \in S(m,n)} \mathrm{sgn}(\sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(m)}) g(v_{\sigma(m+1)}, \dots, v_{\sigma(m+n)}).$$

Example 3.3.12. Let $m = n = 1$. Then for f, g as above, we have

$$\begin{aligned}(f \wedge g)(v_1, v_2) &= \sum_{\sigma \in S(1,1)} \mathrm{sgn}(\sigma) f(v_{\sigma(1)}) g(v_{\sigma(2)}) \\ &= f(v_1)g(v_2) - f(v_2)g(v_1).\end{aligned}$$

Example 3.3.13. For $m = 2$, $n = 1$ we have $S(2, 1) = \{1, (1, 2, 3), (2, 3)\}$. Then for f, g as above we have

$$(f \wedge g)(v_1, v_2, v_3) = f(v_1, v_2)g(v_3) + f(v_2, v_3)g(v_1) - f(v_1, v_3)g(v_2).$$

Proposition 3.3.14. *Let $f \in \mathrm{Alt}^m(V,K)$ and $g \in \mathrm{Alt}^n(V,K)$. Then $f \wedge g \in \mathrm{Alt}^{m+n}(V,K)$.*

Proof. It is clear from the definition that $f \wedge g$ is $(m+n)$-linear, so it only remains to check that it is alternating.

Recall that S_k is generated by transpositions $(j, j+1)$. We have seen above that for any $h \in \mathrm{Alt}^r(V,K)$,

$$h(v_1,\ldots,v_j,v_{j+1},\ldots,v_r) = -h(v_1,\ldots,v_{j+1},v_j,\ldots,v_r).$$

Suppose that h satisfies $h(v_1,\ldots,v_r) = 0$ for all r-tuples with $v_j = v_{j+1}$ for some $1 \leq j \leq r-1$. We claim that this implies that h is alternating. Suppose $v_i = v_j$ for some $i \neq j$ with $i < j$. We can write (i,j) a product of transpositions $(k, k+1)$ so we have

$$\begin{aligned} h(v_1,\ldots,v_j,\ldots,v_i,\ldots,v_r) &= -1h(v_1,\ldots,v_{j+1},v_j,\ldots,v_i,\ldots,v_r) \\ &= h(v_1,v_2,\ldots,v_{j+1},v_{j+2},v_j,\ldots,v_i,\ldots,v_r) \\ &= \cdots = (-1)^{j-i-1}h(v_1,v_2,\ldots,v_{j+1},v_{j+2},\ldots,v_{i-1},v_j,v_i,\ldots,v_r) \\ &= 0 \end{aligned}$$

by assumption. Thus, the claim is satisfied and so it is enough to show $(f \wedge g)(v_1,\ldots,v_{m+n}) = 0$ whenever $v_i = v_{i+1}$ for some $1 \leq i \leq m+n-1$. We show the case that $v_1 = v_2$ as the general case is completely analogous.

Let

$$\begin{aligned} S_{(1,2)} &= \{\sigma \in S(m,n) : \sigma(1) = 1, \sigma(m+1) = 2\} \\ S_{(2,1)} &= \{\sigma \in S(m,n) : \sigma(1) = 2, \sigma(m+1) = 1\} \\ S_0 &= S(m,n) - (S_{(1,2)} \cup S_{(2,1)}). \end{aligned}$$

Suppose that $\sigma \in S_0$. We must have either $v_{\sigma(1)} = v_{\sigma(2)}$ or $v_{\sigma(m+1)} = v_{\sigma(m+2)}$ by the definition of $S(m,n)$ and S_0. Thus, for $\sigma \in S_0$ we have either $f(v_{\sigma(1)}, v_{\sigma(2)},\ldots,v_{\sigma(m)}) = 0$ or $g(v_{\sigma(m+1)}, v_{\sigma(m+2)},\ldots,v_{\sigma(m+n)}) = 0$. Thus, we see that we can ignore these terms in the definition of $f \wedge g$ when $v_1 = v_2$. Thus, in our situation we have

$$(f \wedge g)(v_1,\ldots,v_{m+n}) = \sum_{\sigma \in S_{(1,2)} \cup S_{(2,1)}} \mathrm{sgn}(\sigma) f(v_{\sigma(1)},\ldots,v_{\sigma(m)}) g(v_{\sigma(m+1)},\ldots,v_{\sigma(m+n)}).$$

The transposition $\tau = (1,2)$ gives a bijection $S_{(1,2)} \to S_{(2,1)}$, so we can write

$$\begin{aligned} (f \wedge g)(v_1,\ldots,v_{m+n}) = &\sum_{\sigma \in S_{(1,2)}} \mathrm{sgn}(\sigma) f(v_{\sigma(1)},\ldots,v_{\sigma(m)}) g(v_{\sigma(m+1)},\ldots,v_{\sigma(m+n)}) \\ &- \sum_{\sigma \in S_{(1,2)}} \mathrm{sgn}(\sigma) f(v_{\tau\sigma(1)},\ldots,v_{\tau\sigma(m)}) g(v_{\tau\sigma(m+1)},\ldots,v_{\tau\sigma(m+n)}). \end{aligned}$$

Since $\sigma \in S_{(1,2)}$, we have $\sigma(1) = 1$ and $\sigma(m+1) = 2$ and so $\tau\sigma(1) = 2$ and $\tau\sigma(m+1) = 1$ and $\tau\sigma(j) = \sigma(j)$ for $j \notin \{1, m+1\}$. However, $v_1 = v_2$ so we see that

$$
\begin{aligned}
f(v_{\sigma(1)},\ldots,v_{\sigma(m)})g(v_{\sigma(m+1)},\ldots,v_{\sigma(m+n)}) &= f(v_1,v_{\sigma(2)},\ldots,v_{\sigma(m)})g(v_2,v_{\sigma(m+2)},\ldots,v_{\sigma(m+n)})\\
&= f(v_1,v_{\tau\sigma(2)},\ldots,v_{\tau\sigma(m)})g(v_2,v_{\tau\sigma(m+2)},\ldots v_{\tau\sigma(m+n)})\\
&= f(v_2,v_{\tau\sigma(2)},\ldots,v_{\tau\sigma(m)})g(v_1,v_{\tau\sigma(m+2)},\ldots v_{\tau\sigma(m+n)}).
\end{aligned}
$$

This shows that the remaining terms in $f \wedge g$ cancel. The same argument for general $v_i = v_{i+1}$ combined with the observation at the beginning of the proof shows that $f \wedge g \in \mathrm{Alt}^{m+n}(V,K)$. □

Exercise 3.3.15. Given $c \in K$, $f_1, f_2 \in \mathrm{Alt}^m(V,K)$, $g_1, g_2 \in \mathrm{Alt}^n(V,K)$, then

1. $(f_1 + f_2) \wedge g_1 = (f_1 \wedge g_1) + (f_2 \wedge g_1)$
2. $(cf_1) \wedge g_1 = c(f_1 \wedge g_1) = f_1 \wedge (cg_1)$
3. $f_1 \wedge (g_1 + g_2) = (f_1 \wedge g_1) + (f_2 \wedge g_2)$.

Lemma 3.3.16. *Let $f \in \mathrm{Alt}^m(V,K)$, $g \in \mathrm{Alt}^n(V,K)$. Then*

$$f \wedge g = (-1)^{mn} g \wedge f.$$

Proof. Define $\tau \in S(m+n)$ by

$$
\begin{aligned}
\tau(1) = m+1, \tau(2) = m+2, \ldots, \tau(n) = m+n\\
\tau(n+1) = 1, \tau(n+2) = 2, \ldots, \tau(m+n) = m.
\end{aligned}
$$

Note that $\mathrm{sgn}(\tau) = (-1)^{mn}$. The map

$$
\begin{aligned}
S(m,n) &\to S(n,m)\\
\sigma &\mapsto \sigma \circ \tau
\end{aligned}
$$

is a bijection. Observe that

$$g(v_{\sigma\tau(1)},\ldots,v_{\sigma\tau(n)}) = g(v_{\sigma(m+1)},\ldots,v_{\sigma(m+n)})$$

and

$$f(v_{\sigma\tau(n+1)},\ldots,v_{\sigma\tau(m+n)}) = f(v_{\sigma(1)},\ldots,v_{\sigma(m)}).$$

Thus, we have

$$
\begin{aligned}
(g \wedge f)(v_1,\ldots,v_{m+n}) &= \sum_{\sigma \in S(n,m)} \mathrm{sgn}(\sigma) g(v_{\sigma(1)},\ldots,v_{\sigma(n)}) f(v_{\sigma(n+1)},\ldots,v_{\sigma(m+n)})\\
&= \sum_{\sigma \in S(m,n)} \mathrm{sgn}(\sigma\tau) g(v_{\sigma\tau(1)},\ldots,v_{\sigma\tau(n)}) f(v_{\sigma\tau(n+1)},\ldots,v_{\sigma\tau(m+n)})\\
&= (-1)^{mn} \sum_{\sigma \in S(m,n)} \mathrm{sgn}(\sigma) g(v_{\sigma(m+1)},\ldots,v_{\sigma(m+n)}) f(v_{\sigma(1)},\ldots,v_{\sigma(m)})\\
&= (-1)^{mn} (f \wedge g)(v_1,\ldots,v_{m+n}).
\end{aligned}
$$

□

We leave the messy and tedious proof of the following lemma as an exercise. It follows along the same reasoning as in the previous few results.

Lemma 3.3.17. *Let $f \in \mathrm{Alt}^m(V,K)$, $g \in \mathrm{Alt}^n(V,K)$, and $h \in \mathrm{Alt}^r(V,K)$. Then one has*

$$f \wedge (g \wedge h) = (f \wedge g) \wedge h.$$

Definition 3.3.18. A vector space A over K is a *K-algebra* if there is an associative bilinear map

$$m : A \times A \to A.$$

We say the algebra is *unitary* if there exists an element $1_A \in A$ so that $m(1_A, a) = a = m(a, 1_A)$ for all $a \in A$.

Definition 3.3.19. Let $\{A_k\}$ be a sequence of K-vector spaces along with an associative bilinear map

$$m : A_k \times A_l \to A_{k+l}$$

for all k, l. Such a sequence is called a *graded K-algebra*. The elements of A_k are said to have *degree k*.

Recall that $\mathrm{Alt}^0(V,K) = K$. Using this, if we define

$$c \wedge f = cf$$

for $c \in \mathrm{Alt}^0(V,K)$, $f \in \mathrm{Alt}^n(V,K)$, then we have the following theorem.

Theorem 3.3.20. *The sequence $\{\mathrm{Alt}^k(V,K)\}_{k=0}^{\infty}$ along with the wedge product map forms a graded algebra.*

The following lemma will be useful in determining a basis of $\mathrm{Alt}^k(V,K)$.

Lemma 3.3.21. *Let $f_1, \ldots, f_n \in \mathrm{Alt}^1(V,K)$. Then*

$$(f_1 \wedge \cdots \wedge f_n)(v_1, \ldots, v_n) = \det \begin{pmatrix} f_1(v_1) & \cdots & f_1(v_n) \\ \vdots & \ddots & \vdots \\ f_n(v_1) & \cdots & f_n(v_n) \end{pmatrix}.$$

Proof. The case $n = 2$ is clear as we have

$$\begin{aligned}(f_1 \wedge f_2)(v_1, v_2) &= f_1(v_1)f_2(v_2) - f_1(v_2)f_2(v_1) \\ &= \det \begin{pmatrix} f_1(v_1) & f_1(v_2) \\ f_2(v_1) & f_2(v_2) \end{pmatrix}.\end{aligned}$$

We now proceed by induction on n. Assume the result is true for all $2 \leq k < n$. Observe we have

$$\begin{aligned} f_1 \wedge (f_2 \wedge \cdots \wedge f_n)(v_1, \ldots, v_n) &= \sum_{\sigma \in S(1,n-1)} \mathrm{sgn}(\sigma) f_1(v_{\sigma(1)})(f_2 \wedge \cdots \wedge f_n)(v_{\sigma(2)}, \ldots, v_{\sigma(n)}) \\ &= \sum_{j=1}^{n} (-1)^{j+1} f_1(v_j)(f_2 \wedge \cdots \wedge f_n)(v_1, \ldots, \hat{v}_j, \ldots, v_n) \end{aligned}$$

where $(v_1, \ldots, \hat{v}_j, \ldots, v_n)$ indicates the $(n-1)$-tuple with v_j removed.

On the other hand, we have

$$\begin{aligned}
\det \begin{pmatrix} f_1(v_1) & \cdots & f_1(v_n) \\ \vdots & \ddots & \vdots \\ f_n(v_1) & \cdots & f_n(v_n) \end{pmatrix} &= f_1(v_1) \det \begin{pmatrix} f_2(v_2) & \cdots & f_2(v_n) \\ \vdots & \ddots & \vdots \\ f_n(v_2) & \cdots & f_n(v_n) \end{pmatrix} \\
&\quad + \cdots + (-1)^{n+1} f_1(v_n) \det \begin{pmatrix} f_2(v_1) & \cdots & f_2(v_{n-1}) \\ \vdots & \ddots & \vdots \\ f_n(v_1) & \cdots & f_n(v_{n-1}) \end{pmatrix} \\
&= \sum_{j=1}^{n} (-1)^{j+1} f_1(v_j)(f_2 \wedge \cdots \wedge f_n)(v_1, \ldots, \hat{v}_j, \ldots, v_n)
\end{aligned}$$

by our induction hypothesis. Thus, we have the result. □

Corollary 3.3.22. *Forms $f_1, \ldots, f_n \in \mathrm{Alt}^1(V, K)$ are linearly independent if and only if $f_1 \wedge \cdots \wedge f_n \neq 0$.*

Proof. First, suppose that $f_1, \ldots, f_n$ are linearly dependent. Thus, for some $1 \leq j \leq n$ we can write

$$f_j = \sum_{i \neq j} a_i f_i$$

for some $a_i \in K$. For ease of notation we take $j = n$. We have

$$f_1 \wedge \cdots \wedge f_n = \sum_{i=1}^{n-1} a_i (f_1 \wedge \cdots \wedge f_{n-1} \wedge f_i).$$

Thus,

$$\begin{aligned}
(f_1 \wedge \cdots \wedge f_n)(v_1, \ldots, v_n) &= \sum_{i=1}^{n-1} a_i (f_1 \wedge \cdots \wedge f_{n-1} \wedge f_i)(v_1, \ldots, v_n) \\
&= \sum_{i=1}^{n-1} a_i \det \begin{pmatrix} f_1(v_1) & \cdots & f_1(v_n) \\ \vdots & \ddots & \vdots \\ f_{n-1}(v_1) & \cdots & f_{n-1}(v_n) \\ f_i(v_1) & \cdots & f_i(v_n) \end{pmatrix} \\
&= 0
\end{aligned}$$

since each determinant has a repeated row.

Conversely, suppose now that $f_1, \ldots, f_n$ are linearly independent. Then for each j, there exists v_j so that

$$f_i(v_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\det(f_i(v_j)) = 1$ and so it must be the case that $f_1 \wedge \cdots \wedge f_n \neq 0$ by Lemma 3.3.21. □

Recall that given a vector space V over a field K with basis $e_1, \ldots, e_n$, one can define the dual basis $\varepsilon_1, \ldots, \varepsilon_n$ of $\mathrm{Alt}^1(V, K) = \mathrm{Hom}_K(V, K)$ by

$$\varepsilon_i(e_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.3.23. *Let $e_1, \ldots, e_n$ be a basis of V and $\varepsilon_1, \ldots, \varepsilon_n$ the dual basis of $\mathrm{Alt}^1(V, K)$. Then*

$$\{\varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(n)}\}_{\sigma \in S(n, m-n)}$$

is a basis of $\mathrm{Alt}^n(V, K)$. In particular, we see that

$$\dim_K \mathrm{Alt}^n(V, K) = \binom{\dim V}{n}.$$

Proof. One can show (and should as an exercise) that for $f \in \mathrm{Alt}^n(V, K)$ one can write

$$f(v_1, \ldots, v_n) \sum_{\sigma \in S(n, m-n)} f(e_{\sigma(1)}, \ldots, e_{\sigma(n)}) \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(n)}(v_1, \ldots, v_n).$$

Thus, our set spans $\mathrm{Alt}^n(V, K)$ so it only remains to show they are linearly independent.

Suppose we have a relation

$$\sum_{\sigma \in S(n, m-n)} a_\sigma \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(n)} = 0$$

for some $a_\sigma \in K$. If we apply this equation to $(e_{\sigma(1)}, \ldots, e_{\sigma(n)})$ and use the fact that

$$\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_n}(e_{j_1}, \ldots, e_{j_n}) = \begin{cases} 0 & \{i_1, \ldots, i_n\} = \{j_1, \ldots, j_n\} \\ \mathrm{sgn}(\sigma) & \text{otherwise} \end{cases}$$

where σ is the permutation that takes $\{i_1, \ldots, i_n\}$ to $\{j_1, \ldots, j_n\}$ we obtain

$$a_\sigma \,\mathrm{sgn}(\sigma) = 0$$

and so $a_\sigma = 0$. Thus, we have the result. □

Suppose now that we have two vector spaces V and W over K and a linear map $T : V \to W$. For each n we have an induced linear map

$$\mathrm{Alt}^n(T) : \mathrm{Alt}^n(W, K) \to \mathrm{Alt}^n(V, K)$$

given by

$$\mathrm{Alt}^n(T)(f)(v_1, \ldots, v_n) = f(T(v_1), \ldots, T(v_n)).$$

One can easily check that this map is well-defined, i.e., that $\mathrm{Alt}^n(T)(f)$ lies in $\mathrm{Alt}^n(V, K)$ for all $f \in \mathrm{Alt}^n(W, K)$.

Note that we have $\mathrm{Alt}^n(\mathrm{id}) = \mathrm{id}$ and given two linear maps $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$, we have

$$\mathrm{Alt}^n(T_2 \circ T_1) = \mathrm{Alt}^n(T_1) \circ \mathrm{Alt}^n(T_2).$$

This gives that $\mathrm{Alt}^n(\star)$ is a contravariant functor.

Theorem 3.3.24. *The characteristic polynomial of a linear endomorphism $T : V \to V$ is given by*

$$\det(T - x) = \sum_{j=0}^{n} (-1)^j \operatorname{tr}(\mathrm{Alt}^{n-j}(T))x^j$$

where $n = \dim_K V$ and tr denotes the trace map.

We leave the proof of this theorem to the reader. We note that if $\dim_K V = n$ and $T : V \to V$ is a linear map, then necessarily $\mathrm{Alt}^n(T) : K \to K$ is a linear map and so must be multiplication by a constant. Using the theorem we have

$$\begin{aligned}\det(T) &= \sum_{i=0}^{n} (-1)^i \operatorname{tr}(\mathrm{Alt}^{n-i}(T))0^i \\ &= \operatorname{tr}(\mathrm{Alt}^n(T)).\end{aligned}$$

Since $\mathrm{Alt}^n(T)$ is a constant, we see that $\operatorname{tr}(\mathrm{Alt}^n(T)) = \mathrm{Alt}^n(T)$ and thus this constant is precisely $\det(T)$. In particular, we see that $\mathrm{Alt}^n(T)$ acts on K by multiplication by $\det(T)$.

We now restrict to the case that $V = \mathbb{R}^n$, $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$, and $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is the dual basis of $\mathrm{Alt}^1(\mathbb{R}^n)$. We will encompass the case that $V = \mathbb{C}^n$ into this framework as we will view $\mathbb{C}$ as a 2-dimensional $\mathbb{R}$-vector space and so view $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ as a $\mathbb{R}$-vector space. Since we will be interested in C^∞ functions instead of holomorphic ones, this is the appropriate framework in which to work anyways. We write $\mathrm{Alt}^n(V, K)$ as simply $\mathrm{Alt}^n(V)$ now since $K = \mathbb{R}$ from here on.

Let $U \subset \mathbb{R}^n$ be an open set unless otherwise noted.

Definition 3.3.25. A *differential m-form* on U is a smooth map $\omega : U \to \mathrm{Alt}^m(\mathbb{R}^n)$.

The set of differential m-forms clearly forms a $\mathbb{R}$-vector space. We denote this vector space by $\Omega^m(U)$. One also inherits the wedge product which can be defined on differential forms point-wise, i.e.,

$$(\omega_1 \wedge \omega_2)(x) = \omega_1(x) \wedge \omega_2(x).$$

Example 3.3.26. Let $m = 0$ so that $\mathrm{Alt}^0(\mathbb{R}^n) = \mathbb{R}$. Thus, we have that $\Omega^0(U)$ is the vector space of all smooth real-valued functions on U, i.e.,

$$\Omega^0(U) = C^\infty(U, \mathbb{R}).$$

Before we proceed any further we recall some material from multivariable calculus. The reader that feels the need for further review or desires to read proofs of the stated results is advised to consult [11].

Let $F : U \to \mathbb{R}^m$ be a function. Recall that we say F is *differentiable at* $x \in U$ if there exists a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ so that

$$\lim_{h \to 0} \frac{|F(x+h) - F(x) - L(h)|}{|h|} = 0.$$

We denote the linear transformation L by $D_x F$. Note that this depends on where the derivative is taken and so we include x in the notation. However, when it is the case that one gets the same map $D_x F$ for all values of x we simply write DF. As this is a linear transformation, we can also view $D_x F$ as a $m \times n$-matrix. We will change our viewpoint depending on the situation. Some useful properties are summarized in the following proposition.

Proposition 3.3.27. *Let $F : U \to \mathbb{R}^m$ be a differentiable map.*

1. *If F is a constant map then $D_x F = 0$ for all $x \in U$.*
2. *If F is a linear map then $DF = F$.*
3. *If $F = (F_1, \ldots, F_m)$ is differentiable at $a \in U$, then $\frac{\partial F_i}{\partial x_j}(a)$ exists for all $1 \leq i \leq m$, $1 \leq j \leq n$ and $D_a F$ is the $m \times n$-matrix given by $\left(\frac{\partial F_i}{\partial x_j}(a)\right)$.*

From this we see that given $\omega \in \Omega^m(U)$, we can differentiate ω at any $x \in U$ to obtain a linear map $D_x \omega : \mathbb{R}^n \to \mathbb{R}^m$.

Recall that we determined in Theorem 3.3.23 that a basis of $\mathrm{Alt}^m(\mathbb{R}^n)$ is given by

$$\{\varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(n)}\}_{\sigma \in S(m,n-m)}.$$

For $I = (\sigma(1), \ldots, \sigma(n))$, we write ε_I for $\varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(n)}$ to make the notation more bearable. Given any $x \in U$ and $\omega \in \Omega^m(U)$ we have $\omega(x) \in \mathrm{Alt}^m(\mathbb{R}^n)$, we can write

$$\omega(x) = \sum_I \omega_I(x) \varepsilon_I$$

where I runs over all tuples $(\sigma(1), \ldots, \sigma(n))$ for $\sigma \in S(m, n-m)$ and $\omega_I \in C^\infty(U, \mathbb{R})$ for all such I. Using this we see that $D_x \omega$ is the $\binom{n}{m} \times n$-matrix $\left(\frac{\partial \omega_I}{\partial x_j}(x)\right)$. In other words, it is the linear map defined by

$$D_x \omega(e_j) = \sum_I \frac{\partial \omega_I}{\partial x_i}(x) \varepsilon_I.$$

Thus, from this we see that for each $x \in U$ we have $D_x \omega$ is a linear map

$$D_x \omega : \mathbb{R}^n \to \mathrm{Alt}^m(\mathbb{R}^n).$$

Furthermore, the map $x \mapsto D_x \omega$ is a smooth map from U to the vector space of linear maps from $\mathbb{R}^n$ to $\mathrm{Alt}^m(\mathbb{R}^n)$.

Definition 3.3.28. The *exterior derivative* $d^m : \Omega^m(U) \to \Omega^{m+1}(U)$ is the linear operator

$$d_x^m \omega(v_1, \ldots, v_{m+1}) = \sum_{j=1}^{m+1} (-1)^{j-1} D_x\omega(v_j)(v_1, \ldots, \hat{v}_j, \ldots, v_{m+1}).$$

This definition is sufficiently complicated that it merits further description. First, note that saying $d^m\omega \in \Omega^{m+1}(U)$ means that it is a map $U \to \mathrm{Alt}^{m+1}(\mathbb{R}^n)$. Thus, for each $x \in U$ we have that $d_x^m \omega \in \mathrm{Alt}^{m+1}(\mathbb{R}^n)$ and as such is a map from $\mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ where there are $(m+1)$-copies of $\mathbb{R}^n$. This explains the left-hand side of the definition. For the right-hand side, first observe that since $\omega \in \Omega^m(U)$, we have that $\omega : U \to \mathrm{Alt}^m(\mathbb{R}^n)$ and so $D_x\omega$ is a map $\mathbb{R}^n \to \mathrm{Alt}^m(\mathbb{R}^n)$. Thus, $D_x\omega(v_j)$ is a map from $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ to $\mathbb{R}$ where there are m-copies of $\mathbb{R}^n$. This explains each of the terms in the definition. Of course, we still must show the exterior derivative is well-defined, i.e., that $d^m\omega$ is in $\Omega^{m+1}(U)$. Let $x \in U$. It is clear from the definition that $d_x^m\omega$ is $(m+1)$-linear, so it only remains to check that it is alternating. Suppose that $v_i = v_{i+1}$ for some i. Then we have:

$$\begin{aligned} d_x^m \omega(v_1, \ldots, v_{m+1}) &= \sum_{j=1}^{m+1} (-1)^{j-1} D_x\omega(v_j)(v_1, \ldots, \hat{v}_j, \ldots, v_{m+1}) \\ &= (-1)^{i-1} D_x\omega(v_i)(v_1, \ldots, \hat{v}_i, \ldots, v_{m+1}) + (-1)^{i} D_x\omega(v_{i+1})(v_1, \ldots, \hat{v}_{i+1}, \ldots, v_{m+1}) \\ &= 0 \end{aligned}$$

where we have used that each $D_x\omega(v_j)$ is alternating and that $v_i = v_{i+1}$. Thus, d^m is well-defined.

Note that for convenience we write merely $d\omega$ when $m = 0$. This will save considerably on notation.

Recall the projection maps $x_i : \mathbb{R}^n \to \mathbb{R}$ defined by $x_i(a_1, \ldots, a_n) = a_i$. Each of these projection maps lies in $\Omega^0(U) = C^\infty(U, \mathbb{R})$. Thus, each $dx_i \in \Omega^1(U)$. Observe that for $v = \sum_{i=1}^n a_i e_i$ and any $x \in U$ we have

$$\begin{aligned} d_x x_j(v) &= \sum_{i=1}^{n} a_i d_x x_j(e_i) \\ &= a_j d_x x_j(e_j) \\ &= a_j. \end{aligned}$$

Thus, $dx_j \in \Omega^1(U)$ is the constant map $U \to \mathrm{Alt}^1(\mathbb{R}^n)$ given by $x \mapsto \varepsilon_j$, i.e., $d_x x_j = \varepsilon_j$ for $1 \leq j \leq n$ and all $x \in U$. This allows us to write the basis of $\mathrm{Alt}^m(\mathbb{R}^n)$ in terms of the images under the exterior derivative of the projection maps. In particular, we will often write ε_I as dx_I.

Consider now the case of $\mathbb{C}^n$ considered as a $\mathbb{R}$-vector space. When dealing with $\mathbb{C}$, it is common to use z as the variable and write $z = x + iy$. Thus, coordinates on $\mathbb{C}^n$ can be given as $z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n$. We know from our work to this point and the fact that $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as $\mathbb{R}$-vector spaces that $\mathrm{Alt}^1(\mathbb{C}^n, \mathbb{R})$ has a basis consisting of $2n$-linear maps $\varepsilon_1, \ldots, \varepsilon_{2n}$. From what we have just shown, these can be identified with the images $dx_1, dy_1, \ldots, dx_n, dy_n$. The complex conjugation map sending $z = x + iy$ to $\overline{z} = x - iy$ is a linear map. It is not hard to show that the projection map $\overline{z}_i : \mathbb{C}^n \to \mathbb{C}$ given by $\overline{z}_i(a_1, \ldots, a_n) = \overline{a}_i$ is a n-linear map. One can also easily show that the set $\{dz_1, \ldots, dz_n, d\overline{z}_1, \ldots, d\overline{z}_n\}$ spans the same set over $\mathbb{R}$ as $\{dx_1, dy_1, \ldots, dx_n, dy_n\}$ and so it is customary to take this as our basis in the case we are working with $\mathbb{C}^n$. One should note here that this depends heavily on the fact that we are working over $\mathbb{R}$ and only are considering smooth functions. One could easily develop this theory over $\mathbb{C}$ with holomorphic functions. In this case the conjugation projections would play no role as they are not holomorphic and so would not enter into the theory.

Let $f \in C^\infty(U, \mathbb{R}) = \Omega^0(U)$. Then we can write $df \in \Omega^1(U)$ as

$$d_x f(v) = \sum_{j=1}^{1} (-1)^{j-1} D_x f(v_j) = D_x f(v).$$

If we write $v = \sum_{i=1}^n a_i e_i$, then we have

$$\begin{aligned} D_x f(v) &= \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) a_i \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) \varepsilon_i(v). \end{aligned}$$

Thus, we have

$$d_x f = D_x f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) dx_i.$$

Exercise 3.3.29. Show that

$$\varepsilon_i \wedge \varepsilon_I = \begin{cases} 0 & k \in I \\ (-1)^r \varepsilon_J & k \notin I \end{cases}$$

where r is defined by $i_r < k < i_{r+1}$ and $J = (i_1, \ldots, i_r, k, i_{r+1}, \ldots, i_m)$.

The following result allows us to calculate exterior derivatives quickly and easily by relating the m^{th} exterior derivative to a wedge product of a basis element and a 0^{th} exterior derivative.

Lemma 3.3.30. *Let $\omega \in \Omega^m(U)$ and write*

$$\omega(x) = \sum_I \omega_I(x) dx_I.$$

Then

$$d_x^m \omega = \sum_I d_x \omega_I \wedge dx_I.$$

Proof. It is enough to show the statement for $\omega(x) = \omega_I(x) dx_I$ since the exterior derivative is a linear map. Recall that we have from our calculation above that for $v = \sum_{i=1}^n a_i e_i$,

$$\begin{aligned} D_x \omega(v) &= \sum_{i=1}^n a_i D_{x_i} \omega(e_i) \\ &= \left(\sum_{i=1}^n a_i \frac{\partial \omega_I}{\partial x_i}(x) \right) dx_I \\ &= (d_x \omega_I(v)) dx_I, \end{aligned}$$

i.e,

$$D_x \omega(v) = (d_x \omega_I(v)) dx_I.$$

Thus we have

$$\begin{aligned} (d_x \omega_I \wedge dx_I)(v_1, \ldots, v_{m+1}) &= \sum_{j=1}^{m+1} (-1)^{j-1} d_x \omega_I(v_j) dx_I(v_1, \ldots, \hat{v}_j, \ldots, v_{m+1}) \\ &= \sum_{j=1}^{m+1} (-1)^{j-1} D_x \omega(v_j)(v_1, \ldots, \hat{v}_j, \ldots, v_{m+1}) \\ &= d_x^m \omega(v_1, \ldots, v_{m+1}). \end{aligned}$$

□

Example 3.3.31. Consider the function $f(x, y) \in \Omega^0(\mathbb{R}^2)$ given by

$$f(x, y) = \cos(xy) + x^2 y.$$

We have

$$\begin{aligned} df &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \\ &= (-y \sin(xy) + 2xy) dx + (-x \sin(xy) + x^2) dy \in \Omega^1(\mathbb{R}^2). \end{aligned}$$

Thus, in the notation above we have

$$\omega_1(x, y) = -y \sin(xy) + 2xy$$

and

$$\omega_2(x, y) = -x \sin(xy) + x^2.$$

Observe we have

$$d^1(df) = d\omega_1 \wedge dx + d\omega_2 \wedge dy.$$

Now

$$d\omega_1 = (-y^2\cos(xy) + 2y)dx + (-\sin(xy) - xy\cos(xy) + 2x)dy$$

and

$$d\omega_2 = (-\sin(xy) - xy\cos(xy) + 2x)dx + (-x^2\cos(xy))dy.$$

Using the fact that $dx \wedge dx = dy \wedge dy = 0$ we have

$$d\omega_1 \wedge dx = (-\sin(xy) - xy\cos(xy) + 2x)dy \wedge dx$$

and

$$d\omega_2 \wedge dy = (-\sin(xy) - xy\cos(xy) + 2x)dx \wedge dy.$$

Now use the fact that $dx \wedge dy = -dy \wedge dx$ to see that

$$d^1(df) = 0$$

The fact that $d^{m+1}(d^m\omega) = 0$ holds in far greater generality than given in the previous example. Without this fact we would be unable to develop a theory of cohomology.

Proposition 3.3.32. *For $m \geq 0$ the composition*

$$\Omega^m(U) \xrightarrow{d^m} \Omega^{m+1}(U) \xrightarrow{d^{m+1}} \Omega^{m+2}(U)$$

is identically 0, i.e., the collection $\Omega^*(U) = \{\Omega^m(U), d^m\}$ forms a chain complex.

Proof. First, suppose that $\omega(x) = \omega_I(x)dx_I \in \Omega^m(U)$. We have

$$\begin{aligned} d^m_x\omega &= d_x\omega_I \wedge dx_I \\ &= \left(\sum_{i=1}^n \frac{\partial \omega_I}{\partial x_i}(x)dx_i\right) \wedge dx_I \\ &= \sum_{i=1}^n \frac{\partial \omega_I}{\partial x_i}(x)(dx_i \wedge dx_I). \end{aligned}$$

Recall that $dx_i \wedge dx_i = 0$ and $dx_i \wedge dx_j = -dx_j \wedge dx_i$. Using these relations we have

$$\begin{aligned} d^{m+1}(d^m\omega) &= \sum_{i,j=1}^n \frac{\partial^2 \omega_I}{\partial x_j \partial x_i} dx_j \wedge (dx_i \wedge dx_I) \\ &= \sum_{i<j} \left(\frac{\partial^2 \omega_I}{\partial x_j \partial x_i} - \frac{\partial^2 \omega_I}{\partial x_i \partial x_j}\right) dx_j \wedge dx_i \wedge dx_I \\ &= 0 \end{aligned}$$

using the equality of mixed partial derivatives.

Suppose now that we have $\omega(x) = \sum_I \omega_I(x) dx_I \in \Omega^m(U)$. Then

$$d^m \omega = \sum_I d(\omega_I dx_I)$$

and

$$\begin{aligned} d^{m+1}(d^m \omega) &= \sum_I d^{m+1}(d^m \omega_I dx_I) \\ &= \sum_I 0 \\ &= 0. \end{aligned}$$

□

Observe that we have seen that given $\omega_1 \in \Omega^{m_1}(U)$ and $\omega_2 \in \Omega^{m_2}(U)$, that we can define $\omega_1 \wedge \omega_2 \in \Omega^{m_1+m_2}(U)$ by setting

$$(\omega_1 \wedge \omega_2)(x) = \omega_1(x) \wedge \omega_2(x).$$

This also shows that given $f \in C^\infty(U, \mathbb{R}) = \Omega^0(U)$, we have

$$\begin{aligned} (f\omega_1 \wedge \omega_2)(x) &= f(x)\omega_1(x) \wedge \omega_2(x) \\ &= (f(x) \wedge \omega_1(x)) \wedge \omega_2(x) \\ &= f(x) \wedge (\omega_1(x) \wedge \omega_2(x)) \\ &= f(x)(\omega_1(x) \wedge \omega_2(x)) \end{aligned}$$

and so $f\omega_1 \wedge \omega_2 = f(\omega_1 \wedge \omega_2)$. Similarly we have $f(\omega_1 \wedge \omega_2) = \omega_1 \wedge f\omega_2$. Thus, we have $f\omega_1 \wedge \omega_2 = \omega_1 \wedge f\omega_2$. This observation allows us to prove the following proposition.

Proposition 3.3.33. *Let $\omega_1 \in \Omega^{m_1}(U)$ and $\omega_2 \in \Omega^{m_2}(U)$. Then*

$$d^{m_1+m_2}(\omega_1 \wedge \omega_2) = d^{m_1}\omega_1 \wedge \omega_2 = (-1)^{m_1}\omega_1 \wedge d^{m_2}\omega_2.$$

Proof. We again use linearity of the exterior derivative to reduce to the case that $\omega_1 = f dx_I$ and $\omega_2 = f dx_J$. Then we have

$$\omega_1 \wedge \omega_2 = fg(dx_I \wedge dx_J).$$

Thus,

$$\begin{aligned} d^{m_1+m_2}(\omega_1 \wedge \omega_2) &= d(fg) \wedge dx_I \wedge dx_J \\ &= ((df)g + f(dg)) \wedge dx_I \wedge dx_J \\ &= (df)g \wedge dx_I \wedge dx_J + f(dg) \wedge dx_I \wedge dx_J \\ &= df \wedge dx_I \wedge g dx_J + (-1)^{m_1} f dx_I \wedge dg \wedge dx_J \\ &= d^{m_1}\omega_1 \wedge \omega_2 + (-1)^{m_1}\omega_1 \wedge d^{m_2}\omega_2. \end{aligned}$$

□

Theorem 3.3.34. *The map $d^m : \Omega^m(U) \to \Omega^{m+1}(U)$ for $m \geq 0$ is the unique linear operator satisfying*

1. *$f \in \Omega^0(U)$, then $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$;*
2. *$d^{m+1} \circ d^m = 0$;*
3. *for $\omega_i \in \Omega^{m_i}(U)$, $d^{m_1+m_2}(\omega_1 \wedge \omega_2) = d^{m_1}\omega_1 \wedge \omega_2 + (-1)^{m_1}\omega_1 \wedge d^{m_2}\omega_2$.*

We have shown that the exterior derivatives satisfy the above theorem. We leave the proof of uniqueness to the reader.

We are finally ready to define the de Rham cohomology groups. Those familiar with cohomology will have recognized the chain complex of differential forms and the necessary properties we have been pursuing up to this point.

Definition 3.3.35. The m^{th} de Rham cohomology group of U is the quotient space

$$\mathrm{H}^m_{\mathrm{dR}}(U) = \frac{\ker(d^m : \Omega^m(U) \to \Omega^{m+1}(U))}{\operatorname{im}(d^{m-1} : \Omega^{m-1}(U) \to \Omega^m(U))}.$$

We now check that this more general definition arising from the chain complex of differential forms agrees with what we defined in § 3.1. Restrict to the case that $U \subset \mathbb{R}^2$ for this. Let $f(x,y) \in \Omega^0(U)$. Then we have

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= \operatorname{grad}(f) \cdot (dx, dy). \end{aligned}$$

Let $\omega \in \Omega^1(U)$. Write $\omega = f dx + g dy$. Then

$$\begin{aligned} d^1\omega &= df \wedge dx + dg \wedge dy \\ &= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) \wedge dx + \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right) \wedge dy \\ &= \frac{\partial f}{\partial y}dy \wedge dx + \frac{\partial g}{\partial x}dx \wedge dy \\ &= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy \\ &= \operatorname{curl}(f)(dx \wedge dy). \end{aligned}$$

For $\omega \in \Omega^2(U)$, we can write $\omega = f(dx \wedge dy)$. We have

$$\begin{aligned} d^1\omega &= df \wedge dx \wedge dy \\ &= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) \wedge dx \wedge dy \\ &= 0. \end{aligned}$$

Note that we can use the fact that $d^1 \circ d = 0$ to recover the standard fact that $\operatorname{curl}(\operatorname{grad} f) = 0$.

Exercise 3.3.36. Work out the situation for $U \subset \mathbb{R}^3$ and use the fact that $d^2 \circ d^1 = 0$ to show the standard fact that $\mathrm{Div}(\mathrm{curl}\,\mathbf{F}) = 0$.

We refer to the forms $\omega \in \Omega^m(U)$ with $d^m\omega = 0$ as *closed m-forms* and to the forms in $d^{m-1}(\Omega^{m-1}(U))$ as the *exact m-forms*. In this terminology, the cohomology group $\mathrm{H}^m_{\mathrm{dR}}(U)$ measures the failure of the closed forms to be exact.

Note that a closed form $\omega \in \Omega^m(U)$ defines an element $[\omega] \in \mathrm{H}^m_{\mathrm{dR}}(U)$. We see that $[\omega_1] = [\omega_2]$ precisely if $\omega_1 - \omega_2 \in d^{m-1}(\Omega^{m-1}(U))$, i.e., there is a $\omega' \in \Omega^{m-1}(U)$ so that $\omega_1 - \omega_2 = d^{m-1}\omega'$.

It is clear that we have $\mathrm{H}^m_{\mathrm{dR}}(U) = 0$ for $m < 0$ and that $\mathrm{H}^0_{\mathrm{dR}}(U)$ is the kernel of the map

$$d : C^\infty(U, \mathbb{R}) \to \Omega^1(U),$$

i.e., the space of maps f with $\frac{\partial f}{\partial x_i} = 0$ for $i = 1, \ldots, n$. Thus, $\mathrm{H}^0_{\mathrm{dR}}(U)$ is the space of locally constant maps and as in § 3.1 we have the following result.

Proposition 3.3.37. *The number of connected components of U is precisely the dimension of $\mathrm{H}^0_{\mathrm{dR}}(U)$ as a $\mathbb{R}$-vector space.*

Exercise 3.3.38. Let $U_1, \ldots, U_r$ be open sets in $\mathbb{R}^n$ with $U_i \cap U_j = \emptyset$ for all $i \neq j$. Show that

$$\mathrm{H}^k_{\mathrm{dR}}(U_1 \cup \cdots \cup U_r) = \mathrm{H}^k_{\mathrm{dR}}(U_1) \oplus \cdots \oplus \mathrm{H}^k_{\mathrm{dR}}(U_r).$$

Let $\omega_i \in \Omega^{m_i}(U)$. We can define an associative, bilinear, and anti-commutative map called the cup product

$$\cup : \mathrm{H}^{m_1}_{\mathrm{dR}}(U) \times \mathrm{H}^{m_2}_{\mathrm{dR}}(U) \to \mathrm{H}^{m_1+m_2}_{\mathrm{dR}}(U)$$

by setting

$$[\omega_1] \cup [\omega_2] = [\omega_1 \wedge \omega_2].$$

We must check that this is well-defined. Observe that given closed forms ω_1 and ω_2, we have that

$$\begin{aligned} d^{m_1+m_2}(\omega_1 \wedge \omega_2) &= d^{m_1}\omega_1 \wedge \omega_2 + (-1)^{m_1}\omega_1 \wedge d^{m_2}\omega_2 \\ &= 0 + 0 = 0. \end{aligned}$$

Thus, we have that if ω_1 and ω_2 are closed, so is $\omega_1 \wedge \omega_2$. Now let $\omega_1 + d^{m_1-1}\eta_1$ and $\omega_2 + d^{m_2-1}\eta_2$ be different representatives of $[\omega_1]$ and $[\omega_2]$ respectively. We have

$$(\omega_1+d^{m_1-1}\eta_1)\wedge(\omega_2+d^{m_2-1}\eta_2) = \omega_1\wedge\omega_2+\omega_1\wedge d^{m_2-1}\eta_2+d^{m_1-1}\eta_1\wedge\omega_2+d^{m_1-1}\eta_1\wedge d^{m_2-1}\eta_2.$$

Observe that

$$\begin{aligned} &d^{m_1+m_2-1}(\eta_1 \wedge \omega_2 + (-1)^{m_1}\omega_1 \wedge \eta_2 + \eta_1 \wedge d^{m_2-1}\eta_2) \\ &\quad = d^{m_1+m_2-1}(\eta_1 \wedge \omega_2) + (-1)^{m_1} d^{m_1+m_2-1}(\omega_1 \wedge \eta_2) + d^{m_1+m_2-1}(\eta_1 \wedge d^{m_2-1}\eta_2) \\ &\quad = d^{m_1-1}\eta_1 \wedge \omega_2 + (-1)^{m_1-1}\eta_1 \wedge d^{m_2}\omega_2 + (-1)^{m_1} d^{m_1}\omega_1 \wedge \eta_2 \\ &\qquad + (-1)^{2m_1}\omega_1 \wedge d^{m_2-1}\eta_2 + d^{m_1-1}\eta_1 \wedge d^{m_2-1}\eta_2 + \eta_1 \wedge d^{m_2} \circ d^{m_2-1}\eta_2 \\ &\quad = d^{m_1-1}\eta_1 \wedge \omega_2 + \omega_1 \wedge d^{m_2-1}\eta_2 + d^{m_1-1}\eta_1 \wedge d^{m_2-1}\eta_2. \end{aligned}$$

Thus we have

$$\begin{aligned}[][\omega_1 + d^{m_1-1}\eta_1] \cup [\omega_2 + d^{m_2-1}\eta_2] &= [\omega_1 \wedge \omega_2 + d^{m_1+m_2-1}(\eta_1 \wedge \omega_2 + (-1)^{m_1}\omega_1 \wedge \eta_2 + \eta_1 \wedge d^{m_2-1}\eta_2)] \\ &= [\omega_1 \wedge \omega_2] \\ &= [\omega_1] \cup [\omega_2]\end{aligned}$$

and so the cup product is well-defined.

Note that the cup product makes $\mathrm{H}^*_{\mathrm{dR}}(U)$ into a graded algebra. The existence of a cup product is a feature of general cohomology theory as we will see later.

Let $U_1 \subset \mathbb{R}^{n_1}$ and $U_2 \subset \mathbb{R}^{n_2}$ be open sets and let $\phi : U_1 \to U_2$ be a smooth map. Recall that given a linear map $T : V_1 \to V_2$, for each $m \geq 0$ we associated to T a linear map

$$\mathrm{Alt}^m(T) : \mathrm{Alt}^m(V_2) \to \mathrm{Alt}^m(V_1).$$

From this we showed that the map $V \mapsto \mathrm{Alt}^m(V)$ is a contravariant functor. We would like to have the same type of result in this case. Thus, we need to define a linear map $\mathrm{H}^*_{\mathrm{dR}}(\phi)$ and show it has the required properties. We begin by showing that $U \mapsto \Omega^*(U)$ is a contravariant functor. Our first step in constructing the linear map $\mathrm{H}^m_{\mathrm{dR}}(\phi)$ is to construct an induced map on the level of differential forms. We start with the case $m = 0$ so we need an induced map

$$\Omega^0(\phi) : C^\infty(U_2, \mathbb{R}) \to C^\infty(U_1, \mathbb{R}).$$

Let $f \in C^\infty(U_2, \mathbb{R})$. Observe that we have

$$U_1 \xrightarrow{\phi} U_2 \xrightarrow{f} \mathbb{R},$$

so it is natural to define $\Omega^0(\phi)(f) = f \circ \phi$.

Consider now the general case. Let $\omega \in \Omega^m(U_2)$. Thus, ω is a smooth map from U_2 to $\mathrm{Alt}^m(\mathbb{R}^{n_2})$. We define a map

$$\begin{aligned} U_1 &\to \mathrm{Alt}^m(\mathbb{R}^{n_2}) \\ x &\mapsto \omega(\phi(x)). \end{aligned}$$

We now need to define a map from $\mathrm{Alt}^m(\mathbb{R}^{n_2})$ to $\mathrm{Alt}^m(\mathbb{R}^{n_1})$. The map $D_x\phi : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ induces a map

$$\mathrm{Alt}^m(D_x\phi) : \mathrm{Alt}^m(\mathbb{R}^{n_2}) \to \mathrm{Alt}^m(\mathbb{R}^{n_1}).$$

Thus, we define the map $\Omega^m(\phi) : \Omega^m(U_2) \to \Omega^m(U_1)$ by

$$\Omega^m(\phi)(\omega)_x = \mathrm{Alt}^m(D_x\phi) \circ \omega(\phi(x)).$$

We call $\Omega^m(\phi)$ the pullback map of ϕ. One should note that often $\Omega^m(\phi)$ is written as simply ϕ^* and the m is then to be understood from context.

We now must check that $\Omega^m(\phi)$ has the desired properties. Note that the chain rule gives that for $\phi : U_1 \to U_2$ and $\psi : U_2 \to U_3$ we have

$$D_x(\psi \circ \phi) = D_{\phi(x)}(\psi) \circ D_x(\phi).$$

Thus, for $\omega \in \Omega^m(U_3)$ and $x \in U_1$ we have

$$\begin{aligned}\Omega^m(\psi \circ \phi)(\omega)_x &= \mathrm{Alt}^m(D_x(\psi \circ \phi)) \circ \omega(\psi \circ \phi(x)) \\ &= \mathrm{Alt}^m(D_{\phi(x)}(\psi) \circ D_x(\phi)) \circ \omega(\psi \circ \phi(x)) \\ &= \mathrm{Alt}^m(D_x(\phi)) \circ \mathrm{Alt}^m(D_{\phi(x)}(\psi)) \circ \omega(\psi \circ \phi(x))\end{aligned}$$

where we have used that $\mathrm{Alt}^m(T \circ S) = \mathrm{Alt}^m(S) \circ \mathrm{Alt}^m(T)$. We know that

$$\mathrm{Alt}^m(D_{\phi(x)}(\psi)) : \mathrm{Alt}^m(\mathbb{R}^{n_3}) \to \mathrm{Alt}^m(\mathbb{R}^{n_2})$$

and so $\mathrm{Alt}^m(D_{\phi(x)}(\psi)) \circ \omega(\psi \circ \phi(x)) \in \mathrm{Alt}^m(\mathbb{R}^{n_2})$. Thus we have

$$\begin{aligned}\Omega^m(\psi \circ \phi)(\omega)_x &= \mathrm{Alt}^m(D_x(\phi))(\mathrm{Alt}^m(D_{\phi(x)}(\psi)) \circ \omega(\psi \circ \phi(x))) \\ &= \Omega^m(\phi) \circ \Omega^m(\psi)(\omega)_x\end{aligned}$$

as desired. Similarly one can check that $\Omega^m(\mathrm{id}) = \mathrm{id}_{\Omega^m(U)}$.

Exercise 3.3.39.

1. Show that $\Omega^0(\psi \circ \phi) = \Omega^0(\phi) \circ \Omega^m(\psi)$.
2. If $\phi : U_1 \to U_2$ is the inclusion map, then $\Omega^m(\phi)(\omega) = \omega \circ \phi$ for any m.

Example 3.3.40. Let $\phi : U_1 \to U_2$ be as above and consider the map $dx_i \in \Omega^1(U_2)$, i.e., the map that sends $x \in U_2$ to $\varepsilon_i \in \mathrm{Alt}^1(\mathbb{R}^{n_2})$. Recall that given a map $T : V_1 \to V_2$, the induced map on the spaces of alternating forms is given by

$$\mathrm{Alt}^m(T)(f)(v_1, \ldots, v_m) = f(T(v_1), \ldots, T(v_m)).$$

We apply this to our situation to see that for $x \in U_1$ and $v = \sum_{i=1}^{n_1} a_i e_i \in \mathbb{R}^{n_1}$, we have

$$\Omega^1(\phi)(dx_i)_x(v) = (dx_i)_{\phi(x)}(D_x\phi(v)) = \varepsilon_i(D_x\phi(v))$$

where we use that dx_i is constant. We have

$$D_x\phi(v) = \sum_{k=1}^{n_2}\left(\sum_{j=1}^{n_1}\frac{\partial \phi_k}{\partial x_j}(x)a_j\right)e_k$$

where $\phi = (\phi_1, \ldots, \phi_{n_2})$. Thus,

$$\begin{aligned}\Omega^1(\phi)(dx_i)_x(v) &= \varepsilon_i(D_x\phi(v)) \\ &= \varepsilon_i\left(\sum_{k=1}^{n_2}\left(\sum_{j=1}^{n_1}\frac{\partial \phi_k}{\partial x_j}(x)a_j\right)e_k\right) \\ &= \sum_{j=1}^{n_1}\frac{\partial \phi_i}{\partial x_j}(x)a_j \\ &= \sum_{j=1}^{n_1}\frac{\partial \phi_i}{\partial x_j}(x)\varepsilon_j(v) \\ &= d_x\phi_i(v).\end{aligned}$$

So we have

$$\Omega^1(\phi)(dx_i) = d\phi_i.$$

Theorem 3.3.41. *Let $U_i \in \mathbb{R}^{n_i}$ be open sets for $i = 1, 2$ and $\phi : U_1 \to U_2$ be a smooth map. Then we have*

1. $\Omega^{m_1+m_2}(\phi)(\omega \wedge \tau) = \Omega^{m_1}(\phi)(\omega) \wedge \Omega^{m_2}(\phi)(\tau)$ *for* $\omega \in \Omega^{m_1}(U_2)$ *and* $\tau \in \Omega^{m_2}(U_2)$*;*
2. $\Omega^0(\phi)(f) = f \circ \phi$ *for* $f \in \Omega^0(U_2)$*;*
3. $d^{m_1}\Omega^{m_1}(\phi)(\omega) = \Omega^{m_1+1}(d^{m_1}\omega)$.

Conversely, if $T^m : \Omega^m(U_2) \to \Omega^m(U_1)$ is a collection of linear maps satisfying the above properties, then $T^m = \Omega^m(\phi)$.

Proof. We leave the case of $m_1 m_2 = 0$ as an exercise, so assume $m_1 > 0$ and $m_2 > 0$. Let $x \in U_1$ and let $v_1, \ldots, v_{m_1+m_2}$ be vectors in $\mathbb{R}^{n_1}$. Then we have

$$\begin{aligned}
\Omega^{m_1+m_2}(\phi)(\omega \wedge \tau)(v_1, \ldots, v_{m_1+m_2}) &= (\omega \wedge \tau)_{\phi(x)}(D_x\phi(v_1), \ldots, D_x\phi(v_{m_1+m_2})) \\
&= \sum \operatorname{sgn}(\sigma) \left[\omega_{\phi(x)}(D_x\phi(v_{\sigma(1)}), \ldots, D_x\phi(v_{\sigma(m_1)}))\right] \\
&\quad \cdot \left[\tau_{\phi(x)}(D_x\phi(v_{\sigma(m_1+1)}), \ldots, D_x\phi(v_{\sigma(m_1+m_2)}))\right] \\
&= \sum \operatorname{sgn}(\sigma)\Omega^{m_1}(\phi)(\omega)_x(v_{\sigma(1)}, \ldots, v_{\sigma(m_1)})\Omega^{m_2}(\phi)(\tau)_x(v_{\sigma(m_1+1)}, \ldots, v_{\sigma(m_1+m_2)}) \\
&= (\Omega^{m_1}(\phi)(\omega)_x \wedge \Omega^{m_2}(\phi)(\tau)_x)(v_1, \ldots, v_{m_1+m_2}),
\end{aligned}$$

which gives the first statement.

We have already shown the second statement, so it only remains to prove the third. First we consider the case that $f \in \Omega^0(U_2)$. We wish to show that

$$d\Omega^0(\phi)(f) = \Omega^1(\phi)(df).$$

Recall that we have

$$df = \sum_{j=1}^{n_2} \frac{\partial f}{\partial x_j} dx_j = \sum_{j=1}^{n_2} \frac{\partial f}{\partial x_j} \wedge dx_j.$$

We can use the first two properties now to conclude that

$$\begin{aligned}
\Omega^1(\phi)(df) &= \sum_{j=1}^{n_2} \left(\Omega^0(\phi) \left(\frac{\partial f}{\partial x_j} \right) \wedge \Omega^1(\phi)(dx_j) \right) \\
&= \sum_{j=1}^{n_2} \left(\frac{\partial f}{\partial x_j} \circ \phi \wedge \Omega^1(\phi)(dx_j) \right) \\
&= \sum_{j=1}^{n_2} \left(\frac{\partial f}{\partial x_j} \circ \phi \wedge \left(\sum_{i=1}^{n_1} \frac{\partial \phi_j}{\partial x_i} dx_i \right) \right) \\
&= \sum_{j=1}^{n_2} \sum_{i=1}^{n_1} \left(\frac{\partial f}{\partial x_j} \circ \phi \right) \left(\frac{\partial \phi_j}{\partial x_i} \right) dx_i \\
&= \sum_{i=1}^{n_1} \left(\sum_{j=1}^{n_2} \left(\frac{\partial f}{\partial x_j} \circ \phi \right) \frac{\partial \phi_j}{\partial x_i} \right) dx_i \\
&= \sum_{i=1}^{n_1} \left(\frac{\partial (f \circ \phi)}{\partial x_i} \right) dx_i \\
&= d(f \circ \phi) \\
&= d(\Omega^0(\phi)(f)).
\end{aligned}$$

This gives the third statement for the case $m = 0$. For the general case we again use linearity to reduce to the case $\omega = f dx_I = f \wedge dx_I$. Recall that we have $d^{m_1}\omega = df \wedge dx_I$. Thus,

$$\begin{aligned}
\Omega^{m+1+1}(\phi)(d^{m_1}\omega) &= \Omega^{m_1+1}(\phi)(df \wedge dx_I) \\
&= \Omega^1(\phi)(df) \wedge \Omega^{m_1}(\phi)(dx_I) \\
&= d(\Omega^0(\phi)(f)) \wedge \Omega^{m_1}(\phi)(dx_I).
\end{aligned}$$

Observe we have

$$\begin{aligned}
d^{m_1}\Omega^{m_1}(\phi)(dx_I) &= d^{m_1}(\Omega^1(\phi)(dx_{i_1}) \wedge \cdots \wedge \Omega^1(\phi)(dx_{i_{m_1}})) \\
&= \sum_{j=1}^{m_1} (-1)^{j-1} \Omega^1(\phi)(dx_{i_1}) \wedge \cdots \wedge d^1(\Omega^1(\phi)(dx_{i_j})) \wedge \cdots \wedge \Omega^1(\phi)(dx_{i_{m_1}}) \\
&= 0
\end{aligned}$$

since $\Omega^1(\phi)(dx_i) = d\phi_i$ and $d^1 \circ d = 0$. Thus,

$$\begin{aligned}
\Omega^{m_1+1}(\phi)(d^{m_1}\omega) &= d(\Omega^0(\phi)(f)) \wedge \Omega^{m_1}(\phi)(dx_I) \\
&= d^{m_1}(\Omega^0(\phi)(f) \wedge \Omega^{m_1}(\phi)(dx_I)) \\
&= d^{m_1}(\Omega^{m_1}(\phi)(\omega)).
\end{aligned}$$

We leave the proof of the uniqueness to the following exercise. □

Exercise 3.3.42. 1. Prove the first statement in Theorem 3.3.41 in the case that $m_1 m_2 = 0$.

2. Prove the uniqueness claimed in Theorem 3.3.41.

Example 3.3.43. Let $\phi : (a, b) \to U$ be a smooth curve with $U \subset \mathbb{R}^n$. Write $\phi = (\phi_1, \ldots, \phi_n)$. Let $\omega \in \Omega^1(U)$ be given by

$$\omega = f_1 dx_1 + \cdots f_n dx_n.$$

Then we have

$$\begin{aligned}
\Omega^1(\phi)(\omega) &= \sum_{i=1}^n \Omega^0(\phi)(f_i) \wedge \Omega^1(\phi)(dx_i) \\
&= \sum_{i=1}^n \Omega^0(\phi)(f_i)\Omega^1(\phi)(dx_i) \\
&= \sum_{i=1}^n \Omega^0(\phi)(f_i) d^1(\Omega^0(\phi)(x_i)) \\
&= \sum_{i=1}^n (f_i \circ \phi) d^1(\Omega^0(\phi)(x_i)) \\
&= \sum_{i=1}^n (f_i \circ \phi) d\phi_i \\
&= \sum_{i=1}^n (f_i \circ \phi) \frac{d\phi_i}{dt} dt \\
&= \langle f(\phi(t)), \phi'(t) \rangle dt
\end{aligned}$$

where in this case we write $\langle \, , \rangle$ for the inner product on $\mathbb{R}^n$. One should compare this with the integrand for a line integral.

Exercise 3.3.44. Let $\phi : U_1 \to U_2$ be a smooth map. Show that

$$\Omega^n(\phi)(dx_1 \wedge \cdots \wedge dx_n) = \det(D_x \phi)(dx_1 \wedge \cdots \wedge dx_n).$$

We can now define the linear map $\mathrm{H}^m_{\mathrm{dR}}(\phi)$ for $\phi : U_1 \to U_2$ a smooth map where $U_i \subset \mathbb{R}^{n_i}$ are open sets. For $[\omega] \in \mathrm{H}^m_{\mathrm{dR}}(U_2)$, set

$$\mathrm{H}^m_{\mathrm{dR}}(\phi)([\omega]) = [\Omega^m(\phi)(\omega)].$$

As usual, we must show this map is well-defined. First we must show that if ω is closed, then $\Omega^m(\phi)(\omega)$ is closed as well. This is shown in the third part of Theorem 3.3.41. Now let $\omega + d^{m-1}\eta$ be another representative of $[\omega]$. We have

$$\begin{aligned}
\mathrm{H}^m_{\mathrm{dR}}(\phi)([\omega + d^{m-1}\eta]) &= [\Omega^m(\phi)(\omega + d^{m-1}\eta)] \\
&= [\Omega^m(\phi)(\omega) + \Omega^m(\phi)(d^{m-1}\eta)] \\
&= [\Omega^m(\phi)(\omega) + d^{m-1}(\Omega^{m-1}(\phi)(\eta))] \\
&= [\Omega^m(\phi)(\omega)] \\
&= \mathrm{H}^m_{\mathrm{dR}}(\phi)(\omega)
\end{aligned}$$

and so $\mathrm{H}^m_{\mathrm{dR}}(\phi)$ is well-defined. Observe that for $[\omega_1] \in \mathrm{H}^{m_1}_{\mathrm{dR}}(U_2)$, $[\omega_2] \in \mathrm{H}^{m_2}_{\mathrm{dR}}(U_2)$ we have

$$\begin{aligned}
\mathrm{H}^{m_1+m_2}_{\mathrm{dR}}(\phi)([\omega_1] \cup [\omega_2]) &= [\Omega^{m_1+m_2}(\phi)(\omega_1 \wedge \omega_2)] \\
&= [\Omega^{m_1}(\phi)(\omega_1) \wedge \Omega^{m_2}(\phi)(\omega_2)] \\
&= [\Omega^{m_1}(\phi)(\omega_2)] \cup [\Omega^{m_2}(\phi)(\omega_2)] \\
&= \mathrm{H}^{m_1}_{\mathrm{dR}}(\phi)([\omega_1]) \cup \mathrm{H}^{m_2}_{\mathrm{dR}}(\phi)([\omega_2])
\end{aligned}$$

and so $\mathrm{H}^*_{\mathrm{dR}}(\phi)$ is a homomorphism of graded algebras from $\mathrm{H}^*_{\mathrm{dR}}(U_2)$ to $\mathrm{H}^*_{\mathrm{dR}}(U_1)$.

We finish this section by proving a more general version of Theorem 3.1.5 by constructing a chain homotopy between the identity map and 0.

Theorem 3.3.45. *(Poincare's Lemma) Let $U \subset \mathbb{R}^n$ be a star-shaped open set. Then $\mathrm{H}^m_{\mathrm{dR}}(U) = 0$ for $m > 0$ and $\mathrm{H}^0_{\mathrm{dR}}(U) = \mathbb{R}$.*

Proof. For simplicity we assume U is star-shaped with respect to 0. We claim there is a linear map

$$\Psi^m : \Omega^m(U) \to \Omega^{m-1}(U)$$

so that

$$d^{m-1}\Psi^m + \Psi^{m+1}d^m = \mathrm{id}$$

when $m > 0$ and

$$\Psi^1 d = \mathrm{id} - e$$

where $e(\omega) = \omega(0)$ for $\omega \in \Omega^0(U)$. Granting the existence of this map, observe that we have for a closed form $\omega \in \Omega^m(U)$,

$$d^{m-1}\Psi^m(\omega) = \omega$$

since $\Psi^{m+1}(d^m\omega) = \Psi^{m+1}(0) = 0$. Thus, $[\omega] = [d^{m-1}\Psi^m(\omega)] = [0]$ for any closed form ω with $m > 0$.

For $m = 0$ we have

$$\omega - \omega(0) = \Psi^1(d\omega) = 0$$

since we are assuming ω is a closed form. Thus, ω is a constant. Hence, if we can show the claim we are done.

We begin by constructing a map

$$\widetilde{\Psi}^m : \Omega^m(U \times \mathbb{R}) \to \Omega^{m-1}(U).$$

Let $\omega \in \Omega^m(U \times \mathbb{R})$. We can write

$$\omega = \sum_I f_I(x,t)dx_I + \sum_J g_J(x,t)dt \wedge dx_J$$

where $I = (i_1, \ldots, i_m)$ and $J = (j_1, \ldots, j_{m-1})$. Define

$$\widetilde{\Psi}^m(\omega) = \sum_J \left(\int_0^1 g_J(x,t)dt \right) dx_J.$$

Note that we have

$$d^{m-1}\widetilde{\Psi}^m(\omega) = \sum_{J,k} \left(\int_0^1 \frac{\partial g_J}{\partial x_k}(x,t)dt \right) dx_k \wedge dx_J$$

and since

$$d^m\omega = \sum_{I,i} \frac{\partial f_I(x,t)}{\partial x_i} dx_i \wedge dx_I + \sum_{J,j} \frac{\partial g_J(x,t)}{\partial x_j} dx_j \wedge dt \wedge dx_J,$$

we have

$$\widetilde{\Psi}^{m+1}(d^m\omega) = \sum_I \left(\int_0^1 \frac{\partial f_I}{\partial t}(x,t)dt \right) dx_I - \sum_{J,j} \left(\int_0^1 \frac{\partial g_J}{\partial x_j} dt \right) dx_j \wedge dx_J.$$

Thus,

$$\begin{aligned} d^{m-1}\widetilde{\Psi}^m(\omega) + \widetilde{\Psi}^{m+1}(d^m\omega) &= \sum_I \left(\int_0^1 \frac{\partial f_I}{\partial t}(x,t)dt \right) dx_I \\ &= \sum_I f_I(x,1)dx_I - \sum_I f_I(x,0)dx_I. \end{aligned} \tag{3.1}$$

Define $\psi(t)$ to be a smooth function so that

$$\begin{aligned} \psi(t) &= 0 \quad \text{if } t \leq 0 \\ \psi(t) &= 1 \quad \text{if } t \geq 1 \\ 0 \leq \psi(t) &\leq 1 \quad \text{otherwise} \end{aligned}$$

Set

$$\begin{aligned} \phi : U \times \mathbb{R} &\to U \\ \phi(x,t) &= \psi(t)x. \end{aligned}$$

This is well-defined because U is star-shaped.

Define

$$\Psi^m(\omega) = \widetilde{\Psi}(\Omega^m(\phi)(\omega))$$

with $\widetilde{\Psi}^m$ defined as above. Write $\omega = \sum_I h_I(x)dx_I$. Observe that we have

$$\begin{aligned} \Omega^1(dx_i) &= d\phi_i \\ &= d((\psi(t)x)_i) \\ &= x_i\psi'(t)dt + \psi(t)dx_i. \end{aligned}$$

Thus,

$$\begin{aligned}
\Omega^m(\phi)(\omega) &= \Omega^m(\phi)(\sum_I h_I(x)dx_I) \\
&= \sum_I \Omega^m(\phi)(h_I(x)dx_I) \\
&= \sum_I \Omega^0(\phi)(h_I(x)) \wedge \Omega^m(\phi)(dx_I) \\
&= \sum_I (h_I(\psi(t)x)(d\psi(t)x_{i_1} + \psi(t)dx_{i_1}) \wedge \cdots \wedge (d\psi(t)x_{i_m} + \psi(t)dx_{i_m})).
\end{aligned}$$

In our notation, we have

$$\sum_I f_I(x,t)dx_I = \sum_I h_I(\psi(t)x)\psi(t)^m dx_I.$$

Applying equation (3.1) to this case we have

$$\begin{aligned}
d^{m-1}\Psi^m(\omega) + \Psi^{m+1}(d^m\omega) &= d^{m-1}\widetilde{\Psi}^m(\Omega^m(\phi)(\omega) + \widetilde{\Psi}^{m+1}(d^m\Omega^m(\phi)(\omega)) \\
&= \sum_I f_I(x,1)dx_I - \sum_I f_I(x,0)dx_I \\
&= \sum_I h_I(\psi(1)x)\psi(1)^m dx_I - \sum_I h_I(\psi(0)x)\psi(0)^m dx_I \\
&= \sum_I h_I(x)dx_I \\
&= \omega.
\end{aligned}$$

If $m = 0$, we have that $d^{m-1} = 0$ and so our equation above reads

$$\begin{aligned}
\Psi^1(d\omega) &= \sum_I h_I(\psi(1)x)dx_I - \sum_I h_I(\psi(0)x)dx_I \\
&= \omega - \omega(0),
\end{aligned}$$

as claimed. This completes the proof of the result. □

3.4 Calculations and Applications of de Rham Cohomology in $\mathbb{R}^n$

In this section we will develop more general ways to calculate de Rham cohomology groups. In particular, we will prove the exactness of the Mayer-Vietoris sequence, a very powerful tool for computing examples. Up to this point we have only been able to calculate $\mathrm{H}^m_{\mathrm{dR}}(U)$ for U a star-shaped open set in $\mathbb{R}^n$. This calculation will turn out to be very useful in more complicated examples as we will shortly see.

Note that throughout this section U and V denote open sets in Euclidean space unless noted otherwise.

Before we proceed, we need to introduce partitions of unity. These are important for working with smooth functions and we will encounter them often in this section.

Let $U \subset \mathbb{R}^n$ be any set, not necessarily open. Let $f : U \to \mathbb{R}$ be a function. The *support of f in U* is the set

$$\operatorname{supp}_U(f) = \operatorname{Cl}(\{x \in U : f(x) \neq 0\}).$$

If U happens to be open then $U - \operatorname{supp}_U(f)$ is the largest open subset of U on which f vanishes.

We state the following standard fact from analysis without proof. One can see Chapter 1, § 8 of [6] for a proof of this fact.

Theorem 3.4.1. *Let $U \subset \mathbb{R}^n$ be open and $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of U. There exist smooth functions*

$$f_i : U \to [0, 1]$$

satisfying

1. $\operatorname{supp}_U(f_i) \subset U_i$ *for every* $i \in I$;
2. *Every $x \in U$ has a neighborhood V on which only finitely many of the f_i do not vanish;*
3. *For every $x \in U$, $\sum_{i \in I} f_i(x) = 1$.*

The functions $\{f_i\}_{i \in I}$ are called a *partition of unity*. The terminology arises from the third property listed in the theorem.

The following theorem is a precursor to the Mayer-Vietoris sequence.

Theorem 3.4.2. *Let $U_1, U_2 \subset \mathbb{R}^n$ be open sets. Set $U = U_1 \cup U_2$. Write*

$$i_k : U_k \hookrightarrow U$$

and

$$j_k : U_1 \cap U_2 \hookrightarrow U_k$$

be he natural inclusion maps for $k = 1, 2$. We have the following short exact sequence

$$0 \longrightarrow \Omega^m(U) \xrightarrow{i^m} \Omega^m(U_1) \oplus \Omega^m(U_2) \xrightarrow{j^m} \Omega^m(U_1 \cap U_2) \longrightarrow 0$$

where $i^m(\omega) = (\Omega^m(i_1)(\omega), \Omega^m(i_2)(\omega))$ and $j^m(\omega_1, \omega_2) = \Omega^m(j_1)(\omega_1) - \Omega^m(j_2)(\omega_2)$.

Proof. Recall that given open sets V_1, V_2 and a smooth map $\phi : V_1 \to V_2$, we defined

$$\Omega^m(\phi) : \Omega^m(V_2) \to \Omega^m(V_1)$$

by

$$\Omega^m(\phi)(\omega) = \sum_I (f_I \circ \phi) d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_m}$$

for

$$\omega = \sum_I f_I dx_I.$$

We have also shown that if ϕ is an inclusion map of open sets, then $\phi_i(x) = x_i$ and so

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_m} = dx_{i_1} \wedge \cdots \wedge dx_{i_m}.$$

Thus, for ϕ an inclusion map we have

$$\Omega^m(\phi)(\omega) = \sum_I (f_I \circ \phi) dx_I.$$

We can now apply these results to our inclusions i_k and j_k.

Suppose there is a $\omega \in \Omega^m(U)$ so that $i^m(\omega) = 0$, i.e., $(\Omega^m(i_1)(\omega), \Omega^m(i_2)(\omega)) = (0, 0)$. Thus, we must have

$$\Omega^m(i_k)(\omega) = \sum_I (f_I \circ i_k) dx_I = 0.$$

However, this is the case if and only if $f_I \circ i_k = 0$ for all I since the dx_I form a basis. Thus, $f_I \circ i_1 = 0 = f_I \circ i_2$ for all I and so $f_I = 0$ on U. Thus, $\omega = 0$ and so i^m is injective.

The next step is to show that

$$\ker(j^m) = \operatorname{im}(i^m).$$

Let $\omega \in \Omega^m(U)$. Define $j : U_1 \cap U_2 \hookrightarrow U$ to be the natural inclusion map. Observe that $j = i_k \circ j_k$ for $k = 1, 2$. We have

$$\begin{aligned}
j^m(i^m(\omega)) &= j^m((\Omega^m(i_1)(\omega), \Omega^m(i_2)(\omega)) \\
&= \Omega^m(j_1)(\Omega^m(i_1)(\omega)) - \Omega^m(j_2)(\Omega^m(i_2)(\omega)) \\
&= \Omega^m(j)(\omega) - \Omega^m(j)(\omega) \\
&= 0.
\end{aligned}$$

Thus, we have that $\operatorname{im}(i^m) \subset \ker(j^m)$.

Let $(\omega_1, \omega_2) \in \Omega^m(U_1) \oplus \Omega^m(U_2)$ so that $j^m(\omega_1, \omega_2) = 0$. Write $\omega_1 = \sum_I f_I dx_I$ and $\omega_2 = \sum_I g_I dx_I$. Since $j^m(\omega_1, \omega_2) = 0$, we must have $\Omega^m(j_1)(\omega_1) = \Omega^m(j_2)(\omega_2)$. Thus, we must have

$$\sum_I (f_I \circ j_1) dx_I = \sum_I (g_I \circ j_2) dx_I,$$

i.e.,

$$f_I \circ j_1 = g_I \circ j_2$$

for all I. This statement is equivalent to $f_I(x) = g_I(x)$ for all $x \in U_1 \cap U_2$. Define

$$h_I(x) = \begin{cases} f_I(x) & x \in U_1 \\ g_I(x) & x \in U_2. \end{cases}$$

Note that $h_I(x)$ is well-defined since f_I and g_I agree on $U_1 \cap U_2$ and it is clearly a smooth function. We have $i^m(\sum_I h_I dx_I) = (\omega_1, \omega_2)$. Thus, $\ker(j^m) \subset \mathrm{im}(i^m)$ and so we have equality.

It remains to show that j^m is surjective. Let p_1, p_2 be a partition of unity of U with support $\{U_1, U_2\}$. Let $f : U_1 \cap U_2 \to \mathbb{R}$ be a smooth function. We extend f to a function on U_1 and a function on U_2 via the partition of unity. Set

$$f_2(x) = \begin{cases} -f(x)p_1(x) & x \in U_1 \cap U_2 \\ 0 & x \in U_2 - \mathrm{supp}_U(p_1). \end{cases}$$

This is smooth because $\mathrm{supp}_U(p_1) \cap U_2 \subset U_1 \cap U_2$. Similarly, define

$$f_1(x) = \begin{cases} f(x)p_2(x) & x \in U_1 \cap U_2 \\ 0 & x \in U_1 - \mathrm{supp}_U(p_2). \end{cases}$$

We have $f(x) = f_1(x) - f_2(x)$ for all $x \in U_1 \cap U_2$.

Now for $\omega = \sum_I f_I dx_I \in \Omega^m(U_1 \cap U_2)$, we can apply the construction to each $f_I : U_1 \cap U_2 \to \mathbb{R}$ giving functions $f_{I,k} : U_k \to \mathbb{R}$ and differential forms $\omega_k = \sum_I f_{I,k} dx_I \in \Omega^m(U_k)$ for $k = 1, 2$. So by construction we have $j^m(\omega_1, \omega_2) = \omega$ and so j^m is indeed surjective. □

From this the proof of the Mayer-Vietoris sequence follows easily.

Theorem 3.4.3. *(Mayer-Vietoris sequence) Let $U_1, U_2 \subset \mathbb{R}^n$ be open sets, $U = U_1 \cup U_2$, and i^m, j^m defined as in Theorem 3.4.2. There exists a long exact sequence of vector spaces*

$$\cdots \longrightarrow \mathrm{H}^m_{\mathrm{dR}}(U) \xrightarrow{\mathrm{H}^m_{\mathrm{dR}}(i^m)} \mathrm{H}^m_{\mathrm{dR}}(U_1) \oplus \mathrm{H}^m_{\mathrm{dR}}(U_2) \xrightarrow{\mathrm{H}^m_{\mathrm{dR}}(j^m)} \mathrm{H}^m_{\mathrm{dR}}(U_1 \cap U_2) \xrightarrow{\partial^m} \mathrm{H}^{m+1}_{\mathrm{dR}}(U) \longrightarrow \cdots.$$

Proof. We have that i^* and j^* are chain maps and so we can apply Theorem 3.4.2 to get a short exact sequence of chain complexes. Theorem 3.2.3 along with Exercise 3.2.7 finish the result. □

Exercise 3.4.4. Work out the map ∂^m explicitly in this case.

Exercise 3.4.5. Let U_1 and U_2 be disjoint open sets in $\mathbb{R}^n$. Then

$$i : \mathrm{H}^m_{\mathrm{dR}}(U_1 \cup U_2) \xrightarrow{\simeq} \mathrm{H}^m_{\mathrm{dR}}(U_1) \oplus \mathrm{H}^m_{\mathrm{dR}}(U_2).$$

Example 3.4.6. We return to the example of $\mathbb{R}^2 - 0$ that was examined in § 3.1. Set

$$U_1 = \mathbb{R}^2 - \{(x, y) : x \geq 0, y = 0\}$$

and

$$U_2 = \mathbb{R}^2 - \{(x, y) : x \leq 0, y = 0\}.$$

Note that U_1 and U_2 are each star-shaped. Furthermore, we have

$$U_1 \cup U_2 = \mathbb{R}^2 - 0$$

and

$$U_1 \cap U_2 = \{(x, y) : y > 0\} \cup \{(x, y) : y < 0\} = \mathbb{R}^2_+ \sqcup \mathbb{R}^2_-.$$

Note each term of the disjoint union in $U_1 \cap U_2$ is star-shaped and so we have

$$\begin{aligned} \mathrm{H}^m_{\mathrm{dR}}(U_1 \cap U_2) &= \mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^2_+ \sqcup \mathbb{R}^2_-) \\ &= \mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^2_+) \oplus \mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^2_-) \\ &= \begin{cases} \mathbb{R} \oplus \mathbb{R} & m = 0 \\ 0 & m \neq 0. \end{cases} \end{aligned}$$

Thus, we are set up perfectly to apply Mayer-Vietoris. Applying this we have for $m > 0$

$$0 \xrightarrow{j^m} \mathrm{H}^m_{\mathrm{dR}}(U_1 \cap U_2) \xrightarrow{\partial^m} \mathrm{H}^{m+1}_{\mathrm{dR}}(\mathbb{R}^2 - 0) \longrightarrow 0$$

and so

$$\mathrm{H}^m_{\mathrm{dR}}(U_1 \cap U_2) \cong \mathrm{H}^{m+1}_{\mathrm{dR}}(\mathbb{R}^2 - 0).$$

Thus, for $m \geq 2$ we have

$$\mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^2 - 0) = 0.$$

Consider now the case when $m = 1$. In this case Mayer-Vietoris gives the exact sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{i^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{j^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\partial^0} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{R}^2 - 0) \longrightarrow 0.$$

The fact that $\ker(i^0) = 0$ gives $\mathrm{im}(i^0) = \mathbb{R}$. Thus, we have $\ker(j^0) = \mathbb{R}$ and so we must have

$$\partial^0 : \mathbb{R} \oplus \mathbb{R} / \mathrm{im}(j^0) \cong \mathrm{H}^1_{\mathrm{dR}}(\mathbb{R}^1 - 0),$$

i.e.,

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{R}^2 - 0) \cong \mathbb{R}.$$

Thus, using that we already know $\mathrm{H}^0_{\mathrm{dR}}(\mathbb{R}^2 - 0)$ we have shown that

$$\mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^2 - 0) \cong \begin{cases} \mathbb{R} & m = 0 \\ \mathbb{R} & m = 1 \\ 0 & m > 1. \end{cases}$$

Proposition 3.4.7. *Let $U \subset \mathbb{R}^n$ be an open set and assume that U is covered by finitely many star-shaped open sets $U_1, \ldots, U_r$. Then $\mathrm{H}^m_{\mathrm{dR}}(U)$ is finitely generated.*

Proof. We proceed by induction on r. If $r = 1$ the result is clear by Theorem 3.3.45. Suppose we have the result for all $k \leq r-1$. Set $V = U_1 \cup \cdots \cup U_{r-1}$ so we have $U = V \cup U_r$. The result is clear for $m = 0$ so we assume $m \geq 1$. We have using Mayer-Vietoris

$$\cdots \longrightarrow \mathrm{H}^{m-1}_{\mathrm{dR}}(V \cap U_r) \xrightarrow{\partial^m} \mathrm{H}^m_{\mathrm{dR}}(U) \xrightarrow{i^m} \mathrm{H}^m_{\mathrm{dR}}(V) \oplus \mathrm{H}^m_{\mathrm{dR}}(U_r) \longrightarrow \cdots .$$

We know that

$$\mathrm{H}^m_{\mathrm{dR}}(U)/\ker(i^m) \cong \mathrm{im}(i^m).$$

However, we have that $\ker(i^m) \cong \mathrm{im}(\partial^m)$ and so we can write

$$\mathrm{H}^m_{\mathrm{dR}}(U) \cong \mathrm{im}(i^m) \oplus \mathrm{im}(\partial^m).$$

Our induction hypothesis gives that

$$V \cap U_r = (U_1 \cap U_r) \cap \cdots \cap (U_{r-1} \cap U_r)$$

and so $\mathrm{H}^{m-1}_{\mathrm{dR}}(V \cap U_r)$, $\mathrm{H}^m_{\mathrm{dR}}(V)$, and $\mathrm{H}^m_{\mathrm{dR}}(U_r)$ are all finitely generated. In particular, $\partial^m(\mathrm{H}^{m-1}_{\mathrm{dR}}(V \cap U_r))$ and $i^m(\mathrm{H}^m_{\mathrm{dR}}(U))$ are finitely generated as a subspace of a finitely generated space is finitely generated and the image of a finitely generated space is finitely generated. Thus, $\mathrm{H}^m_{\mathrm{dR}}(U)$ must be finitely generated as well. □

For cohomology to actually be useful for anything one needs to have topological spaces that "look alike" to have the same cohomology groups and topological spaces that do not to have different cohomology groups. In other words, it is important that we can distinguish between topological spaces if we know each spaces' cohomology groups. In order to see we can do this, we need to define the correct notion of equivalence between topological spaces.

Definition 3.4.8. Let X and Y be topological spaces and let

$$f_i : X \to Y$$

be continuous maps for $i = 0, 1$. We say f_0 is *homotopic* to f_1 if there exists a continuous map

$$F : X \times [0,1] \to Y$$

so that

$$F(x,0) = f_0(x)$$

and

$$F(x,1) = f_1(x)$$

for all $x \in X$. We write $f_0 \simeq f_1$ or $f_0 \simeq_F f_1$ if we want to keep track of the homotopy F.

The way one should view this definition is to view $F(x,t)$ as a family of continuous maps $f_t(x) = F(x,t)$ that continuously deforms f_0 into f_1.

Lemma 3.4.9. *Homotopy is an equivalence relation.*

Proof. Clearly $f \simeq f$ via $F(x,t) = f(x)$ for all t.

Suppose $f \simeq_F g$. Define $G(x,t) = F(x, 1-t)$. Then $g \simeq_G f$.

Finally, suppose $f \simeq_F g$ and $g \simeq_G h$. Define

$$H(x,t) = \begin{cases} F(x,2t) & 0 \leq\leq 1/2 \\ G(x,2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

Then we have $f \simeq_H h$. □

Example 3.4.10. Let $f_i : U \to V$ be continuous maps for $i = 0, 1$ with $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ open sets with V star-shaped with respect to a point y. Define

$$F_i : U \times [0,1] \to V$$

by

$$F_i(x,t) = (1-t)f_i(x) + ty$$

for $i = 0, 1$. It is clear the maps $F_i(x,t)$ are continuous and since V is star-shaped with respect to y they are well-defined as well. Thus, we have each map f_i is homotopic to the constant map sending all of X to the point y. Since homotopy is an equivalence relation, we see that $f_0 \simeq f_1$. The map F_i is called the *straight-line homotopy*. We see that any continuous maps into a star-shaped region are homotopic.

Lemma 3.4.11. *Let X, Y, and Z be topological spaces and let $f_i : X \to Y$, $g_i : Y \to Z$ be continuous maps for $i = 0, 1$. If $f_0 \simeq_F f_1$ and $g_0 \simeq_G g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$.*

Proof. The required homotopy is given by $H(x,t) = G(F(x,t),t)$. □

Definition 3.4.12. A continuous map $f : X \to Y$ is said to be a *homotopy equivalence* if there exists a continuous map $g : Y \to X$ so that $g \circ f \simeq \mathrm{id}_X$ and $g \circ f \simeq \mathrm{id}_Y$. We call g a *homotopy inverse* to f. We say X and Y are *homotopy equivalent* if there is a homotopy equivalence between them. In particular, we say X is *contractible* if there is a homotopy equivalence between X and a one point space.

Since homotopy is an equivalence relation, we can use the above definition to partition topological spaces into homotopy equivalence classes.

Exercise 3.4.13. Show that X is contractible if and only if id_X is homotopic to a constant map.

Lemma 3.4.14. *If X and Y are homeomorphic they are necessarily homotopy equivalent.*

Proof. This fact is obvious from the definition. □

Lemma 3.4.15. *Let $U \subset \mathbb{R}^n$ be an open star-shaped region. Then U is contractible.*

Proof. Let $x \in U$ be the point with which U Is star-shaped with respect to. Let f be the map $f : U \to \{x\}$ that sends everything to x and let $g : \{x\} \hookrightarrow U$ be the natural inclusion. Observe that $f \circ g = \mathrm{id}_{\{x\}}$. It only remains to show that $h = g \circ f \simeq \mathrm{id}_U$. Define

$$F(y,t) = (1-t)x + ty.$$

Since U is star-shaped with respect to x we have

$$F : U \times [0,1] \to U$$

and

$$F(y,0) = g \circ f$$
$$F(y,1) = \mathrm{id}_U\,.$$

Thus, U is contractible. □

Example 3.4.16. Consider the spaces $\mathbb{R}^n - 0$ and S^{n-1}. We claim they are homotopy equivalent. Define $g : \mathbb{R}^n - 0 \to S^{n-1}$ by setting $g(x) = \frac{x}{|x|}$ and define $f : S^{n-1} \hookrightarrow \mathbb{R}^n - 0$ to be the natural inclusion map. Observe that we have $g \circ f : S^{n-1} \to S^{n-1}$ is actually equal to the identity map, so certainly homotopic to it.

Define $G(x,t) : (\mathbb{R}^n - 0) \times [0,1] \to \mathbb{R}^n - 0$ by $G(x,t) = (1-t)x + t\left(\frac{x}{|x|}\right)$. This gives $(f \circ g)(x) = \frac{x}{|x|}$ is homotopy equivalent to $\mathrm{id}_{\mathbb{R}^n - 0}$. Thus, we have $\mathbb{R}^n - 0$ and S^{n-1} are homotopy equivalent.

Note that $\mathbb{R}^n - 0$ is not homeomorphic to S^{n-1}. For example, S^{n-1} is compact and $\mathbb{R}^n - 0$ is not compact. Thus, we see that spaces can be homotopy equivalent without being homeomorphic. Thus, the homotopy equivalence classes are larger than the equivalence classes one obtains by considering spaces as equivalent if they are homeomorphic.

While homotopy provides a way to separate topological spaces into equivalence classes that hopefully will turn out to be more useful than separating them by homeomorphism, it is phrased in terms of continuous maps where we have been working with smooth maps. The next few results will show that working with only smooth maps is "good enough" for what we want to do.

Lemma 3.4.17. *Let $A \subset \mathbb{R}^n \subset V \subset U \subset \mathbb{R}^n$ where U and V are open in $\mathbb{R}^n$ and A is closed in U. Let $h : U \to W$ be a continuous map into an open set $W \subset \mathbb{R}^m$ with smooth restriction to V. For any continuous function $\epsilon : U \to (0,\infty)$ there exists a smooth map $f : U \to W$ satisfying*

1. *$|f(x) - h(x)| \leq \epsilon(x)$ for every $x \in U$;*

2. *$f(x) = h(x)$ for every $x \in A$.*

Proof. If $W \neq \mathbb{R}^m$, we can replace $\epsilon(x)$ by

$$\epsilon_1(x) = \min(\epsilon(x), (1/2)\rho(h(x), \mathbb{R}^m - W))$$

where

$$\rho(y, \mathbb{R}^m - W) = \inf\{|y - z| : z \in \mathbb{R}^m - W\}.$$

Now if $f : U \to \mathbb{R}^m$ satisfies the first condition with ϵ_1 instead of ϵ, we obtain $f(U) \subset W$. Thus, without loss of generality we may assume $W = \mathbb{R}^m$.

The fact that h and ϵ are continuous allows us to find for each point $x_0 \in U - A$ an open neighborhood $U_{x_0} \subset U - A$ of x_0 so that

$$|h(x) - h(x_0)| < \epsilon(x)$$

for all $x \in U_{x_0}$. Consider the open cover of U consisting of V and $\{U_{x_0}\}_{x_0 \in U - A}$. We know there is a partition of unity with respect to this open cover, call it $\{p_{x_0}\}$. Using the properties of a partition of unity we define a smooth function

$$f(x) = p_0(x)h(x) + \sum_{x_0 \in U - A} p_{x_0}(x)h(x_0).$$

Note we also have

$$h(x) = p_0(x)h(x) + \sum_{x_0 \in U - A} p_{x_0}(x)h(x).$$

Thus,

$$f(x) - h(x) = \sum_{x_0 \in U - A} p_{x_0}(x)(h(x_0) - h(x)).$$

Since $\operatorname{supp}_U(p_{x_0}) \subset U_{x_0} \subset U - A$, we have the second part of the theorem. As for the first part, we have

$$\begin{aligned}
|f(x) - h(x)| &\leq \sum_{x_0 \in U - A} p_{x_0}(x)|h(x_0) - h(x)| \\
&\leq \sum_{x_0 \in U - A} p_{x_0}(x)\epsilon(x) \\
&= \left(\sum_{x_0 \in U - A} p_{x_0}(x)\right)\epsilon(x) \\
&= \epsilon(x)
\end{aligned}$$

where we have used that necessarily $x \in U_{x_0}$ in order for $p_{x_0}(x) \neq 0$. □

Proposition 3.4.18. *Let U, V be open in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then we have:*

1. *Every continuous map $h : U \to V$ is homotopic to a smooth map.*

2. *If two smooth maps $f_j : U \to V$, $j = 0, 1$ are homotopic, then there exists a smooth map $F : U \times \mathbb{R} \to V$ with $F(x, j) = f_j(x)$ for $j = 0, 1$ and all $x \in U$.*

Proof. We begin with the first statement. By Lemma 3.4.17 we can approximate h by a smooth map $f : U \to V$. Choose f so that V contains the line segment from $h(x)$ to $f(x)$ for every $x \in U$. The straight-line homotopy then gives $h \simeq f$. This gives the first part.

Let G be a homotopy from f_0 to f_1. Let $\psi : \mathbb{R} \to [0, 1]$ be a continuous function with $\psi(t) = 0$ for $t \leq 1/3$, $\psi(t) = 1$ for $t \geq 2/3$. Define

$$H : U \times \mathbb{R} \to V$$

by

$$H(x, t) = G(x, \psi(t)).$$

We have $h(x, t) = f_0(x)$ for $t \in (-\infty, 1/3]$ and $H(x, t) = f_1(x)$ for $t \in [2/3, \infty)$ so H Is smooth on $(-\infty, 1/3] \cup [2/3, \infty)$. Appealing to Lemma **??** again we can approximate H by a smooth map

$$F : U \times \mathbb{R} \to V$$

so that F and H have the same restriction on $U \times \{0, 1\}$. Thus, for $x \in U$ and $k = 0, 1$ we have $F(x, k) = H(x, k) = f_k(x)$ as desired. □

Thus, we have shown that when working with homotopies, it is enough to work in the setting of smooth maps.

Theorem 3.4.19. *Let $f, g : U \to V$ be smooth maps with $f \simeq_F g$. The induced maps*

$$\Omega^m(f), \Omega^m(g) : \Omega^m(V) \to \Omega^m(U)$$

are chain-homotopic.

Proof. Let $\omega \in \Omega^m(U \times \mathbb{R})$. Recall that we can write

$$\omega = \sum_I f_I(x, t) dx_I + \sum_J g_J(x, t) dt \wedge dx_J.$$

Let $\phi_k : U \hookrightarrow U \times \mathbb{R}$ be the inclusion map given by $\phi_k(x) = (x, k)$ for $k = 0, 1$. Then we have

$$\Omega^m(\phi_k)(\omega) = \sum_I f_I(x, k) dx_I.$$

Note that we have used here that $d(\phi_k)_I = dx_I$ since ϕ_k is an inclusion map and $\Omega^m(\phi_k)(dt \wedge dx_J) = 0$ since the t-component of ϕ_k is constant.

Recall we constructed a linear map

$$\widetilde{\Psi}^m : \Omega^m(U \times \mathbb{R}) \to \Omega^{m-1}(U)$$

so that

$$(d^{m-1}\widetilde{\Psi}^m + \widetilde{\Psi}d^m)(\omega) = \Omega^m(\phi_1)(\omega) - \Omega^m(\phi_0)(\omega). \tag{3.2}$$

Consider the composition

$$U \xrightarrow{\phi_k} U \times \mathbb{R} \xrightarrow{F} V.$$

Then we have $F \circ \phi_0 = f$ and $F \circ \phi_1 = g$. Define

$$\Psi^m : \Omega^m(V) \to \Omega^{m-1}(U)$$

by

$$\Psi^m = \widetilde{\Psi}^m \circ \Omega^m(F).$$

We claim that

$$d^m\Psi^m + \Psi^{m-1}d^m = \Omega^m(g) - \Omega^m(f).$$

To see this, we begin by applying equation (3.2) to $\Omega^m(F)(\omega)$:

$$\begin{aligned}(d^{m-1}\widetilde{\Psi}^m + \widetilde{\Psi}d^m)(\Omega^m(F)(\omega)) &= \Omega^m(\phi_1)(\Omega^m(F)(\omega)) - \Omega^m(\phi_0)(\Omega^m(F)(\omega)) \\ &= \Omega^m(F \circ \phi_1)(\omega) - \Omega^m(F \circ \phi_0)(\omega) \\ &= \Omega^m(g)(\omega) - \Omega^m(f)(\omega).\end{aligned}$$

Now observe that since F is a chain map we can write this as

$$\begin{aligned}d^{m-1}\Psi^m(\omega) + \widetilde{\Psi}^{m+1}(d^m\Omega^m(F)(\omega)) &= d^m\Psi^m(\omega) + \widetilde{\Psi}^{m+1}\Omega^m(F)(d^m\omega) \\ &= d^m\Psi^m(\omega) + \Psi^{m+1}d^m\omega.\end{aligned}$$

Thus, we have a chain homotopy between $\Omega^m(g)$ and $\Omega^m(f)$. □

Note that this results shows that if $f \simeq g$, then the induced maps on cohomology are equal. So, given a continuous map $\phi : U \to V$ we can find a smooth map $f : U \to V$ so that $f \simeq \phi$ and by the previous result the induced map

$$\mathrm{H}^m_{\mathrm{dR}}(f) : \mathrm{H}^m_{\mathrm{dR}}(V) \to \mathrm{H}^m_{\mathrm{dR}}(U)$$

is independent of the choice of f. Thus, given a continuous map $\phi : U \to V$, we define

$$\mathrm{H}^m_{\mathrm{dR}}(\phi) : \mathrm{H}^m_{\mathrm{dR}}(V) \to \mathrm{H}^m_{\mathrm{dR}}(U)$$

by $\mathrm{H}^m_{\mathrm{dR}}(\phi) = \mathrm{H}^m_{\mathrm{dR}}(f)$ where f is any smooth map from U to V with $f \simeq \phi$.

Theorem 3.4.20. *Let U, V, W be open sets in Euclidean spaces.*

1. *If $\phi_0, \phi_1 : U \to V$ are homotopic continuous maps, then*

$$\mathrm{H}^m_{\mathrm{dR}}(\phi_0) = \mathrm{H}^m_{\mathrm{dR}}(\phi_1) : \mathrm{H}^m_{\mathrm{dR}}(V) \to \mathrm{H}^m_{\mathrm{dR}}(U).$$

2. *If $\phi : U \to V$ and $\psi : V \to W$ are continuous, then*

$$\mathrm{H}^m_{\mathrm{dR}}(\psi \circ \phi) = \mathrm{H}^m_{\mathrm{dR}}(\phi) \circ \mathrm{H}^m_{\mathrm{dR}}(\psi) : \mathrm{H}^m_{\mathrm{dR}}(W) \to \mathrm{H}^m_{\mathrm{dR}}(U).$$

3. *If the continuous map $\phi : U \to V$ is a homotopy equivalence, then*

$$\mathrm{H}^m_{\mathrm{dR}}(\phi) : \mathrm{H}^m_{\mathrm{dR}}(V) \to \mathrm{H}^m_{\mathrm{dR}}(U)$$

   is an isomorphism.

Proof. Let $f : U \to V$ be a smooth map with $f \simeq \phi_0$. Since homotopy is an equivalence relation and $\phi_0 \simeq \phi_1$, we must have $f \simeq \phi_1$ as well. Thus, $\mathrm{H}^m_{\mathrm{dR}}(\phi_0) = \mathrm{H}^m_{\mathrm{dR}}(f) = \mathrm{H}^m_{\mathrm{dR}}(\phi_1)$. This gives part (1).

Part (2) is known if ϕ and ψ happen to be smooth. Otherwise, choose $f \simeq \phi$ and $g \simeq \psi$ with f and g smooth. Then we have

$$\psi \circ \phi \simeq g \circ f$$

and so

$$\mathrm{H}^m_{\mathrm{dR}}(\psi \circ \phi) = \mathrm{H}^m_{\mathrm{dR}}(g \circ f) = \mathrm{H}^m_{\mathrm{dR}}(f) \circ \mathrm{H}^m_{\mathrm{dR}}(g) = \mathrm{H}^m_{\mathrm{dR}}(\phi) \circ \mathrm{H}^m_{\mathrm{dR}}(\psi).$$

This gives part (2).

Finally, we prove part (3). Let $\psi : V \to U$ be a homotopy inverse to ϕ. Part (2) gives that $\mathrm{H}^m_{\mathrm{dR}}(\psi)$ is an inverse to $\mathrm{H}^m_{\mathrm{dR}}(\phi)$ and so we have an isomorphism of vector spaces. □

Corollary 3.4.21. *Let $f : U \to V$ be a homeomorphism. Then $\mathrm{H}^m_{\mathrm{dR}}(f) : \mathrm{H}^m_{\mathrm{dR}}(V) \to \mathrm{H}^m_{\mathrm{dR}}(U)$ is an isomorphism of vector spaces.*

Corollary 3.4.22. *Let $U \subset \mathbb{R}^n$ be a contractible open set. One has that $\mathrm{H}^0_{\mathrm{dR}}(U) \cong \mathbb{R}$ and $\mathrm{H}^m_{\mathrm{dR}}(U) = 0$ for $m > 0$.*

Proof. Recall that U being contractible is equivalent to id_U being homotopic to a constant map. Let f be such a constant map, say $f(x) = x_0$ for all $x \in U$. Let $F(x,t)$ be the homotopy between id_U and f. Note that $F(x,t)$ defines a continuous curve in U that connects x_0 and x. Thus, U is path-connected, hence connected. Thus, $\mathrm{H}^0_{\mathrm{dR}}(U) \cong \mathbb{R}$.

If $m > 0$, we know that

$$\Omega^m(f) : \Omega^m(U) \to \Omega^m(U)$$

is 0 because $\Omega^m(f)(\omega)_x = \mathrm{Alt}^m(D_x f)(\omega(f(x)))$ and since f is constant, $D_x f = 0$. Thus, by Theorem 3.4.20 we have

$$\mathrm{H}^m_{\mathrm{dR}}(\mathrm{id}_U) = \mathrm{H}^m_{\mathrm{dR}}(f) = 0.$$

Since $\mathrm{H}^m_{\mathrm{dR}}(\mathrm{id}_U) = \mathrm{id}_{\mathrm{H}^m_{\mathrm{dR}}(U)}$, it must be that $\mathrm{H}^m_{\mathrm{dR}}(U) = 0$. □

The following proposition will be very important in subsequent calculations. It allows us to "move up" in the sense of calculating the higher cohomology groups of Euclidean spaces with a closed set removed if we know the cohomology of a smaller Euclidean space with the same closed set removed. In particular, it will allow us to calculate the cohomology of $\mathbb{R}^n - 0$ for $n \geq 2$ using that we already have the calculation in the case $n = 2$.

Proposition 3.4.23. *Let $A \subsetneq \mathbb{R}^n$ be a closed set. Then we have*

$$\begin{aligned} \mathrm{H}^{m+1}_{\mathrm{dR}}(\mathbb{R}^{n+1} - A) &\cong \mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^n - A) \quad m \geq 1 \\ \mathrm{H}^1_{\mathrm{dR}}(\mathbb{R}^{n+1} - A) &\cong \mathrm{H}^0_{\mathrm{dR}}(\mathbb{R}^n - A)/\mathbb{R} \\ \mathrm{H}^0_{\mathrm{dR}}(\mathbb{R}^{n+1} - A) &\cong \mathbb{R} \end{aligned}$$

where we have identified $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$.

Proof. We view $\mathbb{R}^{n+1}$ as $\mathbb{R}^n \times \mathbb{R}$ and define open subsets of $\mathbb{R}^{n+1}$ by

$$U_1 = \mathbb{R}^n \times (0, \infty) \cup (\mathbb{R}^n - A) \times (-1, \infty)$$

and

$$U_2 = \mathbb{R}^n \times (-\infty, 0) \cup (\mathbb{R}^n - A) \times (-\infty, 1).$$

We have

$$U_1 \cup U_2 = \mathbb{R}^{n+1} - A$$

and

$$U_1 \cap U_2 = (\mathbb{R}^n - A) \times (-1, 1).$$

The reader is urged to draw some examples of this to be sure to understand what the sets U_1 and U_2 look like.

Define $\phi : U_1 \to U_1$ by setting

$$\phi(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n, x_{n+1} + 1).$$

For $x \in U_1$ we have that the line segment from x to $\phi(x)$ is contained inside U_1 and so $\mathrm{id}_{U_1} \simeq \phi$. Given any $\phi(x)$ we have that $\phi(x) \in \mathbb{R}^n \times (0, \infty)$ and since $\mathbb{R}^n \times (0, \infty)$ is star-shaped with respect to $y_0 = (0, \ldots, 0, 1)$, we can connect $\phi(x)$ to y_0 by a straight line. Thus, $\phi(x)$ is homotopic to the constant map sending everything in U_1 to y_0. Combining these homotopies we have id_{U_1} is homotopic to a constant map and so U_1 is contractible. Similarly we have that U_2 is contractible and so

$$\mathrm{H}^m_{\mathrm{dR}}(U_k) = \begin{cases} \mathbb{R} & m = 0 \\ 0 & m > 0 \end{cases}$$

for $k = 1, 2$.

Define $\pi : U1 \cap U_2 \to \mathbb{R}^n - A$ to be the natural projection map. Define $i : \mathbb{R}^n - A \to U_1 \cap U_2$ by $i(x) = (x, 0)$. Then $\pi \circ i = \mathrm{id}_{\mathbb{R}^n - A}$ and $i \circ \pi(x, y) = (x, 0)$, which is homotopic to $\mathrm{id}_{U_1 \cap U_2}$ by straight line homotopy. Thus, we have

$$U_1 \cap U_2 \simeq \mathbb{R}^n - A$$

and so

$$\mathrm{H}^m_{\mathrm{dR}}(U_1 \cap U_2) \cong \mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^n - A).$$

Suppose now that $m \geq 1$. Since U_1 and U_2 are contractible, Mayer-Vietoris gives

$$\mathrm{H}^m_{\mathrm{dR}}(U_1 \cap U_2) \cong \mathrm{H}^{m+1}_{\mathrm{dR}}(U_1 \cup U_2),$$

i.e.,

$$\mathrm{H}^{m+1}_{\mathrm{dR}}(\mathbb{R}^{n+1} - A) \cong \mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^n - A).$$

Mayer-Vietoris also gives the exact sequence

$$0 \longrightarrow \mathrm{H}^0_{\mathrm{dR}}(\mathbb{R}^{n+1}-A) \xrightarrow{\mathrm{H}^0_{\mathrm{dR}}(i)} \mathrm{H}^0_{\mathrm{dR}}(U_1) \oplus \mathrm{H}^0_{\mathrm{dR}}(U_2) \xrightarrow{\mathrm{H}^0_{\mathrm{dR}}(j)} \mathrm{H}^0_{\mathrm{dR}}(\mathbb{R}^n - A) \xrightarrow{\partial^0} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{R}^{n+1}-A) \longrightarrow 0.$$

Recall that $\mathrm{H}^0_{\mathrm{dR}}(U)$ are the locally constant functions defined on U. Since U_1 and U_2 are connected, an element of $\mathrm{H}^0_{\mathrm{dR}}(U_1) \oplus \mathrm{H}^0_{\mathrm{dR}}(U_2)$ is a pair of constant functions a_1, a_2. The image of (a_1, a_2) under the map $\mathrm{H}^0_{\mathrm{dR}}(j)$ is $a_1 - a_2 \in \mathrm{H}^0_{\mathrm{dR}}(U_1 \cap U_2)$. Thus,

$$\ker(\partial^0) = \mathrm{im}(\mathrm{H}^0_{\mathrm{dR}}(j)) = (a_1 - a_2)\mathbb{R} \cong \mathbb{R}.$$

Thus,

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{R}^{n+1} - A) \cong \mathrm{H}^0_{\mathrm{dR}}(R^n - A)/\mathbb{R}.$$

Finally, we have that $\dim_{\mathbb{R}}(\mathrm{im}(\mathrm{H}^0_{\mathrm{dR}}(i))) = \dim_{\mathbb{R}}(\ker(\mathrm{H}^0_{\mathrm{dR}}(j))) = 1$ and so

$$\mathrm{H}^0_{\mathrm{dR}}(\mathbb{R}^{n+1} - A) \cong \mathbb{R}.$$

□

We can now easily calculate the cohomology of punctured Euclidean space.

Theorem 3.4.24. *For $n \geq 2$ we have the following*

$$\mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^n - 0) \cong \begin{cases} \mathbb{R} & m = 0, n-1 \\ 0 & m \neq 0, n-1. \end{cases}$$

Proof. We have already shown the case of $n = 2$ in Example 3.4.6. The general case now follows by induction using Proposition 3.4.23. □

We also note that in the case of $n = 1$ we have

$$\mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^1 - 0) \cong \begin{cases} \mathbb{R} \oplus \mathbb{R} & m = 0 \\ 0 & m \neq 0. \end{cases}$$

We can now use this allow with Theorem 3.4.24 to conclude that Euclidean spaces of different dimensions are not homeomorphic. Though Theorem 3.4.25 certainly would be believable to anyone that had read through Chapter 2, it is not something that is easy to prove! It is a good illustration of the power of cohomology to try and prove it using only the basic tools established in Chapter 2.

Theorem 3.4.25. *If $m \neq n$ then $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$.*

Proof. Suppose there is such a homeomorphism, say $\phi : \mathbb{R}^n \to \mathbb{R}^m$. Without loss of generality we may assume $\phi(0) = 0$ as we can change ϕ by a homotopy to make it so if not. Thus, $\phi : \mathbb{R}^n - 0 \to \mathbb{R}^m - 0$ is a homeomorphism. This implies that all the cohomology groups of $\mathbb{R}^n - 0$ and $\mathbb{R}^m - 0$ must be equal. However, this contradicts Theorem 3.4.24 unless $m = n$. □

Definition 3.4.26. Let X be a topological space. Given a map $f : X \to X$ we call $x \in X$ a *fixed point* of f if $f(x) = x$.

For brevity we denote the closed unit ball $\mathrm{Cl}(B(0,1)) \subset \mathbb{R}^n$ by D^n.

Theorem 3.4.27. *(Brouwer's fixed point theorem) Every continuous map $f : D^n \to D^n$ has a fixed point for $n \geq 1$.*

Fixed point theorems are very useful in many areas of mathematics. For example, one has well-known applications of this theorem to economics and game theory. In fact, recently a paper has been posted that uses this theorem to study methods of counter terrorism! For a fun "party-fact" consequence of this theorem, suppose that you have a cup of coffee and you swirl it around. If you assume that all of the particles on the surface of the coffee remain on the surface, then no matter how much you swirl the coffee around at least one particle will end up in the same place it started!

Before we can prove Theorem 3.4.27, we need the following lemma.

Lemma 3.4.28. *There are no continuous maps $g : D^n \to S^{n-1}$ with $g|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$.*

Proof. The case $n = 1$ is clear so we assume $n \geq 2$. Define $f : \mathbb{R}^n - 0 \to \mathbb{R}^n - 0$ by $f(x) = \frac{x}{|x|}$. Recall that we have seen before that $f \simeq \mathrm{id}_{\mathbb{R}^n - 0}$ by the straight line homotopy. Suppose there is such a g. Then for $0 \leq t \leq$ we have

$$F(x,t) = g(tf(x))$$

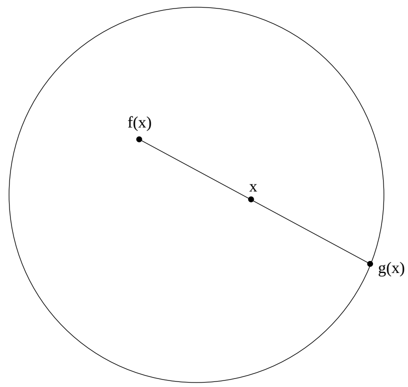
is continuous and

$$\begin{aligned} F(x,0) &= g(0) = \text{constant map} \\ F(x,1) &= g(f(x)) = g\left(\frac{x}{|x|}\right) = \frac{x}{|x|} \end{aligned}$$

where we have used that $g|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$. Thus, F gives a homotopy between f and a constant map, i.e., we see that $\mathrm{id}_{\mathbb{R}^n - 0} \simeq f \simeq$ a constant map. Thus, $\mathbb{R}^n - 0$ is contractible. This is a contradiction as $\mathrm{H}^{n-1}_{\mathrm{dR}}(\mathbb{R}^n - 0) \cong \mathbb{R}$. Thus, there can be no such g. □

We now prove Theorem 3.4.27.

Proof. Suppose that $f(x) \neq x$ for all $x \in D^n$. Thus, x and $f(x)$ determine a line and so we can define $g(x) \in S^{n-1}$ to be the intersection of the ray starting at $f(x)$ going through x and S^{n-1} as pictured:



We can write $g(x) = x+tu$ where $u = \frac{x-f(x)}{|x-f(x)|}$ and $t = -\langle x, u\rangle + \sqrt{1 - |x|^2 + \langle x, u\rangle^2}$ where we again write the inner product on Euclidean space as $\langle\, , \rangle$. However, this gives a continuous map $g : D^n - S^{n-1}$ with $g|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$. This contradicts Lemma 3.4.28 and so it must be that f has a fixed point. □

It is interesting to note that Brouwer was a proponent of only proving theorems with constructive proofs and felt that proofs by contradiction should be avoided at all costs. It is ironic then that his most famous theorem is demonstrated by contraction.

Definition 3.4.29. Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open sets. A smooth map $f : U \to V$ that is bijective and has a smooth inverse function is called a *diffeomorphism.*

The following result is another corollary of Proposition 3.4.23.

Corollary 3.4.30. *Let $A \subsetneq \mathbb{R}^n$ be a closed set. Let*

$$F : \mathbb{R}^{n+1} - A \to \mathbb{R}^{n+1} - A$$

be the diffeomorphism given by

$$F(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n, -x_{n+1}).$$

The induced linear map

$$\mathrm{H}^{m+1}_{\mathrm{dR}}(F) : \mathrm{H}^{m+1}_{\mathrm{dR}}(\mathbb{R}^{n+1} - A) \to \mathrm{H}^{m+1}_{\mathrm{dR}}(\mathbb{R}^{n+1} - A)$$

is multiplication by -1 for $m \geq 0$.

Proof. It was shown in the proof of Proposition 3.4.23 that for $m \geq 1$ we have

$$\partial^m : \mathrm{H}^m_{\mathrm{dR}}(U_1 \cap U_2) \xrightarrow{\simeq} \mathrm{H}^{m+1}_{\mathrm{dR}}(U_1 \cup U_2)$$

is an isomorphism and

$$\partial^0 : \mathrm{H}^0_{\mathrm{dR}}(U_1 \cap U_2) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(U_1 \cup U_2)$$

is a surjection. Thus, to show that $\mathrm{H}^m_{\mathrm{dR}}(F)$ is multiplication by -1, it is enough to show that

$$\mathrm{H}^m_{\mathrm{dR}}(F) \circ \partial^m([\omega]) = -\partial^m([\omega])$$

for $m \geq 0$ and $[\omega] \in \mathrm{H}^m_{\mathrm{dR}}(U_1 \cap U_2)$.

Recall we have the following exact sequence:

$$0 \longrightarrow \Omega^m(U_1 \cup U_2) \xrightarrow{i^m} \Omega^m(U_1) \oplus \Omega^m(U_2) \xrightarrow{j^m} \Omega^m(U_1 \cap U_2) \longrightarrow 0.$$

Thus, given $\omega \in \Omega^m(U_1 \cap U_2)$, there exist ω_1, ω_2 so that

$$\begin{aligned} \omega &= j^m(\omega_1, \omega_2) \\ &= \Omega^m(j_1)(\omega_1) - \Omega^m(j_2)(\omega_2). \end{aligned}$$

Furthermore, recall that

$$\begin{aligned} \partial^m([\omega]) &= [(i^{m+1})^{-1}(d^m_{\Omega^m(U_1)\oplus\Omega^m(U_2)}((j^m)^{-1}(\omega)))] \\ &= [(i^{m+1})^{-1}(d^m_{\Omega^m(U_1)}\omega_1, d^m_{\Omega^m(U_2)}\omega_2)] \\ &= [\tau] \end{aligned}$$

where $\tau \in \Omega^{m+1}(U_1 \cup U_2)$ is so that $\Omega^{m+1}(i_k)(\tau) = d^m_{\Omega^m(U_k)}\omega_k$ for $k = 1, 2$.

Observe we have the following commutative diagrams (where F_0 and F_1 are the restrictions of F):

$$\begin{array}{ccc} \mathbb{R}^{n+1} - A & \xrightarrow{F} & \mathbb{R}^{n+1} - A \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle i_2} \\ U_1 & \xrightarrow{F_1} & U_2 \\ {\scriptstyle j_1}\uparrow & & \uparrow{\scriptstyle j_2} \\ U_1 \cap U_2 & \xrightarrow{F_0} & U_1 \cap U_2 \end{array} \qquad \begin{array}{ccc} \mathbb{R}^{n+1} - A & \xrightarrow{F} & \mathbb{R}^{n+1} - A \\ {\scriptstyle i_2}\uparrow & & \uparrow{\scriptstyle i_1} \\ U_2 & \xrightarrow{F_2} & U_1 \\ {\scriptstyle j_2}\uparrow & & \uparrow{\scriptstyle j_1} \\ U_1 \cap U_2 & \xrightarrow{F_0} & U_1 \cap U_2. \end{array}$$

These diagrams given the following diagrams in differential forms for all $r \geq 0$:

$$\begin{array}{ccc} \Omega^r(\mathbb{R}^{n+1} - A) & \xleftarrow{\Omega^r(F)} & \Omega^r(\mathbb{R}^{n+1} - A) \\ \downarrow{\scriptstyle \Omega^r(i_1)} & & \downarrow{\scriptstyle \Omega^r(i_2)} \\ \Omega^r(U_1) & \xleftarrow{\Omega^r(F_1)} & \Omega^r(U_2) \\ \downarrow{\scriptstyle \Omega^r(j_1)} & & \downarrow{\scriptstyle \Omega^r(j_2)} \\ \Omega^r(U_1 \cap U_2) & \xleftarrow{\Omega^r(F_0)} & \Omega^r(U_1 \cap U_2) \end{array}$$

$$\begin{array}{ccc}
\Omega^r(\mathbb{R}^{n+1}-A) & \xleftarrow{\Omega^r(F)} & \Omega^r(\mathbb{R}^{n+1}-A) \\
\downarrow{\scriptstyle \Omega^r(i_2)} & & \downarrow{\scriptstyle \Omega^r(i_1)} \\
\Omega^r(U_2) & \xleftarrow{\Omega^r(F_2)} & \Omega^r(U_1) \\
\downarrow{\scriptstyle \Omega^r(j_2)} & & \downarrow{\scriptstyle \Omega^r(j_1)} \\
\Omega^r(U_1\cap U_2) & \xleftarrow{\Omega^r(F_0)} & \Omega^r(U_1\cap U_2).
\end{array}$$

Thus, we have

$$\begin{aligned}
-\Omega^m(F_0)(\omega) &= -\Omega^m(F_0)(\Omega^m(j_1)(\omega_1)-\Omega^m(j_2)(\omega_2)) \\
&= \Omega^m(F_0)\circ\Omega^m(j_2)(\omega_2)-\Omega^m(F_0)\circ\Omega^m(j_1)(\omega_1) \\
&= \Omega^m(j_1)\circ\Omega^m(F_1)(\omega_2)-\Omega^m(j_2)\circ\Omega^m(F_2)(\omega_1),
\end{aligned}$$

$$\begin{aligned}
\Omega^{m+1}(i_1)\circ\Omega^{m+1}(F)(\tau) &= \Omega^{m+1}(F_1)\circ\Omega^{m+1}(i_2)(\tau) \\
&= \Omega^{m+1}(F_1)(d^m_{\Omega^m(U_2)}\omega_2) \\
&= d^m_{\Omega^m(U_1)}(\Omega^m(F_1)(\omega_2)),
\end{aligned}$$

and

$$\begin{aligned}
\Omega^{m+1}(i_2)\circ\Omega^{m+1}(F)(\tau) &= \Omega^{m+1}(F_2)\circ\Omega^{m+1}(i_1)(\tau) \\
&= \Omega^{m+1}(F_2)(d^m_{\Omega^m(U_1)}\omega_1) \\
&= d^m_{\Omega^m(U_2)}(\Omega^m(F_2))(\omega_1).
\end{aligned}$$

Combining these results with the definitions we have

$$\begin{aligned}
\partial^m(-[\Omega^m(F_0)(\omega)]) &= \partial^m([\Omega^m(j_1)\circ\Omega^m(F_1)(\omega_2)-\Omega^m(j_2)\circ\Omega^m(F_2)(\omega_1)]) \\
&= [(i^{m+1})^{-1}d^m_{\Omega^m(U_1)\oplus\Omega^m(U_2)}(\Omega^m(F_1)(\omega_2),\Omega^m(F_2)(\omega_1))] \\
&= [(i^{m+1})^{-1}(\Omega^{m+1}(i_1)\circ\Omega^{m+1}(F)(\tau),\Omega^{m+1}(i_2)\circ\Omega^{m+1}(F)(\tau))] \\
&= [\Omega^{m+1}(F)(\tau)].
\end{aligned}$$

Thus,

$$\begin{aligned}
\partial^m\circ \mathrm{H}^m_{\mathrm{dR}}([\omega]) &= \partial^m([\Omega^m(F_0)(\omega)]) \\
&= -[\Omega^{m+1}(F)(\tau)] \\
&= -[\Omega^{m+1}\circ\partial^m(\omega)] \\
&= -\,\mathrm{H}^{m+1}_{\mathrm{dR}}(F)\circ\partial^m([\omega]).
\end{aligned}\tag{3.3}$$

Note that for $\pi : U_1\cap U_2 \to \mathbb{R}^n - A$ the projection map as before, we have $\pi\circ F_0 = \pi$ and so

$$\mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^n-A) \xrightarrow{\mathrm{H}^m_{\mathrm{dR}}(\pi)} \mathrm{H}^m_{\mathrm{dR}}(U_1\cap U_2) \xrightarrow{\mathrm{H}^m_{\mathrm{dR}}(F_0)} \mathrm{H}^m_{\mathrm{dR}}(U_1\cap U_2)$$

is identical to just $\mathrm{H}^m_{\mathrm{dR}}(\pi)$. However, we saw in the proof of Proposition 3.4.23 that $\mathrm{H}^m_{\mathrm{dR}}(\pi)$ is an isomorphism, and so $\mathrm{H}^m_{\mathrm{dR}}(F_0)$ must be the identity map on $\mathrm{H}^m_{\mathrm{dR}}(U_1 \cap U_2)$. Thus, in equation (3.3) we have

$$\partial^m([\omega]) = -\mathrm{H}^{m+1}_{\mathrm{dR}}(F) \circ \partial^m([\omega]),$$

which is exactly what we were trying to show. $\square$

Let $A \in \mathrm{GL}_n(\mathbb{R})$. One has an associated linear endomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$. Moreover, A also defines a diffeomorphism

$$f_A : \mathbb{R}^n - 0 \longrightarrow \mathbb{R}^n - 0$$

in the natural way.

Lemma 3.4.31. *For each $n \geq 2$ the induced map*

$$\mathrm{H}^{n-1}_{\mathrm{dR}}(f_A) : \mathrm{H}^{n-1}_{\mathrm{dR}}(\mathbb{R}^n - 0) \longrightarrow \mathrm{H}^{n-1}_{\mathrm{dR}}(\mathbb{R}^n - 0)$$

is multiplication by $\frac{\det A}{|\det A|}$.

Proof. Let $E_{r,s}$ be the matrix with a 1 in the r^{th} row and s^{th} column and 0's elsewhere. Consider the matrix

$$B = (1_n + cE_{r,s})A$$

for $c \in \mathbb{R}$ and $r \neq s$. The matrix B is thus obtained from A by replacing the r^{th} row by the sum of the r^{th} row and c times the s^{th} row.

We have $f_A \simeq f_B$ via the map

$$F(x,t) = (1_n + ctE_{r,s})Ax.$$

Thus, $\mathrm{H}^{n-1}_{\mathrm{dR}}(f_A) = \mathrm{H}^{n-1}_{\mathrm{dR}}(f_B)$. Note that we also have $\det A = \det B$. Observe that by doing a series of such operations we can put A into the form $\mathrm{diag}(1,1,\ldots,1,\det A)$ where diag denotes a diagonal matrix. Thus, it is enough to prove the theorem for diagonal matrices of this form.

Consider the matrix $\mathrm{diag}(1,\ldots,1,d)$. The map given by

$$F(x,t) = \mathrm{diag}\left(1,\ldots,1,\frac{|d|^k d}{|d|}\right)x$$

gives a homotopy between the map given by

$$\mathrm{diag}\left(1,\ldots,1,\frac{d}{|d|}\right)$$

and

$$\mathrm{diag}(1,\ldots,1,d)$$

and so we reduce to considering

$$\mathrm{diag}(1,\ldots,1,\pm 1).$$

Thus, $\mathrm{H}^{n-1}_{\mathrm{dR}}(f_A)$ is either the identity map or is the map $\mathrm{H}^{n-1}_{\mathrm{dR}}(F)$ given in Corollary 3.4.30 depending upon $\frac{\det A}{|\det A|}$, as claimed. $\square$

Given a point $x \in S^n$, the *tangent space to S^n at x* is defined to be those $y \in R^{n+1}$ so that $\langle x, y \rangle = 0$. We denote the tangent space of S^n at x by $T_x S^n$. A *vector field* on S^n is a continuous map $v : S^n \to \mathbb{R}^{n+1}$ so that $v(x) \in T_x S^n$ for every $x \in S^n$.

Theorem 3.4.32. *The sphere S^n has a tangent vector field v with $v(x) \neq 0$ for all $x \in S^n$ if and only if n is odd.*

Proof. Suppose there is such a vector field. We can extend it to $R^{n+1} - 0$ by setting

$$w(x) = v\left(\frac{x}{|x|}\right).$$

We have that $w(x) \neq 0$ and $\langle w(x), x \rangle = 0$ since $w(x) \in T_x(S^n)$. Define

$$F(x, t) = (\cos \pi t)x + (\sin \pi t)w(x).$$

This is clearly continuous and we have

$$\begin{aligned} F(x, 0) &= x \\ F(x, 1) &= -x. \end{aligned}$$

Furthermore, we claim $F(x, t)$ lies in $\mathbb{R}^{n+1} - 0$ for all $x \in \mathbb{R}^{n+1} - 0$ and $0 \leq t \leq 1$. To see this, observe that we have $\langle F(x, t), x \rangle = (\cos \pi t)\langle x, x \rangle$. So if $\cos \pi t \neq 0$, then $\langle F(x, t), x \rangle \neq 0$ because $x \neq 0$. If $\cos \pi t = 0$, then we must have $t = 1/2$ and then $F(x, 1/2) = w(x) \neq 0$. Thus, we have a homotopy between the identity and the antipodal map. In particular, we must have that the antipodal map f induces the identity map on $\mathrm{H}^n_{\mathrm{dR}}(R^{n+1} - 0) \cong \mathbb{R}$. However, Lemma **??** gives that $\mathrm{H}^n_{\mathrm{dR}}(f)$ is multiplication by $(-1)^{n+1}$. This forces n to be odd.

Conversely, suppose that n is odd. Write $n = 2m - 1$. Define

$$v(x_1, x_2, \ldots, x_{2m-1}, x_{2m}) = (-x_2, x_1, \ldots, -x_{2m}, x_{2m-1}).$$

This is a vector field that satisfies $v(x) \neq 0$ for all $x \in S^n$. □

This theorem has many interesting consequences. For example, one can apply this theorem to anything that can be represented as a vector field on the surface of the Earth. For instance, there is at least one point on the Earth at this moment where there is no wind at all!

The following theorem is purely a point-set topology result. The only tool needed that we did not include in Chapter 2 is the notion of partitions of unity. We could actually strengthen the result by replacing $\mathbb{R}^n$ by any metric space and $\mathbb{R}^m$ by any locally convex topological space.

Theorem 3.4.33. *(Urysohn-Tietze) If $A \subset \mathbb{R}^n$ is closed and $f : A \to \mathbb{R}^m$ is continuous, then there exists a continuous map $g : \mathbb{R}^n \to \mathbb{R}^m$ with $g|_A = f$.*

Proof. For $x \in \mathbb{R}^n$, define

$$\rho(x, A) = \inf_{y \in A} |x - y|.$$

Given any $z \in \mathbb{R}^n - A$ we can define an open neighborhood $U_z \subset \mathbb{R}^n - A$ of z given by

$$U_z = \left\{ x \in \mathbb{R}^n : |x - z| < \frac{1}{2}\rho(z, A) \right\}.$$

The collection of these open sets gives an open cover of $\mathbb{R}^n - A$. Thus, we can use this open cover to define a partition of unity $\{p_z\}$. Define g by

$$g(x) = \begin{cases} f(x) & x \in A \\ \sum_{z \in \mathbb{R}^n - A} p_z(x) f(a(x_0)) & x \in \mathbb{R}^n - A \end{cases}$$

where for $z \in \mathbb{R}^n - A$ we choose $a(z) \in A$ so that

$$|z - a(z)| < 2\rho(z, A).$$

This is a smooth function on $\mathbb{R}^n - A$ because for any x there is a neighborhood V of x so that p_z vanishes on V for all but finitely many z, so g is a finite sum. However, we still must check that g is continuous on the boundary of A.

Given $x \in U_z$ and y on the boundary of A we have

$$\begin{aligned} |y - z| &\leq |y - x| + |x - z| \\ &< |x - y| + \frac{1}{2}\rho(z, A) \\ &\leq |x - y| + \frac{1}{2}|y - z|. \end{aligned}$$

Thus,

$$|y - z| < 2|x - y|$$

for all $x \in U_z$. Since we have

$$\begin{aligned} |z - a(z)| &< 2\rho(z, A) \\ &\leq 2|z - y| \end{aligned}$$

we have for $x \in U_z$ that

$$\begin{aligned} |y - a(z)| &\leq |y - z| + |z - a(z)| \\ &< 3|z - y| \\ &< 6|x - y|. \end{aligned}$$

For $x \in \mathbb{R}^n - A$ we have

$$g(x) - g(y) = \sum_{z \in \mathbb{R}^n - A} p_z(x)(f(a(z)) - f(y))$$

and

$$|g(x) - g(y)| \le \sum_z p_z(x)|f(a(z)) - f(y)| \tag{3.4}$$

where the sum is over those z so that $x \in U_z$.

Now for $\epsilon > 0$, choose $\delta > 0$ so that $|f(b) - f(y)| < \epsilon$ for every $b \in A$ with $|b - y| < 6\delta$. If $x \in \mathbb{R}^n - A$ and $|x - y| < \delta$, then for z with $x \in U_z$ we have $|y - a(z)| < 6\delta$ and $|f(a(z)) - f(y)| < \epsilon$. Thus, using equation (3.4) we have

$$|g(x) - g(y)| \le \sum_z p_z(x)\epsilon = \epsilon$$

and so g is continuous on the boundary of A as well. □

Corollary 3.4.34. *Let $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ be closed sets and $\phi : A \to B$ a homeomorphism. There is a homeomorphism $h : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ so that $h(x, 0_m) = (0_n, \phi(x))$ for all $x \in A$.*

Proof. Theorem 3.4.33 allows us to extend ϕ to a continuous map $g : \mathbb{R}^n \to \mathbb{R}^m$. Define a map

$$h_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$$

by

$$h_1(x, y) = (x, y + g(x)).$$

One can easily check that this is a homeomorphism as it is clearly continuous and the inverse map is given by

$$h_1^{-1}(x, y) = (x, y - g(x)).$$

We can also extend ϕ^{-1} to a continuous map

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

and define a homeomorphism

$$h_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$$

by

$$h_2(x, y) = (x + f(x), y).$$

Set $h = h_2^{-1} \circ h_1$. Then for $x \in A$ we have

$$\begin{aligned} h(x, 0_m) &= h_2^{-1}(x, g(x)) \\ &= h_2^{-1}(x, \phi(x)) \\ &= (x - f(\phi(x)), \phi(x)) \\ &= (0, \phi(x)) \end{aligned}$$

as claimed. □

Corollary 3.4.35. *If $\phi : A \to B$ is a homeomorphism between closed sets in $\mathbb{R}^n$, then ϕ extends to a homeomorphism $\phi' : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$.*

Proof. Combine the homeomorphism of Corollary 3.4.34 with the homeomorphism that switches the factors. $\square$

Theorem 3.4.36. *Let $A \subsetneq \mathbb{R}^n$, $B \subsetneq \mathbb{R}^n$ be closed sets and assume A is homeomorphic to B via ϕ. Then*

$$\mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^n - A) \cong \mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^n - B)$$

for all $m \geq 0$.

Proof. We begin by applying Proposition 3.4.23 inductively to conclude that for all $m > 0$ and all $r \geq 0$ we have

$$\begin{aligned} \mathrm{H}^{m+r}_{\mathrm{dR}}(\mathbb{R}^{n+r} - A) &\cong \mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^n - A) \\ \mathrm{H}^{r}_{\mathrm{dR}}(\mathbb{R}^{n+r} - A) &\cong \mathrm{H}^0_{\mathrm{dR}}(\mathbb{R}^n - A)/\mathbb{R} \end{aligned}$$

and similarly for B. Now we apply Corollary 3.4.35 to see that $\phi'|_{\mathbb{R}^{2n}-A}$ gives a homeomorphism between $\mathbb{R}^{2n} - A$ and $\mathbb{R}^{2n} - B$. Thus, we have

$$\begin{aligned} \mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^n - A) &\cong \mathrm{H}^{m+n}_{\mathrm{dR}}(\mathbb{R}^{2n} - A) \\ &\cong \mathrm{H}^{m+n}_{\mathrm{dR}}(\mathbb{R}^{2n} - B) \\ &\cong \mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^n - B) \end{aligned}$$

for all $m > 0$ and

$$\begin{aligned} \mathrm{H}^0_{\mathrm{dR}}(\mathbb{R}^n - A)/\mathbb{R} &\cong \mathrm{H}^n_{\mathrm{dR}}(\mathbb{R}^{2n} - A) \\ &\cong \mathrm{H}^n_{\mathrm{dR}}(\mathbb{R}^{2n} - B) \\ &\cong \mathrm{H}^0_{\mathrm{dR}}(\mathbb{R}^n - B)/\mathbb{R}. \end{aligned}$$

$\square$

Note that this result shows that if A and B are homeomorphic closed sets in $\mathbb{R}^n$ then $\mathbb{R}^n - A$ and $\mathbb{R}^n - B$ have the same number of connected components.

Example 3.4.37. Let $X \subset \mathbb{R}^3$ be a subset that is homeomorphic to S^1. Such a subset is called a knot; two examples are pictured below.

Figure 3.1: Trefoil knot

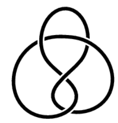


Figure 3.2: Listing knot

We can use Theorem 3.4.36 to calculate $\mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^3 - X)$. This theorem shows it is enough to calculate $\mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^3 - S^1)$.

We can write

$$\mathbb{R}^2 - S^1 = \mathrm{Int}(D^2) \sqcup (\mathbb{R}^2 - D^2)$$

and so

$$\mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^2 - S^1) = \mathrm{H}^m_{\mathrm{dR}}(\mathrm{Int}(D^2)) \oplus \mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^2 - D^2).$$

We know that $\mathrm{Int}(D^2)$ is star-shaped and $\mathbb{R}^2 - D^2$ is homeomorphic to $\mathbb{R}^2 - 0$. Thus we have

$$\mathrm{H}^m_{\mathrm{dR}}(\mathrm{Int}(D^2)) \cong \begin{cases} \mathbb{R} & m = 0 \\ 0 & m > 0 \end{cases}$$

and

$$\mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^2 - D^2) \cong \begin{cases} \mathbb{R} & m = 0 \\ \mathbb{R} & m = 1 \\ 0 & m \geq 2. \end{cases}$$

Thus,

$$\mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^2 - S^1) \cong \begin{cases} \mathbb{R}^2 & m = 0 \\ \mathbb{R} & m = 1 \\ 0 & m \geq 2. \end{cases}$$

We now apply Proposition 3.4.23 to see that

$$\mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^3 - X) \cong \begin{cases} \mathbb{R} & 0 \leq m \leq 2 \\ 0 & m > 2. \end{cases}$$

Exercise 3.4.38. Do the analogous calculation for $X \subset \mathbb{R}^n$ homeomorphic to S^k for $1 \leq k \leq n-2$.

The following theorem, at least in the case of a curve in the plane, was so obvious that for many years no one bothered to write down a precise statement, let along a proof. In the case of a smooth closed curve in the plane it is an easy result in vector calculus as one can employ an argument using normal vectors. However, if one considers the curve given by the von Koch snowflake fractal, it is easy to see such an argument will not work in general.

Theorem 3.4.39. *(Jordan-Brouwer separation theorem) Let $n \geq 2$ and $X \subset \mathbb{R}^n$ be homeomorphic to S^{n-1}. Then*

1. *The space $\mathbb{R}^n - X$ has two connected components. One of the components is bounded and the other is unbounded.*

2. *The set of boundary points for the connected components is given by X.*

Proof. First, observe that since X is homeomorphic to S^{n-1} it is necessarily compact. Since we are in Euclidean space, this gives that X is closed. Applying Theorem 3.4.36 we see that $\mathbb{R}^n - X$ has two connected components because $\mathbb{R}^n - S^{n-1}$ has two connected components, namely, $\text{Int}(D^n) = \{x \in \mathbb{R}^n : |x| < 1\}$ and $W = \{x \in \mathbb{R}^n : |x| > 1\}$.

Let $r = \max_{x \in X} |x|$. This is well defined because the map $x \mapsto |x|$ is a continuous function and X is compact. The set $rW = \{x \in \mathbb{R}^n : |x| > r\}$ is a connected set and so much be contained in one of the connected components of $\mathbb{R}^n - X$. The component that contains rW is then clearly unbounded. We also see that the other component must be contained in the set $\{x \in \mathbb{R}^n : |x| \leq r\}$ and so is bounded. This gives the first part of the theorem.

Let $x \in X$ and let U be an open set containing x in $\mathbb{R}^n$. We want to show that U intersects each of the connected components of $\mathbb{R}^n - X$ so that it is a boundary point. Let $A = X - (X \cap U)$. This is a closed subset of X and so is homeomorphic to a proper closed subset of S^{n-1}, call it B. It is easy to see that $\mathbb{R}^n - B$ is connected (it is path-connected in particular), so we must have $\mathbb{R}^n - A$ is connected as well. Let U_1 and U_2 be the connected components of $\mathbb{R}^n - X$ with U_2 the unbounded component. Given any $x_1 \in U_1$ and $x_2 \in U_2$, there is a continuous path $\gamma : [0,1] \to \mathbb{R}^n - A$ so that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. It is then clear that $\gamma^{-1}(X)$ is nonempty and so the curve given by γ must intersect X. This gives that $\gamma^{-1}(X)$ is a nonempty closed subset of $[0,1]$. Thus, we have that $\gamma^{-1}(X)$ has a largest, y_2, and smallest element, y_1 (possibly equal). Note that $y_1, y_2 \in (0,1)$. We have that $\gamma(y_1)$ and $\gamma(y_2)$ both lie in $X \cap U$ since the curve lies in $\mathbb{R}^n - A$. We have that $\gamma([0, y_1)) \subset U_1$ and $\gamma((y_2, 1]) \subset U_2$ and so there is a $t \in [0, y_1)$ so that

$$\gamma(t_1) \in U_1 \cap U$$

and a $t_2 \in (y_2, 1]$ so that

$$\gamma(t_2) \in U_2 \cap U.$$

Thus, we have the result. □

Exercise 3.4.40. Let $A \subset \mathbb{R}^n$ be homeomorphic to D^k for $k \leq n$. Show that $\mathbb{R}^n - A$ is connected.

Theorem 3.4.41. *Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ an injective continuous map. The image $f(U)$ is open in $\mathbb{R}^n$ and f maps U homeomorphically to $f(U)$.*

Proof. To show that $f : U \to f(U)$ is a homeomorphism, it is enough to show that f takes open sets in U to open sets in $f(U)$. Thus, it is enough to show that $f(U)$ is open in $\mathbb{R}^n$ since then the same argument will work for any open set in U.

Let $B(x_0, \delta) = \{x \in \mathbb{R}^n : |x - x_0| < \delta\}$ be a basis element contained in U. Let $S = \partial B(x_0, \delta)$ be the boundary and $\text{Int}(B(x_0, \delta))$ the interior. Since U can be covered by such discs, it is enough to show that $f(\text{Int}(B(x_0, \delta)))$ is open. The case $n = 1$ follows from calculus so we assume $n \geq 2$.

Note that S is homeomorphic to S^{n-1}, as is $f(S)$. Let U_1 and U_2 be the connected components of $\mathbb{R}^n - f(S)$ with U_1 bounded and U_2 unbounded. The previous exercise gives that $\mathbb{R}^n - f(B(x_0, \delta))$ is connected and since it is disjoint from $f(S)$, it must be contained in U_1 or U_2. Now $f(D)$ is necessarily bounded as it is compact in $\mathbb{R}^n$, so we must have $\mathbb{R}^n - f(B(x_0, \delta))$ is unbounded and so contained in U_2. Thus, $f(S) \cup U_1 \subset \mathbb{R}^n - U_2 \subset f(B(x_0, \delta))$. Thus, $U_1 \subset f(\mathrm{Int}(B(x_0, \delta)))$. Since $\mathrm{Int}(B(x_0, \delta))$ is connected, we have that $f(\mathrm{Int}(B(x_0, \delta)))$ is connected and so is equal to U_1. Thus, it is open as claimed. □

Corollary 3.4.42. *Let $U \subset \mathbb{R}^n$ have the subspace topology and suppose that U is homeomorphic to an open set in $\mathbb{R}^n$. Then U is open in $\mathbb{R}^n$.*

Corollary 3.4.43. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be nonempty open sets. If U is homeomorphic to V, then $m = n$.*

Proof. Assume without loss of generality that $m < n$. Let $V \subset \mathbb{R}^m \subsetneq \mathbb{R}^n$, so V is a subset of $\mathbb{R}^n$ as well. The previous corollary then gives that V is open in $\mathbb{R}^n$. However, this contradicts V being contained in $\mathbb{R}^m$. □

Proposition 3.4.44. *Let $X \subset \mathbb{R}^n$ be homeomorphic to S^{n-1} with $n \geq 2$. Let U_1 and U_2 be the bounded and unbounded connected components of $\mathbb{R}^n - X$. We have*

$$\mathrm{H}^m_{\mathrm{dR}}(U_1) \cong \begin{cases} \mathbb{R} & m = 0 \\ 0 & m > 0 \end{cases}$$

and

$$\mathrm{H}^m_{\mathrm{dR}}(U_2) \cong \begin{cases} \mathbb{R} & m = 0, n-1 \\ 0 & m \neq 0, n-1. \end{cases}$$

Proof. The case of $m = 0$ follows immediately from Theorem 3.4.39. Assume $m > 0$. Set $U = \mathbb{R}^n - D^n$. Then we have isomorphisms

$$\begin{aligned} \mathrm{H}^m_{\mathrm{dR}}(U_1) \oplus \mathrm{H}^m_{\mathrm{dR}}(U_2) &\cong \mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^n - X) \\ &\cong \mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^n - S^{n-1}) \\ &\cong \mathrm{H}^m_{\mathrm{dR}}(\mathrm{Int}(D^n)) \oplus \mathrm{H}^m_{\mathrm{dR}}(U) \\ &\cong \mathrm{H}^m_{\mathrm{dR}}(U). \end{aligned}$$

The natural inclusion map $\iota : U \hookrightarrow \mathbb{R}^n - 0$ is a homotopy equivalence where the inverse is given by

$$f(x) = \left(\frac{|x| + 1}{|x|} \right) x.$$

These are homotopic to the identity by the straight line homotopy. Thus, we have

$$\mathrm{H}^m_{\mathrm{dR}}(\iota) : \mathrm{H}^m_{\mathrm{dR}}(\mathbb{R}^n - 0) \longrightarrow \mathrm{H}^m_{\mathrm{dR}}(U)$$

is an isomorphism. So we have

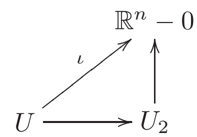
$$\mathrm{H}^m_{\mathrm{dR}}(U) \cong \begin{cases} \mathbb{R} & m = 0, n-1 \\ 0 & m \neq 0, n-1. \end{cases}$$

Thus, if $m \neq 0, n-1$ then we have

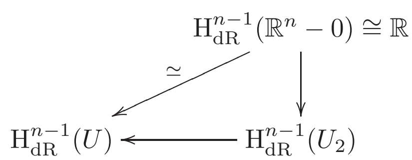
$$\mathrm{H}^m_{\mathrm{dR}}(U_1) = \mathrm{H}^m_{\mathrm{dR}}(U_2) = 0.$$

In general we have that $\dim_{\mathbb{R}} \mathrm{H}^m_{\mathrm{dR}}(U_i)$ is 0 or 1. Thus, it is enough to show that $\mathrm{H}^{n-1}_{\mathrm{dR}}(U_2) \cong \mathbb{R}$.

Assume that $0 \in U_1$. If not, we can translate under straight line homotopy to make it so. Furthermore, assume that $U_1 \cup X \subset D^n$. If this is not the case, we can contract the space under straight line homotopy to make it so. We then have a commutative diagrams of inclusions



Thus, we have



Since $\mathrm{H}^{n-1}_{\mathrm{dR}}(U) \cong \mathbb{R}$ and this isomorphism factors through $\mathrm{H}^{n-1}_{\mathrm{dR}}(U_2)$, it must be the case that $\mathrm{H}^{n-1}_{\mathrm{dR}}(U_2) \neq 0$ and so must be isomorphic to $\mathbb{R}$ as claimed. □

We end this section by calculating the de Rham cohomology of $\mathbb{R}^n$ with r holes.

Theorem 3.4.45. *Let $K_1, \ldots, K_r$ be disjoint compact sets in $\mathbb{R}^n$ with ∂K_j homeomorphic to S^{n-1} for $j = 1, \ldots, r$. Then for $U = \mathbb{R}^n - \bigcup_{j=1}^r K_j$ we have*

$$\mathrm{H}^m_{\mathrm{dR}}(U) \cong \begin{cases} \mathbb{R} & m = 0 \\ \mathbb{R}^r & m = n-1 \\ 0 & m \neq 0, n-1. \end{cases}$$

Proof. We proceed by induction on r. The case of $r = 1$ is given by Proposition 3.4.44. Assume the result is true for

$$U_1 = \mathbb{R}^n - \bigcup_{j=1}^{r-1} K_j.$$

Set $U_2 = \mathbb{R}^n - K_r$. Since the K_j are disjoint we have $U_1 \cup U_2 = \mathbb{R}^n$. Furthermore, $U_1 \cap U_2 = U$. We now apply Mayer-Vietoris.

Let $m = 0$. Then we have

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathrm{H}^0_{\mathrm{dR}}(U) \longrightarrow 0$$

by the induction hypothesis and our previous calculations. Thus, $\mathrm{H}^0_{\mathrm{dR}}(U) \cong \mathbb{R}$ as claimed.

Suppose now that $m > 0$. If $m = n - 1$, we have

$$0 \longrightarrow \mathbb{R}^{r-1} \oplus \mathbb{R} \longrightarrow \mathrm{H}^{n-1}_{\mathrm{dR}}(U) \longrightarrow 0$$

by induction and our previous calculations. If $m \neq 0, n-1$ then we have

$$0 \longrightarrow \mathrm{H}^m_{\mathrm{dR}}(U) \longrightarrow 0.$$

Thus, we have the result. □

3.5 Smooth $\mathbb{R}$-Manifolds

Though we have done many interesting things in this chapter in regards to open subsets of Euclidean space, it is often the case that objects we are interested in studying are not open subsets of Euclidean space. For example, curves are not open subsets but are extremely interesting. In fact, even the sphere which we have looked at often is not an open subset so requires further theory. In this section we set the basic definitions and properties of smooth manifolds. These are spaces that locally look like Euclidean space so will allow us to apply the differential theory we have developed to study them.

Definition 3.5.1. Let M be a Hausdorff space with a countable basis. We call M a *topological manifold* if there exists $n \geq 0$ so that for each $x \in M$ there is an open neighborhood U of x and a homeomorphism $\varphi : U \to \mathbb{R}^n$. The number n is referred to as the *dimension* of M and we will refer to M as a n-manifold if we need to specify the dimension.

Recall that for any $x \in \mathbb{R}^n$ and $\epsilon > 0$, the open ball $B(x, \epsilon) \subset \mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$. Thus, it is equivalent to the above definition to have a homeomorphism $\varphi : U \to B(x, \epsilon)$. In fact, it is enough to have a homeomorphism $\varphi : U \to W$ for W open in $\mathbb{R}^n$.

Definition 3.5.2. Let M be a topological manifold of dimension n.

1. A *chart* (U, φ) on M is a homeomorphism $\varphi : U \to W$ where $U \in \mathcal{T}_M$ and $W \in \mathcal{T}_{\mathbb{R}^n}$.

2. A *local parameterization* around the point $x \in M$ is a homeomorphism $\phi : W \to U$ where $W \in \mathcal{T}_{\mathbb{R}^n}$ and $U \in \mathcal{T}_M$ is a neighborhood of x.

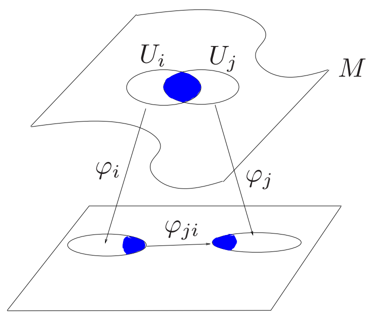
3. A system $A = \{\varphi_i : U_i \to W_i : i \in I\}$ of charts is called an *atlas* provided $\{U_i\}_{i \in I}$ covers M.

4. We say the atlas A is *smooth* when all of the maps

$$\varphi_{ji} := \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are smooth. These maps are called *transition functions.*

We illustrate the transition functions as follows:



Definition 3.5.3. A *smooth structure* on a topological manifold M is a maximal atlas $\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$, i.e., it is an atlas satisfying that if (U, φ) is a chart such that $\varphi \circ \varphi_i^{-1}$ and $\varphi_i \circ \varphi^{-1}$ are smooth for all $i \in I$, then $(U, \varphi) \in \mathcal{A}$.

One should note that given any atlas A, there is a unique smooth structure $\mathcal{A}$ containing A. Namely, set

$$\mathcal{A} = \{(U, \varphi) : \ \varphi \circ \varphi_i^{-1} \text{ and } \varphi_i \circ \varphi^{-1} \text{ are smooth for all } \varphi_i \in A\}.$$

Thus, we do not need to give the entire smooth structure in general as we can specify an atlas and then know that there is a unique smooth structure that contains it. In general when we refer to a chart we mean a chart in the smooth structure.

Definition 3.5.4. A *smooth manifold* is a pair $(M, \mathcal{A})$ consisting of a topological manifold M and a smooth structure $\mathcal{A}$ on M.

As we will only be interested in smooth manifolds, from this point on when we write "manifold" or n-manifold it should be understood that we are working with smooth manifolds. We will drop the atlas $\mathcal{A}$ from the notation for M much as we do not include $\mathcal{T}_M$ in the notation when working with a topological space M. The smooth structure should be clear from the context.

Exercise 3.5.5. Show that if M is a compact manifold it is not possible to give a smooth atlas on M consisting of only one chart.

Example 3.5.6. The sphere S^n is a n-manifold. To see this, we define an atlas with $2(n+1)$ charts $(U_{\pm j}, \varphi_{\pm j})$ where

$$U_{+j} = \{(x_1, \dots, x_{n+1}) \in S^n : x_j > 0\}$$
$$U_{-j} = \{(x_1, \dots, x_{n+1}) \in S^n : x_j < 0\}$$

and

$$\varphi_{\pm j} : U_{\pm j} \to B(0,1) \subset \mathbb{R}^n$$
$$(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, \widehat{x}_j, \ldots, x_{n+1}).$$

The inverse maps are given by

$$\varphi_{\pm j}^{-1}(y_1, \ldots, y_n) = (y_1, \ldots, y_{j-1}, \pm\sqrt{1 - (y_1^2 + \cdots + y_n^2)}, y_j, \ldots, y_n).$$

This gives that we have the necessary charts. It remains to see that the transition functions are smooth. This is straightforward and left as an exercise.

Example 3.5.7. Recall that in § 2.9 we defined $\mathbb{RP}^n$ as a quotient space. We can also view this as a quotient of S^n by setting $\mathbb{RP}^n$ to be the set of equivalence classes $[x] = \{x, -x\}$ for $x \in S^n$. One should check that this is equivalent to the definition given in § 2.9. We have the quotient map

$$\pi : S^n \to \mathbb{RP}^n$$

given by $\pi(x) = [x]$. Thus, U is open in $\mathbb{RP}^n$ if and only if $\pi^{-1}(U)$ is open in S^n. We can use this along with the previous example to put a smooth structure on $\mathbb{RP}^n$. Note that we have $\pi(U_{\pm j})$ are all open in $\mathbb{RP}^n$ and $\pi(U_{+j}) = \pi(U_{-j})$ for all j. Thus, we can set

$$U_j = \pi(U_{\pm j}).$$

Observe that $\pi^{-1}(U_j) = U_{+j} \cup U_{-j}$ and $U_{+j} \cap U_{-j} = \emptyset$. We have that

$$\pi : U_{+j} \to U_j$$

is a homeomorphism. We can define

$$\varphi_j := \varphi_{+j} \circ \pi^{-1} : U_i \to B(0,1) \subset \mathbb{R}^n.$$

The U_j cover $\mathbb{RP}^n$ and the φ_j are smooth maps giving charts. Thus, $\mathbb{RP}^n$ is a n-manifold.

Example 3.5.8. Consider the figure in $\mathbb{R}^2$ given by $M = \{(\sin 2t, \sin t) : 0 \leq t \leq 2\pi\}$. This is not a smooth manifold as there is no chart around 0.

Exercise 3.5.9. Given a topological space M, it is possible to put different smooth structures on M. For example, if $M = \mathbb{R}$ we can use a single chart given by $U = \mathbb{R}$ and $\varphi = \text{id}$ to put a smooth structure on $\mathbb{R}$. One also has that $U = \mathbb{R}$ and $\varphi : U \to \mathbb{R}$ defined by $\varphi(x) = x^3$ is a chart giving a smooth structure on $\mathbb{R}$. Show that these are not equivalent smooth structures.

Exercise 3.5.10. Let M be a m-manifold and N a n-manifold. Show that $M \times N$ is a $(m+n)$-manifold.

Definition 3.5.11. Let N be a subset of a n-manifold M. We say N is a *smooth submanifold of dimension* k if for every $x \in N$ there is a chart $\varphi : U \to W$ on M with $W \in \mathcal{T}_{\mathbb{R}^n}$ so that $x \in U$ and $\varphi(U \cap N) = W \cap \mathbb{R}^k$ where $\mathbb{R}^k \subset \mathbb{R}^n$ is the standard subspace topology.

As with manifolds, we will only be interested in smooth submanifolds so when we use the term "submanifold", it is understood that it is smooth.

Example 3.5.12. For $k \leq n$ we have $\mathbb{R}^k$ is a submanifold of $\mathbb{R}^n$ of dimension k.

Example 3.5.13. The sphere S^n is a submanifold of $\mathbb{R}^{n+1}$ of dimension $n-1$.

Let M and N be smooth manifolds of dimensions m and n respectively. Let $f : M \to N$ be a continuous map. Let $x \in M$. We say f *is smooth at* x if there exist charts $\varphi_1 : U_1 \to W_1$ and $\varphi_2 : U_2 \to W_2$ with $x \in U_1$, $f(x) \in U_2$ so that $\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(f^{-1}(U_2)) \to W_2$ is smooth in a neighborhood of $\varphi_1(x)$. In terms of a diagram, we have

$$\begin{array}{ccc} U_1 & \xrightarrow{\quad f \quad} & U_2 \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle h_2} \\ W_1 & \xrightarrow{\varphi_2 \circ f \circ \varphi_1^{-1}} & W_2. \end{array}$$

Basically what we are saying is that when we map down to Euclidean space the map there should be smooth. If f is smooth at every $x \in M$ we say that f is *smooth*.

Note that since transition functions are by definition smooth, the definition of smooth given above does not depend on the charts chosen in the smooth structures for M and N. Thus, once a smooth structure $\mathcal{A}$ has been chosen for M, we know which functions on M are smooth functions.

Definition 3.5.14. Let M and N be manifolds and $f : M \to N$ a homeomorphism. We say f is a *diffeomorphism* if f is smooth and has a smooth inverse.

One should note here that by definition the charts in our smooth structure are diffeomorphisms. This is important to note in terms of rectifying our definition of smooth manifold with those given in books that begin by assuming that M is a subset of Euclidean space. In that definition, it is assumed from the start that the charts are diffeomorphisms. Clearly such a definition does not work in our case as there is a priori no notion of smoothness without first specifying a smooth structure.

Exercise 3.5.15. Let T be the torus defined in § 2.9.

1. Show that T is diffeomorphic to $S^1 \times S^1$ and so conclude that T is a 2-manifold.

2. Show directly from the definition that T is a 2-manifold.

Consider a point $x \in \mathbb{R}^n$ and let $v = (v_1, \ldots, v_n)$ be a vector. We can view v as an operator on functions f that are differentiable in a neighborhood of x, in particular, we set $v(f)$ to be the directional derivative of f at x in the the direction of v, i.e.,

$$v(f) = v_1 \left.\frac{\partial f}{\partial x_1}\right|_x + \cdots + v_n \left.\frac{\partial f}{\partial x_n}\right|_x.$$

One can easily check that this operation satisfies

$$v(f + cg) = v(f) + cv(g)$$

and

$$v(fg) = f(x)v(g) + g(x)v(f)$$

for any $c \in \mathbb{R}$ and any functions f and g that are differentiable at x. Such operators are familiar in algebra and are known as *linear derivations.* In general we have the following definition.

Definition 3.5.16. Let S be a ring and M a S-module. A map $d : S \to M$ is a *derivation* if it satisfies

$$d(fg) = fd(g) + gd(f)$$

for all $f, g \in S$. If S happens to be a R-algebra, we say a derivation d is a *R-linear derivation*, or just *linear derivation* if it is a map of R-modules.

Exercise 3.5.17. Fit the above specific example of a linear derivation into the general definition just give.

Observe that the operation of taking a derivative at a point is purely a local operation, so it can be characterized by "zooming in" at the point in question. Before we state exactly what this means in our situation, we recall the following definition from algebra.

Definition 3.5.18. Let I be a nonempty set with a partial order $\leq$. For each $i \in I$, let G_i be an additive abelian group. Suppose for every pair $i, j \in I$ with $i \leq j$ there is a map $\rho_{ij} : A_i \to A_j$ so that

1. $\rho_{jk} \circ \rho_{ij} = \rho_{ik}$ whenever $i \leq j \leq k$ and
2. $\rho_{ii} = 1$ for all $i \in I$.

Let H be the disjoint union of all the G_i. Define an equivalence relation $\sim$ on H by setting $g \sim h$ if and only if there exists a k with $i, j \leq k$ and $\rho_{ik}(g) = \rho_{jk}(h)$ for $g \in G_i$, $h \in G_j$. The set of equivalence classes is called the *direct limit of the* G_i and is denoted $\varinjlim_i G_i$.

Now consider a manifold M of dimension n and let $x \in M$. For an open set $U \subset M$ that contains x, let $C^\infty(U)$ denote the set of smooth real-valued functions defined on U. These are groups under addition and we can put a natural order on the open sets by inclusion, which gives maps between the spaces by restriction, i.e., if $U \subset V$, we have $\rho_U^V : C^\infty(V) \to C^\infty(U)$. This allows us to consider the direct limit of the spaces $C^\infty(U)$. Set

$$C_x^\infty = \varinjlim_U C^\infty(U).$$

We will return to this notion in Chapter 5 when we introduce stalks. As this definition can be a bit unwieldy to work with, we boil it down into more familiar terms. Let f and g be C^∞ functions defined on open sets U and V of x. We say that f and g have the same *germ at* x if there is an open neighborhood $W \subset U \cap V$ containing x on which f and g agree. This gives an equivalence relation on the space of C^∞ functions on neighborhoods of x. In fact, this set of equivalence classes is precisely the space C_x^∞. We will denote the germ of f at x by f_x. Note that any germ f_x at x has a well-defined value at the point x, namely, choose any $g \in C^\infty(U)$ that represents the germ and set $f_x(x) = g(x)$.

Definition 3.5.19. A *tangent vector* v *at the point* $x \in M$ is a linear derivation of the algebra C_x^∞. We denote the set of tangent vectors to M at x by $T_x(M)$ and call it the *tangent space*.

Given $v, w \in T_x(M)$, a germ f_x, and a constant $c \in \mathbb{R}$, we can define

$$(v + w)(f_x) = v(f_x) + w(f_x)$$

and

$$(cv)(f_x) = c(v(f_x)).$$

This makes the tangent space $T_x(M)$ into a $\mathbb{R}$-vector space.

Let $\mathfrak{m}_x$ denote the subset of C_x^∞ of germs that vanish at x. It is easy to check that $\mathfrak{m}_x$ is an ideal. We write $\mathfrak{m}_x^k$ to denote the k^{th} power of the ideal. We then have the following very useful result.

Proposition 3.5.20. *The tangent space* $T_x(M)$ *is naturally isomorphic to* $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$.

Proof. First, let $v \in T_x(M)$. Observe that v is a function on $\mathfrak{m}_x$. Furthermore, using the fact that v is a linear derivation gives that v vanishes on $\mathfrak{m}_x$ and so we have a natural map from $T_x(M)$ into $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$.

Now let $\phi \in (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$. Let $f_x(x)$ denote the germ with the constant value $f(x)$. We define a tangent vector associated to ϕ by setting

$$v_\phi(f_x) = \phi((f_x - f_x(x))\mathfrak{m}_x^2)$$

for $f_x C_x^\infty$. Of course we must check that this is actually a tangent vector. The fact that the map is linear is straight forward and can be checked as an exercise. We show that it is a derivation. Let f_x and g_x be germs at x. We have

$$\begin{aligned} v_\phi(f_x g_x) &= \phi((f_x g_x - f_x(x) g_x(x))\mathfrak{m}_x^2) \\ &= \phi((((f_x - f_x(x))(g_x - g_x(x)) + f_x(x)(g_x - g_x(x)) + (f_x - f_x(x))g_x(x))\mathfrak{m}_x^2) \\ &= \phi((f_x - f_x(x))(g_x - g_x(x))\mathfrak{m}_x^2) + f_x(x)\phi((g_x - g_x(x))\mathfrak{m}_x^2) + g_x(x)\phi((f_x - f_x(x))\mathfrak{m}_x^2) \\ &= f_x(x)v_\phi(g_x) + g_x(x)v_\phi(f_x). \end{aligned}$$

Since we now have a mapping from $T_x(M)$ to $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ and vice versa, it only remains to check that they are inverses. We leave this as an exercise. □

The reason that this proposition is particularly useful is that it allows us a convenient way of showing that the dimension of the tangent space as a $\mathbb{R}$-vector space is the same as the dimension of the manifold. Before proving that, we need the following result from multivariable calculus.

Lemma 3.5.21. *Let $k \geq 2$, U a convex open set in $\mathbb{R}^n$ around x, and $g \in C^k(U)$. Then for each $y \in U$, we have*

$$g(y) = g(x)+\sum_{i=1}^{n} \left.\frac{\partial g}{\partial x_i}\right|_x (x_i(y)-x_i(x))+\sum_{i,j}(x_i(y)-x_i(x))(x_j(y)-x_j(x)) \int_0^1 (1-t) \left.\frac{\partial^2 g}{\partial x_i \partial x_j}\right|_{x+t(y-p)} dt.$$

In particular, if $g \in C^\infty(U)$, then the second summation determines an element of $\mathfrak{m}_x^2$ since the integral as a function of y is $C^\infty(U)$.

One can see [13] for a proof of this lemma. We apply it to prove the following.

Theorem 3.5.22. *Let M be a smooth manifold of dimension n and let $x \in M$. Then*

$$\dim_{\mathbb{R}}(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee = n.$$

Proof. Let $x \in M$ and let (U, ϕ) be a chart around x. Let $y_1, \ldots, y_n$ be the coordinate functions, i.e., $y_i = x_i \circ \phi$. Let f_x be a germ at x and let $f \in C^\infty(U)$ be a representative of f_x. We apply the result in Lemma 3.5.21 to the function $f \circ \phi^{-1}$ and then compose with ϕ to obtain:

$$f = \sum_{i=1}^{n} \left.\frac{\partial(f \circ \phi^{-1})}{\partial x_i}\right|_{\phi(x)} (y_i - y_i(x)) + \sum_{i,j}(y_i - y_i(x))(y_j - y_j(x))h$$

in a neighborhood of x with h a smooth function. Thus we have

$$f_x = \sum_{i=1}^{n} \left.\frac{\partial(f \circ \phi^{-1})}{\partial x_i}\right|_{\phi(x)} (y_i - y_i(x)) \pmod{\mathfrak{m}_x^2}.$$

Thus, we see that the cosets $(y_i - y_i(x))\mathfrak{m}_x^2$ spans $\mathfrak{m}_x/\mathfrak{m}_x^2$. Thus we see that $\dim_{\mathbb{R}}(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee \leq n$. We now must show that these are linearly independent. Suppose that we have

$$\sum_{i=1}^{n} a_i(y_i - y_i(x)) \in \mathfrak{m}_x^2.$$

We must show that the a_i are all 0. Observe that we have

$$\sum_{i=1}^{n} a_i(y_i - y_i(x)) \circ \phi^{-1} = \sum_{i=1}^{n}(x_i - x_i(\phi(x))).$$

Thus, we have

$$\sum_{i=1}^{n} a_i(x_i - x_i(\phi(x))) \in \mathfrak{m}^2_{\phi(x)}.$$

However, this implies that

$$\left.\frac{\partial}{\partial x_j}\right|_{\phi(x)} \left(\sum_{i=1}^{n} a_i(x_i - x_i(\phi(x))\right) = 0.$$

However, we know these are linearly independent because we are just working in Euclidean space now, so it must be that the a_i are all 0 as claimed. □

Corollary 3.5.23. *Let M be a smooth manifold of dimension n and let $x \in M$. Then*

$$\dim_{\mathbb{R}} T_x(M) = n.$$

Let f be a differentiable function on a neighborhood of x. We can view tangent vectors as acting on such functions by setting

$$v(f) = v(f_x).$$

Thus, we see that $v(f) = v(g)$ if there is an open neighborhood of x on which f and g agree. We now define some convenient tangent vectors.

Definition 3.5.24. Let (U, ϕ) be a chart around a point $x \in M$ with coordinate functions $y_1, \ldots, y_n$. For each i, define a tangent vector $(\partial/\partial y_i)|_x \in T_x(M)$ by setting

$$\left(\left.\frac{\partial}{\partial y_i}\right|_x\right)(f) = \left.\frac{\partial(f \circ \phi^{-1})}{\partial x_i}\right|_{\phi(x)}$$

for each function f that is smooth in some neighborhood of x. We often use the notation $\left.\frac{\partial f}{\partial y_i}\right|_x$ to denote $\left(\left.\frac{\partial}{\partial y_i}\right|_x\right)(f)$.

Exercise 3.5.25.

1. Show that $(\partial/\partial y_i)|_x$ are tangent vectors. Furthermore, show they are a basis of $T_x(M)$ and are the dual basis to $\{y_i - y_i(x)\}$ of $\mathfrak{m}_x/\mathfrak{m}_x^2$.
2. If $v \in T_x(M)$, show that
$$v = \sum_{i=1}^{n} v(y_i) \left.\frac{\partial}{\partial y_i}\right|_x.$$
3. Applying the definition about to the coordinate functions $x_1, \ldots, x_n$ on $\mathbb{R}^n$ show that one obtains the normal partial derivative operators $\frac{\partial}{\partial x_i}$.

Exercise 3.5.26. Recall from Exercise 3.5.10 that if M is a m-manifold and N a n-manifold, then $M \times N$ is a $(m+n)$-manifold. Show that

$$T_{(x,y)}(M \times N) \cong T_x(M) \oplus T_y(N).$$

Given a map $f : M \to N$ between manifolds, we now define an induced linear map between the tangent spaces. Let $f : M \to N$ be a smooth map between manifolds and let $x \in M$. We wish to define an element $D_x f(v) \in T_{f(x)}(N)$ for each $v \in T_x(M)$. Since $D_x f(v)$ will be a tangent vector, to define it we need to specify its action on functions that are smooth in a neighborhood of $f(x)$. Let g be such a function. Define $D_x f(v)(g)$ by setting

$$D_x f(v)(g) = v(g \circ f).$$

Exercise 3.5.27. Check that this map lands in $T_{f(x)}(N)$.

Exercise 3.5.28. Let (U, ϕ) and (V, ψ) be charts around x and $f(x)$ respectively with coordinates given by $y_i = x_i \circ \phi$ and $z_i = x_i \circ \psi$. Show that

$$D_x f \left(\left. \frac{\partial}{\partial y_j} \right|_x \right) = \sum_{i=1}^{n} \left. \frac{\partial (z_i \circ f)}{\partial y_j} \right|_x \left. \frac{\partial}{\partial z_i} \right|_{f(x)}.$$

Exercise 3.5.29. Let (U, ϕ) be a chart around x and $y_1, \ldots, y_m$ coordinates. Show that $\{D_x y_i|_x\}$ is a basis of $T_x(M)^\vee$ dual to the basis $\{\partial/\partial y_i|_x\}$ of $T_x(M)$. Thus, if $f : M \to \mathbb{R}$ is a smooth function, apply the above result to show that

$$df = \sum_{i=1}^{m} \frac{\partial f}{\partial y_i} dy_i,$$

as in the case of open sets in Euclidean space.

Exercise 3.5.30. Let M_1, M_2, and M_3 be manifolds with $f : M_1 \to M_2$ and $g : M_2 \to M_3$ smooth maps. Prove the chain rule for the corresponding maps on tangent spaces, namely,

$$D_x(g \circ f) = D_{f(x)} g \circ D_x f.$$

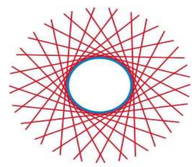
There is a notion called the tangent bundle that packages all of the tangent spaces together at once while still separating them so that they do not intersect. While we will not need the tangent bundle for our purposes, it is an important concept so we include a couple of brief results. One should note that the tangent bundle is a special case of a vector bundle.

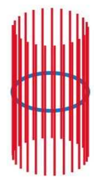
Definition 3.5.31. Let M be a m-manifold. Define the *tangent bundle* TM of M by

$$T(M) = \bigcup_{x \in M} T_x(M).$$

It turns out that the tangent bundle is intrinsic to the manifold itself. This will be shown in Exercise 3.5.34 below.

The tangent bundle of the circle can be pictured as follows where the first picture is the circle in blue with tangent spaces illustrated in red and the second one is the tangent bundle:





Our first step is to show that $T(M)$ is a $2m$-manifold. Observe that there is a natural projection $\pi : T(M) \to M$ given by $\pi(v) = x$ if $v \in T_x(M)$. Let (U, φ) be a chart around $x \in M$ with $\varphi : U \xrightarrow{\simeq} \mathbb{R}^m$ and coordinate functions $x_1, \ldots, x_n$. Set $V = \pi^{-1}(U) \subset T(M)$ and define $\psi : V \to \mathbb{R}^{2m}$ by $\psi(v) = (x_1(\pi(v)), \ldots, x_m(\pi(v)), dx_1(v), \ldots, dx_m(v))$. Now, let $\{(U_i, \varphi_i)\}_{i \in I}$ be an atlas on M and set $V_i = \pi^{-1}(U_i)$. Then given $U \subset T(M)$ we say that U is open in $T(M)$ if and only if $\psi_i(U \cap V_i)$ is open in $\mathbb{R}^{2m}$ for all $i \in I$. This defines a topology on $T(M)$ and $(V_i, \psi_i)_{i \in I}$ defines a smooth structure on $T(M)$.

Exercise 3.5.32. Check that the definition given above for open sets in $T(M)$ satisfies the required properties to give a topology. Check that the transition maps are smooth so that $(V_i, \psi_i)_{i \in I}$ gives a smooth structure on $T(M)$.

Note that one can equivalently write

$$T(M) = \{(x, v) : x \in M, v \in T_x(M)\}.$$

Given a smooth map $f : M \to N$ one has a natural global derivative map $Df : T(M) \to T(N)$ defined by

$$Df(x, v) = (f(x), D_x f(v)).$$

Exercise 3.5.33. Show that Df is a smooth map.

Exercise 3.5.34. Show that if $f : M \to N$ is a diffeomorphism, then $T(M)$ is diffeomorphic to $T(N)$.

In many cases two manifolds may locally look the same, even if they do not globally. As many of the concepts, such as smoothness, are defined locally, it is important to have a notion of when two manifolds look alike locally.

Definition 3.5.35. Let $f : M \to N$ be a smooth map between n-manifolds. We say f is a *local diffeomorphism at $x \in M$* if there exists an open neighborhood U of x and an open neighborhood V of $f(x)$ so that $f|_U : U \to V$ is a diffeomorphism. If f is a local diffeomorphism at each point $x \in M$ we say f is a *local diffeomorphism.*

It is important to note that being a local diffeomorphism is a local property, not a global one. In other words, it is entirely possible for f to be a local diffeomorphism but not a diffeomorphism. For example, consider $M = \mathbb{R}$ and $N = S^1$ with f defined by $f(t) = (\cos t, \sin t)$. Then f is a local diffeomorphism but not a global one.

Recall the inverse function theorem from classical analysis.

Theorem 3.5.36. *Let $U, V \subset \mathbb{R}^n$ be open sets. Let $f : U \to V$ be a smooth map with $D_x f$ an isomorphism. Then f is a local diffeomorphism.*

We will not prove this theorem here as it is a well-known result in analysis. One can see [11] for a proof if it is not a familiar result. We can generalize this theorem to the setting of smooth n-manifolds.

Exercise 3.5.37. Show that if f is a local diffeomorphism at x then $D_x f$ is an isomorphism.

The more remarkable property is that if $D_x f$ is an isomorphism then f is a local diffeomorphism at x. This is nice because checking an isomorphism of vector spaces should be easier than that a function is a local diffeomorphism.

Theorem 3.5.38. *(Inverse Function Theorem) Let M and N be n-manifolds. If $f : M \to N$ is a smooth map so that $D_x f : T_x(M) \to T_{f(x)}(N)$ is an isomorphism, then f is a local diffeomorphism at x.*

Proof. We will use charts along with Theorem 3.5.36 to prove the result. Let (U, φ) and (V, ψ) be charts around x and $f(x)$ respectively. Observe that since $D_x f$ is an isomorphism by assumption and $D_x \varphi$ and $D_{f(x)} \psi$ are isomorphisms by construction, if we set $F = \psi \circ f \circ \varphi^{-1}$ then the fact that the following diagram commutes gives that $D_{\varphi(x)} F$ is an isomorphism.

$$\begin{array}{ccc}
T_x(M) & \xrightarrow{D_x f} & T_{f(x)}(N) \\
{\scriptstyle D_x \varphi}\downarrow & & \downarrow{\scriptstyle D_{f(x)} \psi} \\
\mathbb{R}^n & \xrightarrow{D_{\varphi(x)} F} & \mathbb{R}^n.
\end{array}$$

We now apply Theorem 3.5.36 to see that there exist open neighborhoods W_1 and W_2 of $\varphi(x)$ and $\psi(f(x))$ respectively so that F is a diffeomorphism from W_1 to W_2. We can shrink U, V, W_1 and W_2 if necessary so that $\varphi : U \to W_1$ and $\psi : V \to W_2$ are diffeomorphisms. Thus, we obtain the commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} \\ W_1 & \xrightarrow{F} & W_2. \end{array}$$

Thus, we have that $f|_U$ is a diffeomorphism from U to V as desired. $\square$

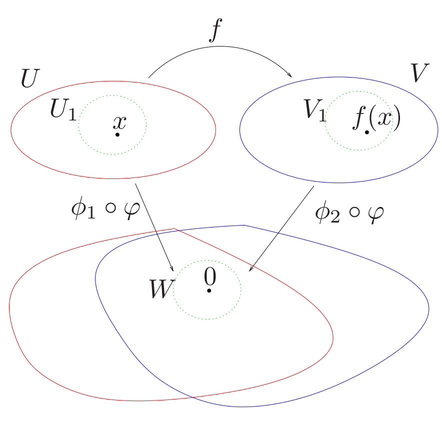
The classical inverse function theorem tells us that if $D_x f$ is an isomorphism then locally f looks like the identity map. In other words, W_1 is an open neighborhood of $x \in \mathbb{R}^n$ and W_2 an open neighborhood of $f(x)$ in $\mathbb{R}^n$, then there exist charts $\phi_1 : W_1 \to W$ and $\phi_2 : W_2 \to W$ so that the following diagram commutes:

$$\begin{array}{ccc} W_1 & \xrightarrow{f} & W_2 \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} \\ W & \xrightarrow{\text{id}} & W. \end{array}$$

To see what this means in our setting, we compose the charts given in the proof of Theorem 3.5.38 with the charts ϕ_1 and ϕ_2 to obtain

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow{\scriptstyle \phi_1 \circ \varphi} & & \downarrow{\scriptstyle \phi_2 \circ \psi} \\ W & \xrightarrow{\text{id}} & W. \end{array}$$

In terms of a picture, we have:



Thus, in a neighborhood of x we have that f looks like the identity map. Note that what is really going on is that when we want to talk about things locally, it is enough to work in a chart. This is the entire point of the definition of a smooth manifold. So the Inverse Function Theorem is saying that we can choose charts so that in these charts the function looks like the identity. This makes working locally, i.e., with these charts, particularly easy.

The Inverse Function Theorem only applies if $\dim M = \dim N$. The natural question to ask is what can we say if $\dim M < \dim N$ or $\dim M > \dim N$? If $f : M \to N$ is a smooth map from a m-manifold to a n-manifold, we have that $D_x f : T_x(M) \to T_{f(x)}(N)$ is a linear map from a vector space of dimension m to a vector space of dimension n. If $m < n$, the best we can hope for is that $D_x f$ is an injection. Similarly, if $m > n$, the best we can hope is that $D_x f$ is a surjection. We begin with the case that $m < n$.

Definition 3.5.39. Let $f : M \to N$ be a smooth map. If $D_x f$ is an injection we say that f is an *immersion at* x. If f is an immersion at every point in M we say f is an *immersion*.

The most basic example of an immersion is the canonical immersion $\mathbb{R}^m \to \mathbb{R}^n$ for $n > m$ given by $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)$. As the Inverse Function Theorem shows us that if $D_x f$ is an isomorphism then f locally looks like the identity map, the Local Immersion Theorem tells us that if f is an immersion at x then f locally looks like the canonical immersion.

Theorem 3.5.40. *(Local Immersion Theorem) Let $f : M \to N$ be an immersion at x. Then there are charts (U, φ) and (V, ψ) around x and $f(x)$ respectively so that the following diagram commutes:*

$$\begin{CD} U @>{f}>> V \\ @V{\varphi}VV @VV{\psi}V \\ W_1 @>{\text{canonical immersion}}>> W_2. \end{CD}$$

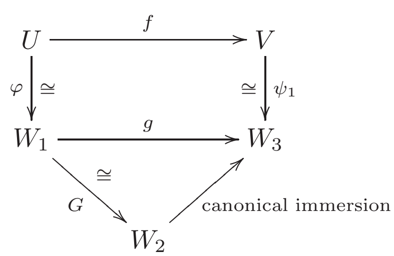
Proof. Let (U_1, φ) and (V_1, ψ_1) be charts around x and $f(x)$ so that the following diagram commutes:

$$\begin{CD} U_1 @>{f}>> V_1 \\ @V{\varphi}VV @VV{\psi_1}V \\ \mathbb{R}^m @>{g}>> \mathbb{R}^n. \end{CD}$$

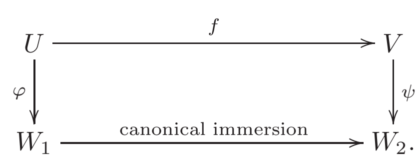
We have that $D_{\varphi(x)} g : \mathbb{R}^m \to \mathbb{R}^n$ is injective. Our goal is to "expand" g so that we can apply the Inverse Function Theorem. Note that we can choose bases of $\mathbb{R}^m$ and $\mathbb{R}^n$ so that $D_{\varphi(x)}g$ is given by the $n \times m$ matrix $\left(\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right)$. Define

$$\begin{aligned} G : \mathbb{R}^m \times \mathbb{R}^{n-m} &\to \mathbb{R}^n \\ (a, b) &\mapsto g(a) + (0, b). \end{aligned}$$

We have that G is clearly a smooth map and $D_{\varphi(x)}G = I_n$ so the Inverse Function Theorem gives that G is a local diffeomorphism. Observe that $g = G \circ (\text{canonical immersion})$. Thus, there exists $W_1 \subset \mathbb{R}^m$, $W_2 \subset \mathbb{R}^m$, and $W_3 \subset \mathbb{R}^n$ so that the following diagram commutes:



where U and V are open subsets of U_1 and V_1 chosen to ensure that the maps φ and ψ_1 are diffeomorphisms. Thus, if we set $\psi = G^{-1} \circ \psi_1$ we have that (V, ψ) is a chart around $f(x)$ so that the following diagram commutes:



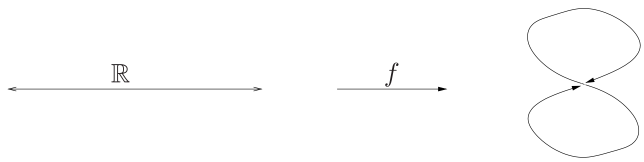
□

It is important to note that this is a local result. As the notion of immersion is an "injectivity" condition, it is natural to inquire if the image of an immersion is a submanifold. Unfortunately this is not true in general.

Exercise 3.5.41. Show that the map from S^1 to $\mathbb{R}^2$ that takes the circle to the figure eight is an immersion but clearly the figure eight is not a submanifold of $\mathbb{R}^2$.

The previous exercise shows that we should require our maps to be injective if we hope to have the image being a submanifold. Even if the map is an injective immersion this is not enough.

Exercise 3.5.42. Show that the image of map f from $\mathbb{R}$ to $\mathbb{R}^2$ defined by



is not a submanifold of $\mathbb{R}^2$.

The previous exercise shows we also need a notion of things that are far apart staying far apart. Fortunately, if we add such a condition and injectivity to our immersion this will be enough!

Definition 3.5.43. A map $f : M \to N$ is called *proper* if the preimage of a compact set is compact. A map that is a proper injective immersion is said to be an *embedding*.

Theorem 3.5.44. *Let $f : M \to N$ be an embedding. Then f maps M diffeomorphically onto a submanifold of N.*

Proof. Our first goal is to show that $f(M)$ is a submanifold of N. Recall that this means for every $f(x) \in f(M)$, there is a chart (V, ψ) in N around $f(x)$ with $\psi : V \xrightarrow{\simeq} W_2 \subset \mathbb{R}^n$ so that $\psi(f(M) \cap V) = W_2 \cap \mathbb{R}^m$ where $\mathbb{R}^m$ sits inside of $\mathbb{R}^n$ via the canonical immersion. Let (U, φ) be a chart around x in M. We claim it is enough to show that $f(U)$ is open in $f(M)$. If $f(U)$ is open in $f(M)$, then we have that since $f(M)$ has the subspace topology in N there is a V' open in N so that $f(U) = f(M) \cap V'$. If (V, ψ) is any chart around $f(x)$ with $\psi : V \xrightarrow{\simeq} W_2$, then we can take for our chart $(V \cap V', \psi|_{V \cap V'})$ with $\psi|_{V \cap V'} : V \cap V' \xrightarrow{\simeq} W_2'$ and see immediately that $\psi|_{V \cap V'}(f(M) \cap V \cap V') = W_2' \cap \mathbb{R}^m$ by applying Theorem 3.5.40 and possibly shrinking our open sets. Thus, in order to see $f(M)$ is a submanifold of N it only remains to show that $f(U)$ is open for any chart (U, φ) around x.

Suppose that there exists a chart (U, φ) so that $f(U)$ is not open in $f(M)$. Then there exists a sequence $\{y_n\}_{n \in \mathbb{N}}$ with $y_n \in f(M) - f(U)$ so that $\{y_n\}$ converges to $y \in f(U)$. Let $x_n \in M$ so that $f(x_n) = y_n$ and $x \in M$ with $f(x) = y$. The fact that f is injective gives that these preimage points are unique. We have that $\{y\} \cup \{y_n\}_{n \in \mathbb{N}}$ is compact along with the fact that f is proper gives that $\{x\} \cup \{x_n\}_{n \in \mathbb{N}}$ is compact as well. Thus, there is a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ that must converge, say to $z \in M$. However, this gives that $\{f(x_{n_k})\}_{k \in \mathbb{N}}$ converges to $f(z)$ and $\{f(x_n)\}_{n \in \mathbb{N}}$ converges to $f(x)$. Again using that f is injective we have that $x = z$. Since U is open, for large enough n we must have $x_n \in U$. However, this contradicts the fact that $f(x_n) = y_n \notin f(U)$. Thus, it must be that $f(U)$ is open. Thus, we have that $f(M)$ is a submanifold.

It only remains to show that $f : M \to f(M)$ is a diffeomorphism. By assumption we have that f is smooth and bijective. This gives an inverse map $f^{-1} : f(M) \to M$. However, since we know that f is a local diffeomorphism, this implies f^{-1} is a local diffeomorphism and so is smooth. Thus, we have a smooth bijective map from M to $f(M)$ with a smooth inverse, thus it is a diffeomorphism. □

The next case to handle is when $f : M \to N$ is a smooth map and $\dim M > \dim N$. In this case $D_x f$ is a linear map from $T_x(M)$ to $T_{f(x)}(N)$ with $\dim_{\mathbb{R}} T_x(M) > \dim_{\mathbb{R}} T_{f(x)}(N)$ so the best we can hope for in this case is for $D_x f$ to be surjective.

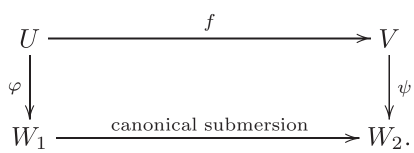
Definition 3.5.45. Let $f : M \to N$ be a smooth map so that $D_x f$ is surjective. We say f is a *submersion at x*. If f is a submersion at each point $x \in M$, we simply say f is a *submersion*.

The canonical submersion is given by the map

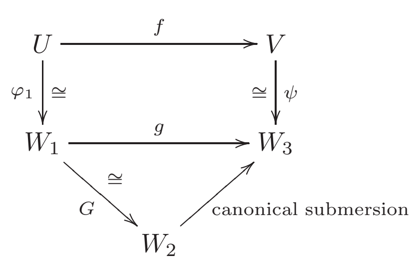
$$\begin{aligned} \mathbb{R}^m &\to \mathbb{R}^n \\ (x_1, \ldots, x_m) &\mapsto (x_1, \ldots, x_n). \end{aligned}$$

As in the case of immersions, it turns out that locally this is the only submersion.

Theorem 3.5.46. *(Local Submersion Theorem) Let $f : M \to N$ be a submersion at x. Then there are charts (U, φ) and (V, ψ) around x and $f(x)$ respectively so that the following diagram commutes:*



Proof. The proof of this theorem is very similar to the proof of the Local Immersion Theorem. We begin by choosing charts (U_1, φ_1) and (V, ψ) around x and $f(x)$ respectively with $\varphi_1 : U_1 \xrightarrow{\simeq} \mathbb{R}^m$ and $\psi : V \xrightarrow{\simeq} \mathbb{R}^n$. Let $g = \psi \circ f \circ \varphi_1^{-1}$. Observe that $D_{\varphi_1(x)} g$ is surjective, so by change of bases we may assume that $D_{\varphi_1(x)} g$ is given by the $n \times m$ matrix $(I_n | 0)$. Define $G : \mathbb{R}^n \to \mathbb{R}^n$ by $G(a) = (g(a), a_{n+1}, \ldots, a_m)$ where $a = (a_1, \ldots, a_m)$. Then we have that $D_{\varphi_1(x)} G = I_m$ and so G is a local diffeomorphism at $\varphi_1(x)$. Observe that $g = G \circ$ (canonical submersion). Thus, there exist open sets W_1, W_2, W_3 so that the following diagram commutes (after possibly shrinking U and V):



Thus, if we set $\varphi = G \circ \varphi_1$ we have the result. ∎

We saw above that by requiring an immersion to be injective and proper we could conclude that the image is a submanifold. In the case of a submersion, we are interested in the preimage of points. In particular, given $f : M \to N$, we would like to be able to determine when $f^{-1}(y)$ is a submanifold of M for $y \in N$. Such results are incredibly important when studying algebraic curves for instance. We will see such an example after the result.

Definition 3.5.47. Let $f : M \to N$ be a smooth map. We say $y \in N$ is a *regular value of* f if $D_x f : T_x(M) \to T_y(N)$ is surjective for every $x \in f^{-1}(y)$.

Theorem 3.5.48. *Let $f : M \to N$ be smooth and $y \in N$ a regular value of f. Then $f^{-1}(y)$ is a submanifold of M with dimension $m - n$.*

Proof. Let $y \in N$ be a regular value and $x \in f^{-1}(y)$. (Note the result is trivially true if $y \notin f(M)$.) We need to show that there is a chart (U, φ) around x in M with $\varphi : U \xrightarrow{\simeq} W_1 \subset \mathbb{R}^m$ so that $\varphi(U \cap f^{-1}(y)) = W_1 \cap \mathbb{R}^{m-n}$.

Since y is a regular value of f, we see that f is a submersion at x and so Theorem 3.5.46 gives charts (U, φ) and (V, ψ) around x and y respectively as in Theorem 3.5.46. Observe that $U \cap f^{-1}(y)$ is an open set in $f^{-1}(y)$ containing x. We must show that $\varphi(U \cap f^{-1}(y)) = W_1 \cap \mathbb{R}^{m-n}$. We may assume that $\psi(y) = 0$. Observe that we have

$$f^{-1}(y) \cap U = \{\varphi^{-1}(0, \ldots, 0, x_{n+1}, \ldots, x_m) : (0, \ldots, 0, x_{n+1}, \ldots, x_m) \in W_1 \cap \mathbb{R}^{m-n}\}.$$

Thus, $\varphi(f^{-1}(y) \cap U) = W_1 \cap \mathbb{R}^{m-n}$ and we have the result. ☐

Corollary 3.5.49. *Let $f : M \to N$ be smooth and set $Z = f^{-1}(y)$ for $y \in N$ a regular value. Then for any $x \in Z$ the kernel of $D_x f : T_x(M) \to T_y(N)$ is $T_x(Z)$.*

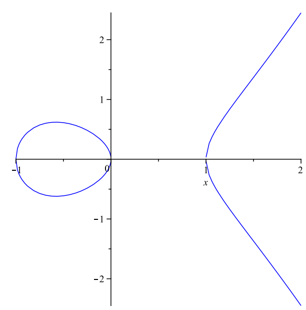
Proof. Note that since $f(Z) = y$, we have $D_x f|_Z = 0$. The face that y is a regular value gives $D_x f : T_x(M) \to T_y(N)$ is onto and so

$$\begin{aligned} \dim_{\mathbb{R}} \ker(D_x f) &= \dim_{\mathbb{R}} T_x(M) - \dim_{\mathbb{R}} T_y(N) \\ &= \dim M - \dim N \\ &= \dim Z. \end{aligned}$$

Thus, $T_x(Z)$ is a subspace of the kernel that has the same dimension as the kernel and so they must be equal. ☐

Example 3.5.50. Consider the map $f : \mathbb{R}^{n+1} \to \mathbb{R}$ given by $f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$. It is clear that this is a smooth map. Observe that for $a = (a_1, \ldots, a_{n+1})$ we have $D_a f = (2a_1, \ldots, 2a_{n+1})$. Thus, for $a \neq 0$ we have that $D_a f$ is surjective and so f is a submersion away from 0. In particular, this shows that $S^n = f^{-1}(1)$ is a n-submanifold of $\mathbb{R}^{n+1}$. This shows how useful this theorem can be as it eliminates having to define charts in many cases.

Example 3.5.51. Let E be the elliptic curve defined by $y^2 = x^3 - x$ along with the usual point at infinity. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = y^2 - x^3 + x$. Observe that for $a = (a_1, a_2)$ we have $D_a f = (-3a_1^2 + 1, 2a_2)$. The only points where $D_a f$ will not be surjective are when $3a_1^2 = 1$ and $2a_2 = 0$. Thus, we must have $a_2 = 0$ and $a_1 = \pm 1/\sqrt{3}$. Observe that $E(\mathbb{R}) = f^{-1}(0)$. We easily see that the points $(\pm 1/\sqrt{3}, 0)$ do not lie in $E(\mathbb{R})$ and so $E(\mathbb{R})$ is a 1-submanifold of $\mathbb{R}^2$. A graph of $E(\mathbb{R})$ is given as follows:



Definition 3.5.52. Let M be a manifold that has a group structure as well. If the multiplication and inversion maps are smooth we call M a *Lie group*.

The theory of Lie groups is a subject unto itself so we only give an example.

Example 3.5.53. Let $\mathrm{M}_n(\mathbb{R})$ denote the set of $n \times n$ matrices with entries in $\mathbb{R}$. It is easy to see that this is diffeomorphic to $\mathbb{R}^{n^2}$ by just listing out the entries of the matrix as a tuple. Let $\mathrm{Skew}_n(\mathbb{R})$ denote the subset of skew-symmetric matrices, i.e., the matrices A in $\mathrm{M}_n(\mathbb{R})$ so that ${}^tA = -A$. This is a manifold of dimension $\frac{n(n-1)}{2}$ (see Exercise 3.5.54.) Define $f : \mathrm{M}_{2n}(\mathbb{R}) \to \mathrm{Skew}_{2n}(\mathbb{R})$ by $f(A) = {}^tA\iota_n A$ where $\iota_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. It is clear that this map is well-defined and smooth. We would like to show that $\mathrm{Sp}_{2n}(\mathbb{R}) = f^{-1}(\iota_n)$ is a submanifold of $\mathrm{M}_{2n}(\mathbb{R})$. First we calculate the derivative of f at a matrix $A \in \mathrm{M}_{2n}(\mathbb{R})$. We have

$$\begin{aligned} D_A f(B) &= \lim_{h\to 0} \frac{f(A + hB) - f(A)}{h} \\ &= \lim_{h\to 0} \frac{{}^t(A + hB)\iota_n(A + hB) - {}^tA\iota_n A}{h} \\ &= \lim_{h\to 0} \frac{{}^tA\iota_n A + h({}^tA\iota_n B) + h({}^tB\iota_n A) + h^2({}^tB\iota_n B) - {}^tA\iota_n A}{h} \\ &= {}^tA\iota_n B + {}^tB\iota_n A \\ &= {}^tA\iota_n B - {}^t({}^tA\iota_n B). \end{aligned}$$

In order to apply Theorem 3.5.48 we must show that $D_A f$ is surjective for each $A \in \mathrm{Sp}_{2n}(\mathbb{R})$, i.e., for each $C \in \mathrm{Skew}_{2n}(\mathbb{R})$ there exists $B \in \mathrm{M}_{2n}(\mathbb{R})$ so that ${}^tA\iota_n B + {}^tB\iota_n A = C$. Observe that since $C \in \mathrm{Skew}_{2n}(\mathbb{R})$ we can write $C = \frac{1}{2}C - \frac{1}{2}({}^tC)$. In particular, if we can find B so that ${}^tA\iota_n B = \frac{1}{2}C$ we will have the surjectivity. Note that ι_n is invertible, in particular, $\iota_n^{-1} = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$. This shows that A is invertible as well with $A^{-1} = \iota_n^{-1}\,{}^tA\iota_n$. Thus, set $B = ({}^tA\iota_n)C$ and we have the result. Thus, $\mathrm{Sp}_{2n}(\mathbb{R})$ is a submanifold of $\mathrm{M}_{2n}(\mathbb{R})$ of dimension $(2n)^2 - \frac{2n(2n-1)}{2} = n(2n-1)$.

Exercise 3.5.54. Show that $\mathrm{Skew}_n(\mathbb{R})$ is a manifold of dimension $\frac{n(n-1)}{2}$. In particular, show it is a submanifold of $\mathrm{M}_n(\mathbb{R})$.

Exercise 3.5.55. Show that matrix multiplication and inversion is smooth on $\mathrm{Sp}_{2n}(\mathbb{R})$ and so $\mathrm{Sp}_{2n}(\mathbb{R})$ is a Lie group.

Exercise 3.5.56. Let $O_n(\mathbb{R})$ be the set of orthogonal matrices, i.e., the $A \in \mathrm{M}_n(\mathbb{R})$ so that ${}^tAA = 1_n$. Show that $O_n(\mathbb{R})$ is a Lie group of dimension $\frac{n(n+1)}{2}$.

These examples show that Theorem 3.5.48 is a very powerful tool for constructing manifolds. It is natural to ask given a smooth map $f : M \to N$, are there regular values of f? If so, are there many of them? Fortunately there are always a large number of regular values. This is given by Sard's theorem, which we state here but omit the proof of. One can find a proof in Chapter 1 of [6].

Theorem 3.5.57. *(Sard's Theorem) Let $f : M \to N$ be a smooth map of manifolds. The set of points of N that are not regular values constitutes a set of measure 0.*

Corollary 3.5.58. *The regular values of any smooth map $f : M \to N$ are dense in Y. In fact, if $f_i : M_i \to N$ is a countable collection of smooth maps, then the points in N that are simultaneously regular values for all f_i are dense in N.*

We end this section by stating Whitney's embedding theorem. This will allow us to assume $M \subset \mathbb{R}^{2m+1}$ for any manifold M. This is useful in many contexts but certainly is not an obvious result. For instance, projective space does not obviously embed into a Euclidean space. A proof of Whitney's embedding theorem can be found in [1].

Theorem 3.5.59. *(Whitney's Embedding Theorem) Let M be a m-manifold. Then there exists an embedding of M into $\mathbb{R}^{2m+1}$.*

In fact, Whitney was able to show that M embeds into $\mathbb{R}^{2m}$, but this result is much more difficult than the one stated. For our purposes the important point is just to note that such an embedding exists. It is also useful to note that the stronger version of the theorem is optimal. In particular, it states that the Klein bottle embeds into $\mathbb{R}^4$. One can see that the Klein bottle does not embed in $\mathbb{R}^3$ which shows the result is optimal. Another easy example would be to see that S^1 embeds into $\mathbb{R}^2$ but not into $\mathbb{R}^1$.

3.6 Differential Forms on Smooth Manifolds

In this section we define differential forms on smooth manifolds and the corresponding de Rham cohomology groups. This will set the stage for § 3.7 when we study integration on smooth manifolds and prove Stokes' theorem in this setting. As in § 3.5, when we refer to a manifold M we will always mean a smooth manifold unless otherwise noted.

Recall that for $U \subset \mathbb{R}^m$ an open set, the space of differential forms $\Omega^k(U)$ consists of smooth functions $\omega : U \to \mathrm{Alt}^k(\mathbb{R}^m)$, i.e., for each $x \in U$ we had $\omega(x) \in \mathrm{Alt}^k(\mathbb{R}^m)$. We want to generalize this to the setting of manifolds. Let M be a m-manifold and let

$$\omega : M \to \mathrm{Alt}^k(T(M))$$

be a function with $\omega(x) \in \mathrm{Alt}^k(T_x(M))$ for each $x \in M$. Let (W, ψ) be a local parameterization of M, i.e., $\psi : W \to U$ is a diffeomorphism between an open set $W \subset \mathbb{R}^n$ and an open set U in M. For any $y \in W$, this gives an isomorphism

$$D_y\psi : \mathbb{R}^m \xrightarrow{\simeq} T_{\psi(y)}(M),$$

which in turn gives an isomorphism

$$\mathrm{Alt}^k(D_y\psi) : \mathrm{Alt}^k(T_{\psi(y)}(M)) \xrightarrow{\simeq} \mathrm{Alt}^k(\mathbb{R}^m).$$

Define

$$\begin{aligned} \Omega^k(\psi)(\omega) = \psi^*(\omega) : W &\longrightarrow \mathrm{Alt}^k(\mathbb{R}^m) \\ y &\mapsto \mathrm{Alt}^k(D_y\psi)(\omega(\psi(y))). \end{aligned}$$

For clarity,

$$\begin{array}{ccccccc} W & \xrightarrow{\psi} & U & \xrightarrow{\omega} & \mathrm{Alt}^k(T_{\psi(y)}(M)) & \xrightarrow{\mathrm{Alt}^k(D_y\psi)} & \mathrm{Alt}^k(\mathbb{R}^m) \\ & & & & & & \\ y & \longmapsto & \psi(y) & \longmapsto & \omega(\psi(y)) & \longmapsto & \mathrm{Alt}^k(D_y\psi)(\omega(\psi(y))). \end{array}$$

Observe that $\psi^*(\omega)$ is a function on Euclidean spaces so we have a notion of smoothness here from classical analysis. We can use this to define the notion of a differential form on M.

Definition 3.6.1. Let $\omega : M \to \coprod_{x \in M} \mathrm{Alt}^k(T_x(M))$ be as above. We say ω is a *smooth differential form on M* if $\psi^*(\omega)$ is smooth for every local parameterization (W, ψ). We denote the set of smooth differential forms by $\Omega^k(M)$.

We will refer to smooth differential forms as just differential forms.

It is useful to note here that the map ψ^* is the generalization of the map $\Omega^k(\psi)$ given in § 3.3 when ψ was then a map between open sets in Euclidean space. The only difference is in the present case our map moves through an open set in M where before all the open sets were in Euclidean space. From now on we write $\Omega^k(\psi)$ as simply ψ^* as it is assumed by this point one has sufficient experience with the notions to be able to understand k from context. One can go through and write everything with $\Omega^k(\psi)$ instead of ψ^* if one prefers, but one should keep in mind that when reading other sources this will be denoted ψ^* there as well.

Exercise 3.6.2. Show that $\Omega^0(M)$ consists of smooth functions.

Exercise 3.6.3. Check that $(\psi_1 \circ \psi_2)^*(\omega) = \psi_2^*(\psi_1^*(\omega))$.

Lemma 3.6.4. *Let M be a manifold and $\{(W_i, \psi_i)\}_{i \in I}$ a family of local parameterizations so that $\cup_{i \in I} \psi_i(W_i) = M$. If $\psi_i^*(\omega)$ is smooth for each i, then ω is smooth.*

Proof. Let (W, ψ) be any local parameterization and let $y \in W$. Since the sets $\psi_i(W_i)$ cover M there is an i so that $\psi(y) \in \psi_i(W_i)$. Let $f := \psi_i^{-1} \circ \psi : \psi^{-1}(\psi_i(W)) \to W_i$. We have that f is a smooth map between open sets in $\mathbb{R}^m$. Observe that if we restrict ψ to $\psi^{-1}(\psi_i(W_i))$, then we have $\psi = \psi_i \circ f$. Thus, using the exercise above we have in a neighborhood of y that

$$\begin{aligned} \psi^*(\omega) &= (\psi_i \circ f)^*(\omega) \\ &= f^*(\psi_i^*(\omega)). \end{aligned}$$

However, by assumption f^* and ψ_i^* are smooth so we get that ψ^* is smooth and thus ω is smooth. □

Note that this result says that we do not have to check smoothness for every local parameterization, it is enough to check it for a collection of parameterizations that cover the manifold.

Our next step is to define the exterior derivative in this setting. As with everything else, we define the exterior derivative

$$d^k : \Omega^k(M) \to \Omega^{k+1}(M)$$

via local parameterizations. Let $\omega \in \Omega^k(M)$ and (W, ψ) be a local parameterization around $x \in M$. Set

$$d_x^k \omega = \mathrm{Alt}^{k+1}((D_y \psi)^{-1}) \circ d_x^k(\psi^* \omega)$$

where $\psi(y) = x$. First, one should check that this indeed maps $\Omega^k(M)$ to $\Omega^{k+1}(M)$. This follows from the definitions of the maps involved. We need to check that this definition is independent of the choice of local parameterization used. Let (W_1, ψ) be a local parameterization for $x \in M$ with $\psi(y) = x$. Any other local parameterization can be given by $\psi \circ f$ where $f : W_2 \to W_1$ is a diffeomorphism, $W_2 \subset \mathbb{R}^m$ open. Let $x_1, \ldots, x_{k+1} \in T_x(M)$ and choose

$v_1, \dots, v_{k+1} \in \mathbb{R}^m$ with $D_y(\psi \circ f)(v_i) = x_i$. Let $y' \in W_2$ so that $f(y') = y$ and define $w_1, \dots, w_{k+1}$ in $\mathbb{R}^m$ so that $D_{y'} f(v_i) = w_i$. We must show

$$d_y^k \psi^*(\omega)(w_1, \dots, w_{k+1}) = d_{y'}^k (\psi \circ f)^*(\omega)(v_1, \dots, v_{k+1}).$$

From a previous exercise we have that

$$(\psi \circ f)^* = f^* \circ \psi^*.$$

We also have from Theorem 3.3.34 that

$$d^k f^*(\tau) = f^* d^k(\tau)$$

where here we want $\tau = \psi^*(\omega)$. Observe that we have

$$\begin{aligned} d_{y'}^k (\psi \circ f)^*(\omega)(v_1, \dots, v_{k+1}) &= d_{y'}^k (f^* \circ \psi^*)(\omega)(v_1, \dots, v_{k+1}) \\ &= d_{y'}^k f^*(\psi^*(\omega))(v_1, \dots, v_{k+1}) \\ &= f^*(d_y^k \psi^*(\omega))(v_1, \dots, v_{k+1}) \\ &= \mathrm{Alt}^{k+1}(D_{y'} f)(d_y^k \psi^*(\omega))(v_1, \dots, v_{k+1}) \\ &= d_y^k \psi^*(\omega)(D_{y'} f(v_1), \dots, D_{y'} f(v_{k+1})) \\ &= d_y^k \psi^*(\omega)(w_1, \dots, w_{k+1}). \end{aligned}$$

Exercise 3.6.5. Check that $d^{k+1} \circ d^k = 0$ for all $k \geq 0$.

Exercise 3.6.6. Show that if M is diffeomorphic to N, then $\Omega^k(M) \cong \Omega^k(N)$ for all $k \geq 0$.

Thus, we have produced a chain complex

$$\cdots \longrightarrow \Omega^{k-1}(M) \xrightarrow{d^{k-1}} \Omega^k(M) \xrightarrow{d^k} \Omega^{k+1}(M) \longrightarrow \cdots.$$

Note that $\Omega^k(M) = 0$ for $k > \dim M$ as then $\mathrm{Alt}^k(T_x M) = 0$ for all $x \in M$. As was shown in § 3.2, a chain complex gives rise to cohomology groups.

Definition 3.6.7. Let M be a manifold. The k^{th} *de Rham cohomology group of* M is defined by

$$\mathrm{H}_{\mathrm{dR}}^k(M) = \mathrm{H}^k(\Omega^*(M)).$$

Exercise 3.6.8. Show that $\mathrm{H}_{\mathrm{dR}}^0(M)$ consists of locally constant functions.

Note that Exercise 3.6.6 shows that if M is diffeomorphic to N then $\mathrm{H}_{\mathrm{dR}}^k(M) \cong \mathrm{H}_{\mathrm{dR}}^k(N)$ as one would expect. It is also important to note that if $U \subset \mathbb{R}^n$ is an open set, if we view U as a n-manifold this definition of de Rham cohomology agrees with what was defined earlier.

Let $f : M \to N$ be a smooth map. For each $k \geq 0$ this induces a map $f^* : \Omega^k(N) \to \Omega^k(M)$ via

$$\Omega^k(f)(\tau)(x) = f^*(\tau)(x) := \mathrm{Alt}^k(D_x f)(\tau(f(x)))$$

for $k > 0$ and

$$\Omega^0(f)(\tau)(x) = f^*(\tau)(x) := \tau(f(x)).$$

For this to be well defined one must check that the image of f^* actually lies in $\Omega^k(M)$. We leave this verification as an exercise to the curious reader. It basically amounts to showing it is smooth by choosing a local parameterization. As in § 3.2, this induces a map on cohomology $\mathrm{H}^k_{\mathrm{dR}}(f) : \mathrm{H}^k_{\mathrm{dR}}(N) \to \mathrm{H}^k_{\mathrm{dR}}(M)$ defined by

$$\mathrm{H}^k_{\mathrm{dR}}(f)([\tau]) = [f^*(\tau)].$$

We can define a wedge product as before since the wedge product on the alternating spaces was done in full generality in § 3.2. Thus, we define $\wedge$: $\Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ by $(\omega \wedge \tau)(x) = \omega(x) \wedge \tau(x)$. As before, we extend this to the cohomology groups in the natural way.

Exercise 3.6.9. Show that for $\omega \in \Omega^k(M)$ and $\tau \in \Omega^l(M)$ we have

$$d^{k+l}(\omega \wedge \tau) = d^k\omega \wedge \tau + (-1)^k \omega \wedge d^l \tau$$
$$\omega \wedge \tau = (-1)^{kl} \tau \wedge \omega.$$

This shows that we have a contravariant functor from the category of manifolds with the morphisms being smooth maps to the category of graded anti-commutative $\mathbb{R}$-algebras.

Let $M = U_1 \cup U_2$ with U_1 and U_2 open sets. As in § 3.4 we have natural inclusion maps $i_k : U_k \hookrightarrow U_1 \cup U_2$ and $j_k : U_1 \cap U_2 \hookrightarrow U_k$. Following the same argument as given in the proof of Theorem 3.4.2 we have the following result.

Theorem 3.6.10. *With the set-up as above we have that the following sequence is exact for each $l \geq 0$:*

$$0 \longrightarrow \Omega^l(M) \xrightarrow{i^l} \Omega^l(U_1) \oplus \Omega^l(U_2) \xrightarrow{j^l} \Omega^l(U_1 \cap U_2) \longrightarrow 0$$

where $i^l(\omega) = (i_1^*(\omega), i_2^*(\omega))$ and $j^l(\omega_1, \omega_2) = j_2^*(\omega_2) - j_1^*(\omega_1)$.

Recall that the only real difficulty in showing the exactness of the sequence in Theorem 3.6.10 is the exactness at the last factor. As in the case of open sets in Euclidean space, we see that given $\omega \in \Omega^l(U_1 \cap U_2)$ and a partition of unity $\{\rho_{U_1}, \rho_{U_2}\}$ subordinate to $\{U_1, U_2\}$, the element $(-\rho_{U_2}\omega, \rho_{U_1}\omega)$ maps to ω.

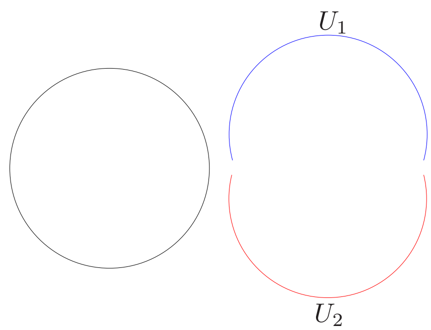
As before, we use this exact sequence to produce a long exact sequence in cohomology.

Theorem 3.6.11. *(Mayer-Vietoris Sequence) With M, U_1, and U_2 as above we have a long exact sequence in cohomology*

$$\cdots \longrightarrow \mathrm{H}^l_{\mathrm{dR}}(M) \xrightarrow{\mathrm{H}^l_{\mathrm{dR}}(i^l)} \mathrm{H}^l_{\mathrm{dR}}(U_1) \oplus \mathrm{H}^l_{\mathrm{dR}}(U_2) \xrightarrow{\mathrm{H}^l_{\mathrm{dR}}(j^l)} \mathrm{H}^l_{\mathrm{dR}}(U_1 \cap U_2) \xrightarrow{\partial^l} \mathrm{H}^{l+1}_{\mathrm{dR}}(U) \longrightarrow \cdots.$$

Exercise 3.6.12. Show that for $\omega \in \Omega^l(U_1 \cap U_2)$, the map ∂^l is given by $\partial^l([\omega]) = [d^l(\rho_{U_2}\omega)]$ on U_1 and $\partial^l([\omega]) = [d^l(\rho_{U_1}\omega)]$ on U_2 where $\{\rho_{U_1}, \rho_{U_2}\}$ is a partition of unity subordinate to $\{U_1, U_2\}$.

Example 3.6.13. We compute the de Rham cohomology of the circle. We choose open U_1 and U_2 in S^1 as in the following picture:



We have $S^1 = U_1 \cup U_2$ and $\mathrm{H}^0_{\mathrm{dR}}(U_1) \oplus \mathrm{H}^0_{\mathrm{dR}}(U_2) \cong \mathbb{R} \oplus \mathbb{R}$ and $\mathrm{H}^0_{\mathrm{dR}}(U_1 \cap U_2) \cong \mathbb{R} \oplus \mathbb{R}$ since we have that $\dim_{\mathbb{R}} \mathrm{H}^0_{\mathrm{dR}}(M)$ is the number of connected components of M as before. We have

$$0 \longrightarrow \mathrm{H}^0_{\mathrm{dR}}(S^1) \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(S^1) \longrightarrow 0$$

where we have used that $\mathrm{H}^1_{\mathrm{dR}}(U_1) \cong \mathrm{H}^1_{\mathrm{dR}}(U_2) = 0$ because U_1 and U_2 are diffeomorphic to $\mathbb{R}$ and $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{R}) = 0$ by Poincare's Lemma. This shows that $\mathrm{H}^1_{\mathrm{dR}}(S^1) \cong \mathbb{R}$ and so we have

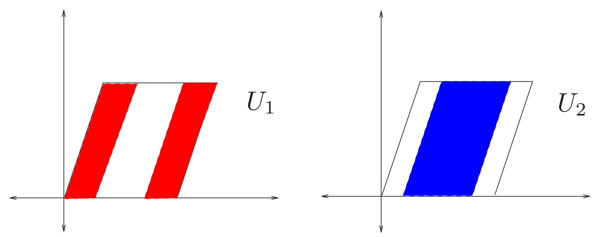
$$\mathrm{H}^k_{\mathrm{dR}}(S^1) \cong \begin{cases} \mathbb{R} & k = 0, 1 \\ 0 & k \geq 2. \end{cases}$$

As in the case of open subsets of $\mathbb{R}^n$, one has the following result in the case of manifolds as well. The proofs are virtually identical to the ones given before. The interested reader can fill in the details by working via charts.

Theorem 3.6.14. *Homotopic maps induce the same map in cohomology. In particular, two manifolds with the same homotopy type have the same de Rham cohomology.*

Example 3.6.15. Let T be the torus. We have seen that this is a 2-manifold. We now calculate the cohomology groups of T. We must make the assumption that $\mathrm{H}^2_{\mathrm{dR}}(T) \cong \mathbb{R}$ for this calculation to work out. We will see in the next section that this is a special case of a much more general result, namely that if M is a compact connected oriented m-manifold, then $\mathrm{H}^m_{\mathrm{dR}}(M) \cong \mathbb{R}$. The torus is such a manifold, so we use this result in our calculation.

Let U be a little more than the upper half of the torus and V be a little more than the bottom half. This can be pictured as follows if we let the top of the torus be where the left and right edges identify.



We have that U_1 and U_2 are diffeomorphic and are given by an annulus. This is clearly seen to be homotopic to S^1, so we know the cohomology of U_1 and U_2. We have that $U_1 \cap U_2$ looks like an annulus inside another annulus. It is clear that $U_1 \cap U_2$ can be written as the disjoint union of open sets W_1 and W_2, each homotopic to S^1. Thus, we have $\mathrm{H}^k_{\mathrm{dR}}(U_1 \cap U_2) \cong \mathrm{H}^k_{\mathrm{dR}}(W_1) \oplus \mathrm{H}^k_{\mathrm{dR}}(W_2) \cong \mathrm{H}^k_{\mathrm{dR}}(S^1) \oplus \mathrm{H}^k_{\mathrm{dR}}(S^1)$. Note that since T is connected, we have that $\mathrm{H}^0_{\mathrm{dR}}(T) \cong \mathbb{R}$. We can now apply Mayer-Vietoris to calculate $\mathrm{H}^k_{\mathrm{dR}}(T)$:

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathrm{H}^1_{\mathrm{dR}}(T) \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathrm{H}^2_{\mathrm{dR}}(T) \to 0.$$

We have that $\mathrm{Im}(\mathrm{H}^0_{\mathrm{dR}}(i^0)) \cong \mathbb{R}$ and so necessarily $\ker(\mathrm{H}^0_{\mathrm{dR}}(j^0)) \cong \mathbb{R}$. This gives that $\mathrm{Im}(\mathrm{H}^0_{\mathrm{dR}}(j^0)) \cong (\mathbb{R} \oplus \mathbb{R})/\mathbb{R} \cong \mathbb{R}$. Thus, $\ker(\partial^0) \cong \mathbb{R}$ and so $\mathrm{Im}(\partial^0) \cong (\mathbb{R} \oplus \mathbb{R})/\mathbb{R} \cong \mathbb{R}$. Thus we see $\dim_{\mathbb{R}} \mathrm{H}^1_{\mathrm{dR}}(T) \geq 1$. We also obtain that $\ker(\mathrm{H}^1_{\mathrm{dR}}(i^1)) \cong \mathbb{R}$.

Now observe that since ∂^1 is surjective and $\mathrm{H}^2_{\mathrm{dR}}(T) \cong \mathbb{R}$, we have $\ker(\partial^1) \cong \mathbb{R}$. Thus, $\mathrm{Im}(\mathrm{H}^1_{\mathrm{dR}}(j^1)) \cong \mathbb{R}$ as well. Then this gives that $\ker(\mathrm{H}^1_{\mathrm{dR}}(j^1)) \cong \mathbb{R}$, which in turn gives $\mathrm{Im}(\mathrm{H}^1_{\mathrm{dR}}(i^1)) \cong \mathbb{R}$. Thus, we have $\mathrm{H}^1_{\mathrm{dR}}(T)/\ker(\mathrm{H}^1_{\mathrm{dR}}(i^1)) \cong \mathrm{Im}(\mathrm{H}^1_{\mathrm{dR}}(i^1))$, i.e., $\mathrm{H}^1_{\mathrm{dR}}(T)/\mathbb{R} \cong \mathbb{R}$. So we must have $\mathrm{H}^1_{\mathrm{dR}}(T) \cong \mathbb{R} \oplus \mathbb{R}$.

Exercise 3.6.16. Calculate the cohomology groups of the "doughtnut with two holes."

Our next step is to discuss differential forms and de Rham cohomology with compact support. Recall that the support of a continuous function f on a topological space X is $\mathrm{Cl}(\{x \in X : f(x) \neq 0\})$. We can now go through and define the differential forms with compact support $\Omega^k_c(M)$ to be the space of differential forms on M that have compact support. Many of the same properties hold. In particular, $\{\Omega^k_c(M)\}_{k \geq 0}$ forms a complex so that we can define the de Rham cohomology groups with compact support $\mathrm{H}^k_{\mathrm{dR},c}(M) = \mathrm{H}^k(\Omega^*_c(M))$. There is one major difference in the case of compact support. Let $f : M \to N$ be smooth function and let $\omega \in \Omega^k_c(N)$. In this situation the map $\Omega^k_c(f)$ is a map $\Omega^k_c(N) \to \Omega^k(M)$. In particular, there is no reason that $\Omega^k_c(f)(\omega)$ should have compact support even if $\omega \in \Omega^k_c(N)$.

Exercise 3.6.17. Let $f : M \times \mathbb{R} \to M$ be the projection map. Show that the of a function with compact support under f does not necessarily have compact support.

This shows that Ω^k_c is not a functor on the category of smooth manifolds and smooth maps. We need to adjust things if we wish this to be a functor.

There are two ways to accomplish this. The first way is to observe that Ω_c^k is a contravariant functor under proper maps. In particular, if we require $f : M \to N$ to be a proper map, then everything works out fine. This is not the approach we will take. The second way is to observe that we can make Ω_c^k into a covariant functor under inclusions of open sets. In particular, if $i : U \hookrightarrow M$ is the inclusion map of an open set $U \in \mathcal{T}_M$ into M, then $i_c^k : \Omega_c^k(U) \to \Omega_c^k(M)$ is the map that sends $\omega \in \Omega_c^k(M)$ to a differential form on M by setting $\omega(x) = 0$ if $x \in M - U$. The reason we choose this second method is that it is the natural way to frame Poincare duality.

Exercise 3.6.18. Show that if $\omega \in \Omega^k(M)$ and $\tau \in \Omega_c^l(M)$, then $\omega \wedge \tau \in \Omega_c^{k+l}(M)$.

Exercise 3.6.19. Show that $\mathrm{H}^0_{\mathrm{dR},c}(\mathbb{R}) = 0$.

Exercise 3.6.20. Show that

$$\mathrm{H}^k_{\mathrm{dR},c}(\text{point}) \cong \begin{cases} \mathbb{R} & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Our next step is to give the Mayer-Vietoris sequence in the setting of compactly supported differential forms. As before, let $M = U_1 \cup U_2$ with U_1 and U_2 open sets. Let $j_k : U_1 \cap U_2 \hookrightarrow U_k$ and $i_k : U_k \hookrightarrow U_1 \cup U_2$ be the inclusion maps as before. We obtain inclusions

$$\begin{array}{ccccc} & & U_1 & & \\ & \overset{j_1}{\nearrow} & & \overset{i_1}{\searrow} & \\ U_1 \cap U_2 & & & & M = U_1 \cup U_2. \\ & \underset{j_2}{\searrow} & & \underset{i_2}{\nearrow} & \\ & & U_2 & & \end{array}$$

Using the convariance of the functor Ω_c^k under inclusion maps we obtain a sequence

$$\begin{array}{ccccc} \Omega_c^k(U_1 \cap U_2) & \xrightarrow{j_c^k} & \Omega_c^k(U_1) \oplus \Omega_c^k(U_2) & \xrightarrow{i_c^k} & \Omega_c^k(M) \\ \omega & \longmapsto & (j_{1,c}^k\omega, -j_{2,c}^k\omega) & & \\ & & (\omega_1, \omega_2) & \longmapsto & i_{2,c}^k\omega_2 + i_{1,c}^k\omega_1. \end{array}$$

Theorem 3.6.21. *The sequence of forms with compact support*

$$0 \longrightarrow \Omega_c^k(U_1 \cap U_2) \xrightarrow{j_c^k} \Omega_c^k(U_1) \oplus \Omega_c^k(U_2) \xrightarrow{i_c^k} \Omega_c^k(M) \longrightarrow 0$$

is exact.

We leave the proof of this theorem as an exercise. The only real difference from the previous proof is that in this case the form $\omega \in \Omega_c^k(M)$ is the image of the form $(\rho_{U_1}\omega, \rho_{U_2}\omega) \in \Omega_c^k(U_1) \oplus \Omega_c^k(U_2)$.

We again obtain a long exact sequence in cohomology.

Theorem 3.6.22. *With $M = U_1 \cup U_2$ as above we have a long exact sequence in cohomology*

$$\cdots \longrightarrow \mathrm{H}^k_{\mathrm{dR},c}(U_1 \cap U_2) \xrightarrow{\mathrm{H}^k_{\mathrm{dR},c}(j_c^k)} \mathrm{H}^k_{\mathrm{dR},c}(U_1) \oplus \mathrm{H}^k_{\mathrm{dR},c}(U_2) \xrightarrow{\mathrm{H}^k_{\mathrm{dR},c}(i_c^k)} \mathrm{H}^k_{\mathrm{dR},c}(M) \xrightarrow{\partial^k} \mathrm{H}^{k+1}_{\mathrm{dR},c}(U_1 \cap U_2) \longrightarrow \cdots.$$

One can explicitly write down the map ∂_c^k here as follows. Let $[\omega] \in \mathrm{H}^k_{\mathrm{dR},c}(U_1 \cup U_2)$ and write $\omega = \omega_1 + \omega_2$ with $\omega_i \in \Omega_c^k(M)$ and $\mathrm{supp}(\omega_i) \subset U_i$. Observe that we have $[d^k\omega] = [d^k\omega_1 + d^k\omega_2]$. However, since ω is a closed form we have that $[d\omega] = 0$. Thus, on $U_1 \cap U_2$ we have $d^k\omega_1$ and $-d^k\omega_2$ are closed form and $[d^k\omega_1]$ and $[-d^k\omega_2]$ agree on $U_1 \cap U_2$. Thus, either of these closed forms represents $\partial_c^k([\omega])$.

One should note that if M is compact, then $\mathrm{H}^k_{\mathrm{dR},c}(M) = \mathrm{H}^k_{\mathrm{dR}}(M)$. In particular, we know the compactly supported cohomology groups of S^1 and T since we calculated $\mathrm{H}^k_{\mathrm{dR}}(S^1)$ and $\mathrm{H}^k_{\mathrm{dR}}(T)$ above. However, if M is not compact the cohomology groups may not be equal. For example, the above exercise gives that $\mathrm{H}^0_{\mathrm{dR},c}(\mathbb{R}) \cong 0$ where $\mathrm{H}^0_{\mathrm{dR}}(\mathbb{R}) \cong \mathbb{R}$.

In order to effectively use Mayer-Vietoris to compute compactly supported cohomology we need an analogue of the Poincare Lemma. We have the following theorem.

Theorem 3.6.23. *(Poincare Lemma for Compact Support) For $n \geq 1$ we have*

$$\mathrm{H}^k_{\mathrm{dR},c}(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & k = n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We begin by showing

$$\mathrm{H}^k_{\mathrm{dR},c}(\mathbb{R}^n \times \mathbb{R}) \cong \mathrm{H}^{k-1}_{\mathrm{dR},c}(\mathbb{R}^n)$$

for all $k \geq 1$. Let $\omega \in \Omega_c^k(\mathbb{R}^n \times \mathbb{R})$. As before, we can write

$$\omega = \sum_I f_I(x,t)dx_I + \sum_J g_J(x,t)dt \wedge dx_J$$

where this time we have that f_I and g_J are compactly supported smooth functions on $\mathbb{R}^n \times \mathbb{R}$. Define

$$\Psi^k : \Omega_c^k(\mathbb{R}^n \times \mathbb{R}) \to \Omega_c^{k-1}(\mathbb{R}^n)$$

by setting

$$\Psi^k(\omega) = \sum_J \left(\int_{\mathbb{R}} g_J(x,t) dt \right) dx_J.$$

Observe that we have

$$d^{k-1}\Psi^k\omega = \sum_{J,j} \int_{\mathbb{R}} \frac{\partial g_J(x,t)}{\partial x_j} dt \wedge dx_j \wedge dx_J.$$

We also have

$$d^k\omega = \sum_I \frac{\partial f_I(x,t)}{\partial t} dt \wedge dx_I - \sum_{J,j} \frac{\partial g_J(x,t)}{\partial x_j} dt \wedge dx_j \wedge dx_J$$

and so

$$\Psi^{k+1}(d^k\omega) = -d^{k-1}\Psi^k\omega.$$

To see this, we have used that f_I has compact support and so

$$\int_{\mathbb{R}} \frac{\partial f_I(x,t)}{\partial t} dt = \lim_{a\to\infty} f(x,a) - \lim_{b\to-\infty} f(x,b) = 0.$$

Thus, we see that Ψ^k is a chain map and so induces a map of cohomology $\mathrm{H}^k_{\mathrm{dR},c}(\Psi^k) : \mathrm{H}^k_{\mathrm{dR},c}(\mathbb{R}^n \times \mathbb{R}) \to \mathrm{H}^{k-1}_{\mathrm{dR},c}(\mathbb{R}^n)$. (Adjust the proof of Lemma 3.2.1 to see this is true.)

Let $\tau = \psi(t)dt$ be a compactly supported 1-form so that $\int_{\mathbb{R}} \psi = 1$. Define $\Phi^k : \Omega^k_c(\mathbb{R}^n) \to \Omega^{k+1}_c(\mathbb{R}^n \times \mathbb{R})$ by sending ω to $\tau \wedge \omega$. Note that the map Φ^k commutes with the exterior derivative so induces a map on cohomology as well. We also have that $\Psi^{k+1} \circ \Phi^k$ is the identity map on $\Omega^k_c(\mathbb{R}^n)$. It is not the case that $\Phi^k \circ \Psi^{k+1}$ is the identity map, but we will show that it is homotopic to the identity so on the level of cohomology is the identity map, which gives that $\mathrm{H}^k_{\mathrm{dR},c}(\mathbb{R}^n \times \mathbb{R}) \cong \mathrm{H}^{k-1}_{\mathrm{dR},c}(\mathbb{R}^n)$.

We now construct a chain homotopy Ξ. In particular, we will show that on $\Omega^k_c(\mathbb{R}^n \times \mathbb{R})$ we have

$$1 - \Phi^k\Psi^{k+1} = \pm(d^k\Xi^{k+1} - \Xi^{k+2}d^{k+1}). \tag{3.5}$$

Set $A(t) = \int_{-\infty}^t \tau$. Define $\Xi^k : \Omega^k_c(\mathbb{R}^n \times \mathbb{R}) \to \Omega^{k-1}_c(\mathbb{R}^n \times \mathbb{R})$ by sending ω to

$$\sum_J \left[\left(\int_{-\infty}^t g_J(x,y)dy \right) dx_J - A(t) \left(\int_{\mathbb{R}} g_J(x,t)dt \right) dx_J \right].$$

Observe that since all of our operators are linear, it is enough to check equation (3.5) on differential forms of the form $f_I(x,t)dx_I$ and $g_J(x,t)dt \wedge dx_J$ that add up to ω. We begin with a form $f_I(x,t)dx_I$. First, we have that $(1 - \Phi^k\Psi^{k+1})(f_I(x,t)dx_I) = f_I(x,t)dx_I$, as can easily be seen from the definitions of Φ^k and Ψ^{k+1}. We now must calculate $(d^k\Xi^{k+1} - \Xi^{k+2}d^{k+1})(f_I(x,t)dx_I)$. We have that $\Xi^{k+1}(f_I(x,t)dx_I) = 0$ by definition, so we have

$$\begin{aligned}(d^k\Xi^{k+1} - \Xi^{k+2}d^{k+1})(f_I(x,t)dx_I) &= -(\Xi^{k+2}d^{k+1})(f_I(x,t)dx_I)\\ &= -\Xi^{k+2}(d^{k+1}(f_I(x,t) \wedge dx_I))\\ &= -\Xi^{k+2}(d^0(f_I(x,t)) \wedge dx_I + f_I(x,t) \wedge d^{k-1}dx_I)\\ &= -\Xi^{k+2}\left(\sum_{i=1}^n \frac{\partial f_I(x,t)}{\partial x_i}dx_i \wedge dx_I + \frac{\partial f_I(x,t)}{\partial t}dt \wedge dx_I + 0\right)\\ &= -\left(\left(\int_\infty^t \frac{\partial f_I(x,y)}{\partial y}\right)dx_I - A(t)\left(\int_{\mathbb{R}} \frac{\partial f_I(x,t)}{\partial t}dt\right)dx_I\right)\\ &= -f_I(x,t)dx_I\end{aligned}$$

where we have used again that f_I is compactly supported. This gives the result for differential forms of the type $f_I(x,t)dx_I$.

Now we consider differential forms of the type $g_J(x,t)dt \wedge dx_J$. In this case we have

$$(1 - \Phi^k\Psi^{k+1})(g_J(x,t)dt \wedge dx_J) = g_J(x,t)dt \wedge dx_J - \tau \wedge \left(\int_{\mathbb{R}} g_J(x,t)dt\right)dx_J.$$

We now compute $(d^k\Xi^{k+1} - \Xi^{k+2}d^{k+1})(g_J(x,t)dt \wedge dx_J)$:

$$\begin{aligned}d^k\Xi^{k+1}(g_J(x,t)dt \wedge dx_J) &= d^k\left(\left(\int_{-\infty}^t g_J(x,y)dy\right) \wedge dx_J - A(t)\left(\int_{\mathbb{R}} g_J(x,t)dt\right) \wedge dx_J\right)\\ &= d^k\left(\int_{-\infty}^t g_J(x,y)dy\right) \wedge dx_J - d^k\left(A(t)\int_{\mathbb{R}} g_J(x,t)dt\right) \wedge dx_J\\ &= \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\int_{-\infty}^t g_J(x,y)dy\right)dx_i \wedge dx_J + \frac{\partial}{\partial t}\left(\int_{-\infty}^t g_J(x,y)dy\right)dt \wedge dx_J\\ &\quad - \tau \wedge \left(\int_{\mathbb{R}} g_J(x,t)dt\right)dx_J - A(t)\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\int_{\mathbb{R}} g_J(x,t)dt\right)dx_i \wedge dx_J\\ &= \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\int_{-\infty}^t g_J(x,y)dy\right)dx_i \wedge dx_J + g_J(x,t)dt \wedge dx_J\\ &\quad - \psi(t)\left(\int_{\mathbb{R}} g_J(x,t)dt\right) \wedge dx_J - A(t)\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\int_{\mathbb{R}} g_J(x,t)dt\right)dx_i \wedge dx_J\end{aligned}$$

$$
\begin{aligned}
-\Xi^{k+2}d^{k+1}(g_J(x,t)dt\wedge dx_J) &= -\Xi^{k+2}\left(d(g_J(x,t))\wedge dt\wedge dx_J\right)\\
&= -\Xi^{k+2}\left(\sum_{i=1}^{n}\frac{\partial g_J(x,t)}{\partial x_i}dx_i\wedge dt\wedge dx_J + \frac{\partial g_J(x,t)}{\partial t}dt\wedge dt\wedge dx_J\right)\\
&= -\Xi^{k+2}\left(\sum_{i=1}^{n}\frac{\partial g_J(x,t)}{\partial x_i}dx_i\wedge dt\wedge dx_J\right)\\
&= \sum_{i=1}^{n}\Xi^{k+2}\left(\frac{\partial g_J(x,t)}{\partial x_i}dt\wedge dx_i\wedge dx_J\right)\\
&= \sum_{i=1}^{n}\left[\left(\int_{-\infty}^{t}\frac{\partial g_J(x,y)}{\partial x_i}dy\right)dx_i\wedge dx_J - A(t)\left(\int_{\mathbb{R}}\frac{\partial g_J(x,t)}{\partial x_i}dt\right)dx_i\wedge dx_J\right]
\end{aligned}
$$

Thus, we see that

$$
\begin{aligned}
(d^k\Xi^{k+1}-\Xi^{k+2}d^{k+1})(g_J(x,t)dt\wedge dx_J) &= g_J(x,t)dt\wedge dx_J - \tau\wedge\left(\int_{\mathbb{R}}g_J(x,t)dt\right)dx_J\\
&= (1-\Phi^k\Psi^{k+1})(g_J(x,t)dt\wedge dx_J)
\end{aligned}
$$

as claimed. Thus, we have the result that

$$
\mathrm{H}^k_{\mathrm{dR},c}(\mathbb{R}^n\times\mathbb{R})\cong\mathrm{H}^{k-1}_{\mathrm{dR},c}(\mathbb{R}^n)
$$

for $k\geq 1$.

We can now finish the result by induction. We wish to calculate $\mathrm{H}^k_{\mathrm{dR},c}(\mathbb{R}^n)$. We begin by showing $\mathrm{H}^1_{\mathrm{dR},c}(\mathbb{R})\cong\mathbb{R}$. Since we already know that $\mathrm{H}^0_{\mathrm{dR},c}(\mathbb{R})=0$, this will give the result for $n=1$. Consider the integration map

$$
\int_{\mathbb{R}}:\Omega^1_c(\mathbb{R})\to\mathbb{R}.
$$

This map is clearly surjective. Let $f\in\Omega^0_c(\mathbb{R})$ be a 0-form and $df=\frac{\partial f}{\partial x}dx$ be the image. Since f has compact support, we can find an interval $[a,b]$ so that $\mathrm{supp}(f)\subsetneq[a,b]$. Thus, we have

$$
\begin{aligned}
\int_{\mathbb{R}}\frac{\partial f}{\partial x}dx &= \int_a^b\frac{\partial f}{\partial x}dx\\
&= f(b)-f(a)\\
&= 0-0\\
&= 0.
\end{aligned}
$$

This shows that $\int_{\mathbb{R}}$ vanishes on the exact 1-forms. Suppose now that $g(x)dx$ lies in the kernel of $\int_{\mathbb{R}}$. Then we have that

$$
f(x)=\int_{-\infty}^{x}g(t)dt
$$

has compact support and $df = g(x)dx$. Thus, we have that the kernel of $\int_{\mathbb{R}}$ is exactly the exact 1-forms and so we have

$$\mathrm{H}^1_{\mathrm{dR},c}(\mathbb{R}) \cong \mathbb{R}.$$

This combined with the isomorphism $\mathrm{H}^k_{\mathrm{dR},c}(\mathbb{R}^n \times \mathbb{R}) \cong \mathrm{H}^k_{\mathrm{dR},c}(\mathbb{R}^n)$ gives that $\mathrm{H}^n_{\mathrm{dR},c}(\mathbb{R}^n) \cong \mathbb{R}$ for all $n \geq 0$. Similarly, using the fact that $\mathrm{H}^0_{\mathrm{dR},c}(\mathbb{R}^k) = 0$ for all $k \geq 1$ combined with the isomorphism and induction gives the rest of the result. □

In the next section we will define integration on manifolds. It is via integration that many of the results calculating the compactly supported de Rham cohomology of manifolds are proven.

3.7 Integration on Manifolds

In this section we generalize integration from $\mathbb{R}^n$ to a certain type of manifold. Before we work on manifolds, we recall some notions for integration in $\mathbb{R}^n$ and open subsets of $\mathbb{R}^n$.

Let $x_1, \ldots, x_n$ be the standard coordinates on $\mathbb{R}^n$. The Riemann integral of a function f on $\mathbb{R}^n$ is defined by

$$\int_{\mathbb{R}^n} f dx_1 \cdots dx_n = \sum_{\Delta x_i \to 0} f \Delta x_1 \cdots \Delta x_n.$$

One learns in analysis class that the Riemann integral can be generalized to Lebesgue integration which allows a larger class of sets and functions to be included in the definition. We now frame integration on $\mathbb{R}^n$ in terms of differential forms.

Let $\omega \in \Omega^n_c(\mathbb{R}^n)$. We have seen that we can uniquely write

$$\omega(x) = f(x) dx_1 \wedge \cdots \wedge dx_n$$

for f a compactly supported smooth function on $\mathbb{R}^n$ with values in $\mathbb{R}$. Note that strictly speaking we have not shown this for compactly supported differential forms, but all of the arguments given clearly work in this case as well. We define

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f dx_1 \wedge \cdots \wedge dx_n = \int_{\mathbb{R}^n} f d\mu_n$$

where $d\mu_n$ denotes Lebesgue measure on $\mathbb{R}^n$.

Let $U \subset \mathbb{R}^n$ be an open set and let $\omega \in \Omega^n_c(U)$. We can again write

$$\omega(x) = f(x) dx_1 \wedge \cdots \wedge dx_n$$

for $f \in \Omega^0_c(U)$. We can smoothly extend f, and so ω, to $\mathbb{R}^n$ by setting $f(x) = 0$ for $x \in \mathbb{R}^n - \mathrm{supp}_U(f)$. Thus, we can define

$$\int_U \omega = \int_{\mathbb{R}^n} \omega$$

where in the second integral it is understood that ω is extended by 0 off of U.

Lemma 3.7.1. *Let U and V be open sets in $\mathbb{R}^n$ with $\phi : U \to V$ a diffeomorphism. Assume that $\det(D_x\phi)$ has constant sign $\delta = \pm 1$ for all $x \in U$. (Note that $\det(D_x\phi)$ is simply the Jacobi determinant.) Then for $\omega \in \Omega_c^n(V)$ we have*

$$\int_U \phi^*\omega = \delta \int_V \omega.$$

Proof. Let $\omega \in \Omega_c^n(V)$ and write $\omega = f dx_1 \wedge \cdots \wedge dx_n = f \wedge dx_1 \wedge \cdots \wedge dx_n$ with $f \in \Omega_c^0(V)$. We have

$$\begin{aligned}
\phi^*(\omega)(x) &= \phi^*(f \wedge dx_1 \wedge \cdots \wedge dx_n)(x) \\
&= \phi^*(f)(x) \wedge \phi^*(dx_1) \wedge \cdots \wedge \phi^*(dx_n) \\
&= f(\phi(x)) \wedge d\phi^*(x_1) \wedge \cdots \wedge d\phi^*(x_n) \\
&= f(\phi(x)) \det(D_x\phi) dx_1 \wedge \cdots \wedge dx_n
\end{aligned}$$

where we have used Exercise 3.3.44. Now observe that $\delta = \frac{\det(D_x\phi)}{|\det(D_x\phi)|}$ and so we have

$$\begin{aligned}
\int_U \phi^*(\omega) &= \int_U \delta f(\phi(x)) |\det(D_x\phi)| dx_1 \wedge \cdots \wedge dx_n \\
&= \delta \int_U f(\phi(x)) |\det(D_x\phi)| d\mu_n \\
&= \delta \int_V f(x) d\mu_n \\
&= \delta \int_V \omega.
\end{aligned}$$

□

Definition 3.7.2. Let $\phi : U \to V$ be a diffeomorphism between open sets in $\mathbb{R}^n$. If $\det(D_x\phi) > 0$ for all $x \in U$ we say that ϕ is *orientation-preserving.*

Note that integrals on open subsets of $\mathbb{R}^n$ are not invariant under diffeomorphisms, but are invariant under orientation-preserving diffeomorphisms.

Definition 3.7.3. Let M be a m-manifold with atlas $\{(U_i, \varphi_i)\}_{i \in I}$. We say that $\{(U_i, \varphi_i)\}$ is *oriented* if the transition functions $\varphi_j \circ \varphi_i^{-1}$ are all orientation preserving. We say M is *orientable* if there is an oriented atlas on M.

Proposition 3.7.4. *A manifold M of dimension m is orientable if and only if there exists $\omega \in \Omega^m(M)$ with $\omega(x) \neq 0$ for all $x \in M$.*

Proof. As in the proof of Lemma 3.7.1, we use Exercise 3.3.44 to note that $\phi : \mathbb{R}^m \to \mathbb{R}^m$ is orientation preserving if and only if $\phi^*(dx_1 \wedge \cdots \wedge dx_m)$ is a positive multiple of $dx_1 \wedge \cdots \wedge dx_m$ at every point.

Suppose that M has an oriented atlas $\{(U_i, \varphi_i)\}_{i \in I}$. Then we know that

$$(\varphi_j \circ \varphi_i^{-1})^*(dx_1 \wedge \cdots \wedge dx_n) = f_{i,j} dx_1 \wedge \cdots \wedge dx_n$$

for some positive function $f_{i,j}$. In particular, we can write

$$\varphi_j^*(dx_1 \wedge \cdots \wedge dx_m) = \varphi_i^*(f_{i,j}) \wedge \varphi_i^*(dx_1 \wedge \cdots \wedge dx_m).$$

Thus, if we set $\omega_j = \varphi_j^*(dx_1 \wedge \cdots \wedge dx_m)$ and $\omega_i = \varphi_i^*(dx_1 \wedge \cdots \wedge dx_m)$, then we have

$$\omega_j = (f_{i,j} \circ \varphi_i)\omega_i$$

where $f_{i,j} \circ \varphi_i$ is everywhere positive.

Let $\{\rho_i\}$ be a partition of unity with respect to the cover $\{U_i\}$ of M. Set $\omega = \sum_i \rho_i \omega_i$. Let $x \in M$. Then we have that for all i where ω_i is defined at x, then ω_i are all positive multiples of each other. Since $\rho_i \geq 0$ and not all ρ_i can vanish at any point, we must have $\omega(x) \neq 0$ for all $x \in M$.

Now suppose that there is a $\omega \in \Omega^m(M)$ so that $\omega(x) \neq 0$ for all $x \in M$. Let $\varphi_i : U_i \to \mathbb{R}^m$ be a chart. Then there exists a nowhere vanishing real-valued smooth function f_i on U_i so that $\varphi_i^*(dx_1 \wedge \cdots \wedge dx_m) = f_i\omega$. Thus, we must have that f_i is either positive everywhere or negative everywhere. If f_i happens to be negative everywhere, we can replace φ_i by $\psi_i = \phi \circ \varphi_i$ where $\phi : \mathbb{R}^m \to \mathbb{R}^m$ is given by $\phi(x_1, \ldots, x_m) = (-x_1, x_2, \ldots, x_m)$. Observe that $\psi_i^*(dx_1 \wedge \cdots \wedge dx_m) = \varphi_i^* \circ \phi^*(dx_1 \wedge \cdots \wedge dx_m) = -\varphi_i^*(dx_1 \wedge \cdots \wedge dx_m) = -f_i\omega$, we see that it is fine to assume that f_i is everywhere positive for all i. Thus, we have that any transition function

$$\varphi_{ji} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

will pull $dx_1 \wedge \cdots \wedge dx_m$ back to a positive multiple of itself. Thus, $\{(U_i, \varphi_i)\}$ is an oriented atlas. □

Definition 3.7.5. Let M be a m-manifold and $\omega \in \Omega^m(M)$ a non-vanishing differential form. We call such a form an *orientation form* on M.

Definition 3.7.6. Let ω_1 and ω_2 be orientation forms on M. We say ω_1 is equivalent to ω_2 if there exists $f \in \Omega^0(M) = C^\infty(M, \mathbb{R})$ so that $\omega_1 = f\omega_2$ and $f(x) > 0$ for all $x \in M$. An equivalence class of orientation forms on M is called an *orientation of* M. We denote it by $[M]$.

One should note that if M if connected, since f must have a constant sign on M we must have that there are only two possible orientations for M. Let ω be an orientation form on M. Let $v_1, \ldots, v_m$ be a basis of T_xM. We say the basis is *positively oriented* if $\omega(x)(v_1, \ldots, v_m) > 0$ and *negatively oriented* if $\omega(x)(v_1, \ldots, v_m) < 0$.

Example 3.7.7. Let $M = \mathbb{R}^m$. Recall that the differential form $dx_1 \wedge \cdots \wedge dx_m$ is constant and non-zero. Thus, this gives an orientation form on $\mathbb{R}^m$. We call this the *standard orientation* of $\mathbb{R}^m$. Under this orientation form the basis $e_1 = (1, 0, \ldots, 0), \ldots, e_m = (0, \ldots, 0, 1)$ is positively oriented. To see this, recall that

$$\begin{aligned} dx_1 \wedge \cdots \wedge dx_m(e_1, \ldots, e_m) &= \det \begin{pmatrix} dx_1(e_1) & \cdots & dx_1(e_m) \\ \vdots & \ddots & \vdots \\ dx_m(e_1) & \cdots & dx_m(e_m) \end{pmatrix} \\ &= \det 1_m \\ &= 1 \end{aligned}$$

where we write 1_m for the m by m identity matrix and we have used that $dx_i(a_1, \ldots, a_m) = a_j$.

Let $\omega \in \Omega_c^m(M)$, $\{(U_i, \varphi_i)\}$ an oriented atlas giving the orientation $[M]$ on M. Suppose that there exists $i \in I$ so that the support of ω lies in U_i. Then it is natural to define the integral of ω over M as

$$\int_{[M]} \omega = \int_{\mathbb{R}^n} (\varphi_i^{-1})^*(\omega).$$

If the support of ω does not happen to lie in a single chart, we must define the integral in terms of a partition of unity. Let $\{\rho_i\}$ be a partition of unity with respect to the cover $\{U_i\}$. Define

$$\int_{[M]} \omega = \sum_i \int_{\mathbb{R}^n} (\varphi_i^{-1})^*(\rho_i \omega).$$

One should note that this is well-defined by the orientability assumption. We will write $\int_M \omega$ for $\int_{[M]} \omega$ when the orientation of M is fixed. Note we also write $\int_{U_i} \rho_i \omega$ for $\int_{\mathbb{R}^n} (\varphi_i^{-1})^*(\rho_i \omega)$ to ease notation.

Proposition 3.7.8. *The definition of $\int_M \omega$ is independent of the choice of atlas in the orientation as well as the choice of partition of unity.*

Proof. Let $\{(V_j, \psi_j)\}$ be another atlas in the orientation and $\{\rho'\}$ a partition of unity with respect to the cover $\{V_j\}$. Note that since $\sum_j \rho' = 1$ we have

$$\sum_i \int_{U_i} \rho_i \omega = \sum_{i,j} \int_{U_i} \rho_i \rho'_j \omega.$$

Now $\rho_i \rho'_j \omega$ has support on $U_i \cap V_j$ and so we have

$$\int_{U_i} \rho_i \rho'_j \omega = \int_{V_j} \rho_i \rho'_j \omega.$$

Thus,

$$\begin{aligned} \sum_i \int_{U_i} \rho_i \omega &= \sum_{i,j} \int_{V_j} \rho_i \rho'_j \omega \\ &= \sum_j \int_{V_j} \rho'_j \omega \end{aligned}$$

where we have used that $\sum \rho_i = 1$. This gives the result. □

Definition 3.7.9. Let $f : M \to N$ be a diffeomorphism. Let M be oriented by ω_M and N oriented by ω_N. We say that f is *orientation preserving* if the orientation form $f^*(\omega_N)$ is equivalent to ω_M. If the orientation form $f^*(\omega_N)$ is equivalent to $-\omega_M$ we say f is *orientation reversing.*

Lemma 3.7.10. *Let M and N be orientable m-manifolds.*

1. *The integral $\int_{[M]} \omega$ changes sign when the orientation of M is reversed.*
2. *If $\omega \in \Omega_c^m(M)$ has support contained in an open set $U \subset M$, then*
$$\int_M \omega = \int_U \omega$$
where U has the orientation induced from M.
3. *If $f : M \to N$ is an orientation preserving diffeomorphism , then we have*
$$\int_N \omega = \int_M f^*(\omega)$$
for $\omega \in \Omega_c^m(N)$.

Proof. This follows immediately from the fact that the results are true for open subsets of Euclidean space along with the fact that we can use a partition of unity to restrict to the case that the support of ω is contained in a coordinate patch. One should fill in the details as an exercise. □

One might note at this point that we are only able to integrate a m-form on a m-manifold. However, we can integrate other forms over submanifolds. Let $N \subset M$ be an oriented n-submanifold. Let $i : N \hookrightarrow M$ be the natural inclusion map. Let $\omega \in \Omega_c^l(M)$. We have a natural "restriction" of ω to N defined by $i^*(\omega)$. For instance, if ω is a 0-form, i.e., a smooth function real-valued function on M, then we have $i^*(\omega)$ is exactly the restriction of the function ω to N. If ω happens to be a n-form that has support that intersects N in a compact set, then we can define the integral of ω over N by setting
$$\int_N \omega = \int_N i^*(\omega).$$
It is understood that if we are integrating a form over a submanifold that we mean the pullback of the form under the inclusion map so we generally drop the i^* from the notation.

Example 3.7.11. Let $\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$ be a smooth 1-form on $\mathbb{R}^3$. We wish to integrate this one form over a simple smooth curve C given by $\gamma : I \to \mathbb{R}^3$ where $I = (a, b) \subset \mathbb{R}$. Assume that ω has compact support when restricted to C, in particular, assume that ω is supported on $\gamma([c,d])$ for some $[c,d] \subset (a,b)$. Thus, we have that γ serves as local coordinates and so we have

$$\begin{aligned}\int_C \omega &= \int_{\mathbb{R}^n} \gamma^*(\omega) \\ &= \int_c^d \gamma^*(\omega).\end{aligned}$$

Write $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$. Recall from Example 3.3.43 that we have

$$\gamma^*(\omega) = \sum_{j=1}^{3} f_j(\gamma(t)) \frac{d\gamma_j}{dt} dt.$$

In particular, we see that in this case we have that $\int_C \omega$ is exactly the line integral of $\mathbf{F} = (f_1, f_2, f_3)$ over the curve C as defined in calculus, i.e.,

$$\int_C \omega = \int_C \mathbf{F} \cdot d\gamma.$$

Example 3.7.12. Let $\omega \in \Omega_c^2(\mathbb{R}^3)$ be given by

$$\omega = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2.$$

Let S be a surface in $\mathbb{R}^3$ given by the graph of a function $G : \mathbb{R}^2 \to \mathbb{R}$ with $x_3 = G(x_1, x_2)$. We now put the integral $\int_S \omega$ into a familiar form from calculus class. The map $h : \mathbb{R}^2 \to S$ given by

$$h(x_1, x_2) = (x_1, x_2, G(x_1, x_2))$$

gives a parameterization of S. We have

$$\begin{aligned}h^*(dx_1 \wedge dx_2) &= h^*(dx_1) \wedge h^*(dx_2) = dx_1 \wedge dx_2, \\ h^*(dx_2 \wedge dx_3) &= h^*(dx_2) \wedge h^*(dx_3) = dx_2 \wedge dG \\ &= dx_2 \wedge \left(\frac{\partial G}{\partial x_1} dx_1 + \frac{\partial G}{\partial dx_2} dx_2 \right) \\ &= -\frac{\partial G}{\partial x_1}(dx_1 \wedge dx_2) \\ h^*(dx_3 \wedge dx_1) &= -\frac{\partial G}{\partial x_2}(dx_1 \wedge dx_2).\end{aligned}$$

Thus, we have

$$\begin{aligned}\int_S \omega &= \int_{\mathbb{R}^2} h^* \omega \\ &= \int_{\mathbb{R}^2} (f_1, f_2, f_3) \cdot \left(-\frac{\partial G}{\partial x_1}, -\frac{\partial G}{\partial x_2}, 1 \right) dx_1 \wedge dx_2.\end{aligned}$$

To rectify this with calculus, observe that if we set $\mathbf{n}(x_1, x_2, x_3) = \left(-\frac{\partial G}{\partial x_1}, -\frac{\partial G}{\partial x_2}, 1\right)$ then we have that $\mathbf{n}(x_1, x_2, x_3)$ is perpendicular to S at each point $(x_1, x_2, x_3) \in S$. Let $\mathbf{u} = \frac{\mathbf{n}}{|\mathbf{n}|}$ be the unit normal vector. Set $\mathbf{F} = (f_1, f_2, f_3)$. Let $dA = |\mathbf{n}| dx_1 \wedge dx_2$ be a 2-form. This form is normally referred to as the area form of the surface S. Our equation then reads

$$\int_S \omega = \int_{\mathbb{R}^2} (\mathbf{F} \cdot \mathbf{u}) dA,$$

which is the familiar integral of a function on a surface from calculus.

We would also like to give a generalization of Stokes' theorem to the setting of oriented manifolds. Of course, in order to make sense of Stokes' theorem we need to introduce manifolds with boundary. Let $\mathbb{H}^m$ be the subset of $\mathbb{R}^m$ given by

$$\mathbb{H}^m = \{(x_1, \ldots, x_m) : x_m \geq 0\}.$$

It is clear that the boundary of $\mathbb{H}^m$ is given by

$$\partial \mathbb{H}^m = \{(x_1, \ldots, x_{m-1}, 0)\}.$$

This is clearly diffeomorphic to $\mathbb{R}^{m-1}$ under the map sending $(x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m-1}$ to $(x_1, \ldots, x_{m-1}, 0) \in \partial \mathbb{H}^m$.

Definition 3.7.13. A *manifold of dimension m with boundary* is given by an atlas $\{(U_i, \varphi_i)\}_{i \in I}$ where U_i is homeomorphic to either $\mathbb{R}^m$ or $\mathbb{H}^m$.

For each point $x \in \partial M$, the tangent space $T_x(\partial M)$ has codimension 1 in $T_x M$. Thus, there are precisely two unit vectors in $T_x M$ that are perpendicular to $T_x(\partial M)$. Let $\psi : W \to M$ be a local parameterization with $\psi(0) = x$ and W open in $\mathbb{H}^m$. We have that the map $(D_0 \psi)^{-1} : T_x X \to \mathbb{R}^m$ carries one of the unit vectors to the unit vector at 0 in $\mathbb{R}^m$ pointing into $\mathbb{H}^m$ and the other to the unit vector at 0 in $\mathbb{R}^m$ pointing outward from $\mathbb{H}^m$. Lemma 3.7.14 shows that this does not depend on the choice of local parameterization. We denote the unit vector in $T_x(X)$ that maps to the outward pointing unit normal vector by n_x and refer to it as the outward normal vector. The orientation on ∂M is given by declaring the sign of a basis $v_1, \ldots, v_{m-1}$ of $T_x(\partial M)$ to be the sign of $\omega(x)(n_x, v_1, \ldots, v_{k-1})$. For $n = 1$ we declare the orientation of the 0-dimensional manifold to be the sign of n_x. Note what is happening here is that the orientation on M induces two possible orientations on ∂M and we are fixing which one we will work with. Recall from calculus class that one always said a curve enclosing an area in $\mathbb{R}^2$ was a positively oriented curved if the area was to the left as one traversed the curve. That is essentially what is happening here, just in more generality. This orientation is the correct one so that Stokes' theorem has the familiar statement from calculus class.

Lemma 3.7.14. *Let $f : \mathbb{H}^n \to \mathbb{H}^n$ be a diffeomorphism with everywhere positive Jacobian determinant. The map f induces a map $\widetilde{f}$ of the boundary of $\mathbb{H}^n$ with itself. The induced map, as a diffeomorphism of $\mathbb{R}^{n-1}$ also has positive Jacobian determinant everywhere.*

Proof. Let x be an interior point of $\mathbb{H}^n$. The Inverse Function Theorem shows that the preimage of x must itself lie on the interior of $\mathbb{H}^n$ as well. Thus, we see that f must map the boundary to the boundary. It remains to check that $\widetilde{f}$ has positive Jacobian determinant.

Consider the case $n = 2$. Write

$$\begin{aligned} x_1 &= f_1(y_1, y_2) \\ x_2 &= f_2(y_1, y_2). \end{aligned}$$

We have that $\widetilde{f}$ is given by

$$x_1 = f_1(y_1, 0).$$

As the Jacobian determinant of f is assumed to be positive, we have

$$\begin{vmatrix} \frac{\partial f_1}{\partial y_1}(y_1, 0) & \frac{\partial f_1}{\partial y_2}(y_1, 0) \\ \frac{\partial f_2}{\partial y_1}(y_1, 0) & \frac{\partial f_2}{\partial y_2}(y_1, 0) \end{vmatrix} > 0.$$

For a point on the boundary we have $f_2(y_1, 0) = 0$ for all y_1. In particular, this gives that $\frac{\partial f_2}{\partial y_1}(y_1, 0) = 0$ for all y_1. The fact that f maps $\mathbb{H}^n$ to itself we must have $\frac{\partial f_2}{\partial y_2}(y_1, 0) > 0$. Therefore, we have

$$\frac{\partial f_1}{\partial y_1}(y_1, 0) > 0,$$

which is what we wanted to show. □

Recall that $\partial\mathbb{H}^m$ is diffeomorphic to $\mathbb{R}^{m-1}$. We have an orientation on $\partial\mathbb{H}^m$ via the induced orientation from $\mathbb{H}^m$ as described above, but we also have an orientation arising from the diffeomorphism with $\mathbb{R}^{k-1}$. However, it is not always the case that these two orientations agree. Let $e_1, \ldots, e_m$ be the standard basis for $\mathbb{R}^m$. We saw above that this is a positively oriented basis with respect to the standard orientation. Furthermore, $e_1, \ldots, e_{m-1}$ is a positively oriented basis for $\mathbb{R}^{m-1}$ with respect to the standard orientation there. The outward normal vector to $\partial\mathbb{H}^m$ is given by $-e_m = (0, \ldots, 0, -1)$. Thus, in the boundary orientation of $\partial\mathbb{H}^m$ induced from the orientation on $\mathbb{H}^m$ the sign of the basis $e_1, \ldots, e_{m-1}$ is the sign of the ordered basis $-e_m, e_1, \ldots, e_{m-1}$ in the standard orientation of $\mathbb{H}^m$. One can easily calculate that the sign is given by $(-1)^m$. Thus, we see that the induced orientation on $\partial\mathbb{H}^m$ differs from the standard orientation on $\mathbb{R}^{m-1}$ by the factor $(-1)^m$.

Theorem 3.7.15. *(Stokes' Theorem) Let M be a compact oriented m-manifold with boundary. If $\omega \in \Omega_c^{m-1}(M)$ and ∂M is given the induced orientation, then*

$$\int_M d^{m-1}\omega = \int_{\partial M} \omega.$$

Proof. First, observe that since each integral is linear, we may assume that the support of ω lies in the image of a single local parameterization (W, ψ) with W open in $\mathbb{R}^m$ or $\mathbb{H}^m$. We treat each case separately.

First, suppose that W is open in $\mathbb{R}^m$. In this case we see the support of ω does not intersect ∂M and so $\int_{\partial M} \omega = 0$. Furthermore, we have

$$\begin{aligned}\int_M d^{m-1}\omega &= \int_W \psi^*(d^{m-1}\omega) \\ &= \int_W d^{m-1}(\psi^*(\omega)).\end{aligned}$$

Write $\tau = \psi^*(\omega)$. As $\tau \in \Omega_c^{m-1}(W)$, we can write

$$\tau = \sum_{i=1}^{m} f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m$$

for some $f_i \in \Omega_c^0(W)$. Thus, we have

$$\begin{aligned}d^{m-1}\tau &= \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_i} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m \\ &= \sum_{i=1}^{m} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_m.\end{aligned}$$

This gives

$$\begin{aligned}\int_{\mathbb{R}^m} d^{m-1}\tau &= \sum_{i=1}^{m} (-1)^{i-1} \int_{\mathbb{R}^m} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_m \\ &= \sum_{i=1}^{m} (-1)^{i-1} \int_{\mathbb{R}^m} \frac{\partial f_i}{\partial x_i} d\mu_m.\end{aligned}$$

We know from analysis that the Lebesgue integral can be computed by a series of integrals over $\mathbb{R}$ in any order. We integrate the i^{th} term first with respect to dx_i. Up to multiplication by $(-1)^{i-1}$ the i^{th} term is given by

$$\begin{aligned}\int_{\mathbb{R}^m} \frac{\partial f_i}{\partial x_i} d\mu_m &= \int_{\mathbb{R}^{m-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i} dx_i \right) dx_1 \cdots \widehat{dx_i} \cdots dx_m \\ &= \int_{\mathbb{R}^{m-1}} \left(\lim_{t\to\infty} f_i(x_1, \ldots, t, \ldots, x_m) - \lim_{s\to-\infty} f_i(x_1, \ldots, s, \ldots, x_m) \right) dx_1 \cdots \widehat{dx_i} \cdots dx_m \\ &= 0\end{aligned}$$

where we have used that f_i has compact support. Since this holds for $i = 1, \ldots, m$, we have $\int_M d^{m-1}\omega = 0$ and so the result holds for U open in $\mathbb{R}^m$.

Now suppose that $W \subset \mathbb{H}^k$ is open. The argument given above works up until the very last step, and even here the only issue is with the dx_m term. In particular, we have

$$\int_M d^{m-1}\omega = (-1)^{m-1} \int_{\mathbb{R}^m} \frac{\partial f_m}{\partial x_m} d\mu_m.$$

We can rewrite this integral as

$$(-1)^{m-1} \int_{\mathbb{R}^m} \frac{\partial f_m}{\partial x_m} d\mu_m = (-1)^{m-1} \int_{\mathbb{R}^{m-1}} \left(\int_0^\infty \frac{\partial f_m}{\partial x_m} dx_m \right) dx_1 \cdots dx_{m-1}.$$

We now use the fact that f is compactly supported to conclude that

$$\begin{aligned} \int_M d^{m-1}\omega &= (-1)^{m-1} \int_{\mathbb{R}^{m-1}} -f_m(x_1, \ldots, x_{m-1}, 0) dx_1 \cdots dx_{m-1} \\ &= (-1)^m \int_{\mathbb{R}^{m-1}} f_m(x_1, \ldots, x_{m-1}, 0) d\mu_{m-1}. \end{aligned}$$

Now consider the integral

$$\int_{\partial M} \omega = \int_{\partial \mathbb{H}^m} \tau.$$

Observe that since we have $x_m = 0$ on $\partial\mathbb{H}^m$ we have $dx_m = 0$ on $\partial\mathbb{H}^m$ as well and so the form

$$f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m = 0$$

except in the case that $i = m$. Thus, the restriction of τ to $\partial\mathbb{H}^m$ is given by

$$\tau|_{\partial\mathbb{H}^m} = f_m(x_1, \ldots, x_{m-1}, 0) dx_1 \wedge \cdots \wedge dx_{m-1}.$$

Recalling that the diffeomorphism between $\partial\mathbb{H}^m$ and $\mathbb{R}^{m-1}$ changes the orientation by $(-1)^m$, we have

$$\begin{aligned} \int_{\partial M} \omega &= \int_{\partial\mathbb{H}^m} f_m(x_1, \ldots, x_{m-1}, 0) dx_1 \wedge \cdots \wedge dx_{m-1} \\ &= (-1)^m \int_{\mathbb{R}^{m-1}} f_m(x_1, \ldots, x_{m-1}, 0) d\mu_{m-1}. \end{aligned}$$

Thus, we have the result in this case as well. □

Exercise 3.7.16. Show that the map

$$\int_M : \Omega_c^m(M) \to \mathbb{R}$$

descends to a map on cohomology.

We will finish this chapter on differential topology by proving a version of Poincare duality and observing a couple of corollaries of it. Before we can give Poincare duality, we need the notion of a good cover of a manifold.

Definition 3.7.17. Let M be a m-manifold. Let $\mathcal{U} = \{U_i\}$ be an open cover of M. We say that $\mathcal{U}$ is a *good cover* if all nonempty finite intersections of elements in $\mathcal{U}$ are diffeomorphic to $\mathbb{R}^m$, i.e., for any $U_{i_1}, \ldots, U_{i_n}$ with $U_{i_1} \cap \cdots \cap U_{i_n} \neq \emptyset$ we have that $U_{i_1} \cap \cdots \cap U_{i_n}$ is diffeomorphic to $\mathbb{R}^m$. If there exists a good cover $\mathcal{U}$ of M so that $\mathcal{U}$ consists of finitely many open sets we saw the cover is of *finite type*.

Theorem 3.7.18. *Every manifold has a good cover. If the manifold happens to be compact then the cover may be chosen to be finite.*

We omit a proof of this theorem. It is not a particularly difficult result, but it uses the result that one can put a Riemannian metric on a manifold and that every point in a Riemannian manifold has a geodesically convex neighborhood. Neither are particularly difficult results, but other than for this result we will not encounter them so for brevity we omit them. To see a proof of this theorem one can consult Theorem 5.1 of [2].

The Mayer-Vietoris sequence allows us to prove the following result.

Proposition 3.7.19. *Let M be a m-manifold of finite type. The de Rham cohomology groups of M are all finite dimensional.*

Proof. First, observe that the Mayer-Vietoris sequence gives the exact sequence

$$\cdots \longrightarrow \mathrm{H}^{k-1}_{\mathrm{dR}}(U_1 \cap U_2) \xrightarrow{\partial^{k-1}} \mathrm{H}^k_{\mathrm{dR}}(U_1 \cup U_2) \xrightarrow{\mathrm{H}^k_{\mathrm{dR}}(i^k)} \mathrm{H}^k_{\mathrm{dR}}(U_1) \oplus \mathrm{H}^k_{\mathrm{dR}}(U_2) \longrightarrow \cdots .$$

From this we have

$$\mathrm{H}^k_{\mathrm{dR}}(U_1 \cup U_2) \cong \ker(\mathrm{H}^k_{\mathrm{dR}}(i^k)) \oplus \mathrm{im}(\mathrm{H}^k_{\mathrm{dR}}(i^k)) \cong \mathrm{im}(\partial^{k-1}) \oplus \mathrm{im}(\mathrm{H}^k_{\mathrm{dR}}(i^k)).$$

Thus, we have that if $\mathrm{H}^k_{\mathrm{dR}}(U_1)$, $\mathrm{H}^k_{\mathrm{dR}}(U_2)$, and $\mathrm{H}^{k-1}_{\mathrm{dR}}(U_1 \cap U_2)$ are all finite dimensional, then $\mathrm{H}^k_{\mathrm{dR}}(U_1 \cup U_2)$ must be as well.

First suppose that M is diffeomorphic to $\mathbb{R}^m$. Poincare's lemma then gives the result. We now proceed by induction on the number of sets in the cover of M. Suppose the cohomology of any manifold covered by at most $n-1$ open sets is finite dimensional. Let M be a manifold covered by $\{U_1, \ldots, U_n\}$. We have that $(U_1 \cup \cdots \cup U_{n-1}) \cap U_n$ has a good cover by the $n-1$ open sets $U_1 \cap U_n, \ldots, U_{n-1} \cap U_n$. Our induction hypothesis gives that $U_1 \cup \cdots \cup U_{n-1}$, U_n, and $(U_1 \cup \cdots \cup U_{n-1}) \cap U_n$ all have finite dimensional cohomology groups. Now the remark above using the Mayer-Vietoris sequence gives the result. ☐

Exercise 3.7.20. Show that if M is of finite type then the groups $\mathrm{H}^k_{\mathrm{dR},c}(M)$ are all finite dimensional.

Let M be of finite type. The *Betti numbers of M* are defined by

$$b_i(M) = \dim_{\mathbb{R}} \mathrm{H}^i_{\mathrm{dR}}(M).$$

The Euler characteristic of M is defined to be

$$\chi(M) = \sum_{i=0}^{m} (-1)^i b_i(M).$$

It turns out that $\chi(M)$ can be computed by studying vector fields. This result is the Poincare-Hopf theorem. This would take us too far afield to prove or even state precisely, but is worth mentioning.

We briefly recap a few linear algebra facts now. Let V be a $\mathbb{R}$-vector space. Recall the dual space to V is given by $V^\vee = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. Let $\langle\, , \rangle : V \otimes W \to \mathbb{R}$ be a pairing of vector spaces. We say the pairing is *nondegenerate* if $\langle v, w\rangle = 0$ for all $v \in V$ implies $w = 0$ and $\langle v, w\rangle = 0$ for all $w \in W$ implies that $v = 0$. Another way to view this is that the pairing is nondegenerate if the map $v \mapsto \langle v, *\rangle$ defines an injection of V into $W^\vee$ and the map $w \mapsto \langle *, w\rangle$ defines an injection of W into $V^\vee$.

Lemma 3.7.21. *Let V and W be finite dimensional $\mathbb{R}$ vector spaces. The pairing*

$$\langle\, , \rangle : V \otimes W \to \mathbb{R}$$

is nondegenerate if and only if the map $v \mapsto \langle v, *\rangle$ defines an isomorphism $V \xrightarrow{\simeq} W^\vee$.

We leave the proof of this lemma as an exercise. It is either familiar from linear algebra or good linear algebra practice.

Exercise 3.7.16 shows that integration descends to a map on cohomology. Let $[\omega] \in \mathrm{H}^k_{\mathrm{dR}}(M)$ and $[\tau] \in \mathrm{H}^{m-k}_{\mathrm{dR},c}(M)$ for M a m-manifold. Then we have $[\omega] \wedge [\tau] = [\omega \wedge \tau] \in \mathrm{H}^m_{\mathrm{dR},c}(M)$. Combining this with Exercise 3.6.18 we see that we have a map

$$\int_M : \mathrm{H}^k_{\mathrm{dR}}(M) \otimes \mathrm{H}^{m-k}_{\mathrm{dR},c}(M) \to \mathbb{R}.$$

Theorem 3.7.22. *(Poincare Duality) The map $\int_M : \mathrm{H}^k_{\mathrm{dR}} \otimes \mathrm{H}^{m-k}_{\mathrm{dR},c}(M) \to \mathbb{R}$ given by integrating the wedge product is a nondegenerate pairing if M is an orientable manifold of finite type. Equivalently, we have that if M is an orientable manifold of finite type then*

$$\mathrm{H}^k_{\mathrm{dR}}(M) \cong (\mathrm{H}^{m-k}_{\mathrm{dR},c}(M))^\vee.$$

Before we can prove this theorem we need two lemmas. The first is the Five Lemma, a proof of which can be found in any book on homological algebra.

Lemma 3.7.23. *Given a commutative diagram of abelian groups and group homomorphisms*

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} & & \\
\cdots & \longrightarrow & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 & \longrightarrow & \cdots
\end{array}$$

in which the rows are exact, if the maps α, β, δ, and ϵ are isomorphisms, then so is γ.

Lemma 3.7.24. *The map*

$$\int_M : \mathrm{H}^k_{\mathrm{dR}}(M) \otimes \mathrm{H}^{m-k}_{\mathrm{dR},c}(M) \to \mathbb{R}.$$

induces a sign-commutative diagram

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\partial^{k-1}} & \mathrm{H}^k_{\mathrm{dR}}(U \cup V) & \xrightarrow{i^k} & \mathrm{H}^k_{\mathrm{dR}}(U) \oplus \mathrm{H}^k_{\mathrm{dR}}(V) & \xrightarrow{j^k} & \mathrm{H}^k_{\mathrm{dR}}(U \cap V) & \xrightarrow{\partial^k} \cdots \\
& \downarrow f_1^k & & \downarrow f_2^k & & \downarrow f_3^k & \\
\cdots \longrightarrow & \mathrm{H}^{m-k}_{\mathrm{dR},c}(U \cup V)^\vee & \xrightarrow{(i_c^{m-k})^\vee} & \mathrm{H}^{m-k}_{\mathrm{dR},c}(U)^\vee \oplus \mathrm{H}^{m-k}_{\mathrm{dR},c}(V)^\vee & \xrightarrow{(j_c^{m-k})^\vee} & \mathrm{H}^{m-k}_{\mathrm{dR},c}(U \cap V)^\vee & \xrightarrow{(\partial_c^{m-k-1})^\vee} \cdots .
\end{array}$$

where to ease notation we write i^k for $\mathrm{H}^k_{\mathrm{dR}}(i^k)$ and similarly for the other maps on cohomology. Note here that sign-commutative means the diagram commutes up to a possible difference in sign when one goes around a square along the different paths.

Proof. This proof is mainly just writing down what each of the maps does. The first step is to determine the vertical maps and the horizontal maps in the last row. The vertical maps are easy to write down. Given $[\omega] \in \mathrm{H}^k_{\mathrm{dR}}(U \cup V)$ we need to associate a linear functional to $[\omega]$. Define $f_1^k([\omega]) \in \mathrm{H}^{m-k}_{\mathrm{dR},c}(U \cup V)^\vee$ by setting

$$f_1^k([\omega])([\tau]) = \int_{U \cup V} [\omega \wedge \tau].$$

The map f_3^k is defined analogously. The map f_2^k is defined by

$$f_2^k([\omega_1], [\omega_2])([\tau_1], [\tau_2]) = \int_U [\omega_1 \wedge \tau_1] + \int_V [\omega_2 \wedge \tau_2].$$

The Mayer-Vietoris sequence for compactly supported de Rham cohomology gives the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^k_{\mathrm{dR},c}(U \cap V) \xrightarrow{\mathrm{H}^k_{\mathrm{dR},c}(j_c^k)} \mathrm{H}^k_{\mathrm{dR},c}(U) \oplus \mathrm{H}^k_{\mathrm{dR},c}(V) \xrightarrow{\mathrm{H}^k_{\mathrm{dR},c}(i_c^k)} \mathrm{H}^k_{\mathrm{dR},c}(U \cup V) \xrightarrow{\partial^k} \mathrm{H}^{k+1}_{\mathrm{dR},c}(U \cap V) \longrightarrow \cdots .$$

Recall that given a long exact sequence of vector spaces

$$\cdots \longrightarrow A_n \xrightarrow{g_n} A_{n+1} \xrightarrow{g_{n+1}} A_{n+2} \longrightarrow \cdots$$

one has a long exact sequence of the dual spaces

$$\cdots \longrightarrow A_{n+2}^\vee \xrightarrow{g_{n+1}^\vee} A_{n+1}^\vee \xrightarrow{g_n^\vee} A_n^\vee \longrightarrow \cdots$$

where the maps $g_n^\vee$ are given by setting $g_n^\vee(\phi) = \phi \circ g_n$. This gives the maps along the bottom row of the diagram.

We will prove the result for the square with the i^k and $(i_c^{m-k})^\vee$ maps and the square with the ∂^k and $(\partial_c^{m-k-1})^\vee$ maps, leaving the square with the j^k and $(j_c^{m-k})^\vee$ maps as an exercise. Let $[\omega] \in \mathrm{H}^k_{\mathrm{dR}}(U \cup V)$. Then we have

$$\begin{aligned} f_2^k(i^k([\omega]))([\tau_1],[\tau_2]) &= \int_U [i_U^*(\omega) \wedge \tau_1] + \int_V [i_V^*(\omega) \wedge \tau_2] \\ &= \int_U [\omega \circ i_U \wedge \tau_1] + \int_V [\omega \circ i_V \wedge \tau_2]. \end{aligned}$$

We have that the map i_c^{m-k} takes $([\tau_1],[\tau_2])$ to $[\tau_1 \circ i_U + \tau_2 \circ i_V]$. Thus we have

$$\begin{aligned} (i_c^{m-k})^\vee(f_1^k([\omega]))([\tau_1],[\tau_2]) &= f_1^k([\omega])([\tau_1 \circ i_U] + [\tau_2 \circ i_V]) \\ &= \int_{U \cup V} [\omega \wedge \tau_1 \circ i_U] + [\omega \wedge \tau_2 \circ i_V] \\ & \int_U [\omega \circ i_U \wedge \tau_1] + \int_V [\omega \circ i_V \wedge \tau_2]. \end{aligned}$$

Thus, we have the result for the first square.

The square with the ∂^k and $(\partial_c^{m-k-1})^\vee$ is given by

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{dR}}(U \cap V) & \xrightarrow{\partial^k} & \mathrm{H}^{k+1}_{\mathrm{dR}}(U \cup V) \\ \downarrow{\scriptstyle f_3^k} & & \downarrow{\scriptstyle f_1^{k+1}} \\ \mathrm{H}^{m-k}_{\mathrm{dR},c}(U \cap V)^\vee & \xrightarrow{(\partial_c^{m-k-1})^\vee} & \mathrm{H}^{m-k-1}_{\mathrm{dR},c}(U \cup V)^\vee. \end{array}$$

Now let $[\omega] \in \mathrm{H}^k_{\mathrm{dR}}(U \cap V)$. Recall that $\partial^k([\omega])$ satisfies that $\partial^k([\omega])|_U = [-d^k(\rho_V \omega)]$ and $\partial^k([\omega])|_V = [d^k(\rho_U \omega)]$. Since $[\omega] \in \mathrm{H}^k_{\mathrm{dR}}(U \cap V)$, we have that $\partial^k([\omega])$ has support contained in $U \cap V$, we can write for $[\tau] \in \mathrm{H}^{m-k-1}_{\mathrm{dR},c}(U \cup V)$

$$\begin{aligned} f_1^{k+1}(\partial^k[\omega])([\tau]) &= \int_{U \cup V} [\partial^k(\omega) \wedge \tau] \\ &= \int_{U \cap V} [-d^k(\rho_V \omega) \wedge \tau]. \end{aligned}$$

Observe that we have $d^k(\rho_V \omega) = d\rho_V \wedge \omega + \rho_V \wedge d^k\omega$. Since $[d^k\omega] = 0$ by definition of the cohomology group, and $[\rho_V \wedge d^k\omega] = [\rho_V] \wedge [d^k\omega]$, we have that $[d^k(\rho_V \omega)] = [d\rho_V \wedge \omega]$. Thus, we obtain

$$f_1^{k+1}(\partial^k[\omega])([\tau]) = -\int_{U \cap V} [d(\rho_V)\omega \wedge \tau].$$

Recall that the map $\partial_c^{m-k-1} : \mathrm{H}^{m-k-1}_{\mathrm{dR},c}(U \cup V) \to \mathrm{H}^{m-k}_{\mathrm{dR},c}(U \cap V)$ is given by $[\tau] \mapsto [d^{m-k-1}(\rho_V \tau)]$. As above, we have that $[d^{m-k-1}(\rho_V \tau)] = [d(\rho_V)\tau]$.

Thus, for $\omega \in \mathrm{H}^k_{\mathrm{dR}}(U \cap V)$ we have

$$\begin{aligned}(\partial_c^{m-k-1})^\vee(f_3^k([\omega]))([\tau]) &= \int_{U\cap V}[\omega \wedge \partial_c^{m-k-1}\tau] \\ &= \int_{U\cap V}[\omega \wedge d(\rho_V)\tau] \\ &= (-1)^k \int_{U\cap V}[d(\rho_V)\omega \wedge \tau].\end{aligned}$$

This gives the sign-commutativity of this square as well. □

We can now prove Poincare duality.

Proof. (Proof of Theorem 3.7.22) Note that Lemma 3.7.23 along with Lemma 3.7.24 shows that if we know Poincare duality for U, V, and $U \cap V$ then we will have it for $U \cup V$. We use induction on the cardinality of the good cover of M. If M is diffeomorphic to $\mathbb{R}^m$, then Poincare duality follows from the fact that Poincare's lemma gives

$$\mathrm{H}^k_{\mathrm{dR}}(\mathbb{R}^m) \cong \begin{cases} \mathbb{R} & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{H}^k_{\mathrm{dR},c}(\mathbb{R}^m) \cong \begin{cases} \mathbb{R} & k = m \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have Poincare duality in this case. Suppose now that Poincare duality holds for any manifold having a good cover with $n-1$ sets. Let M be a manifold that has a good cover with n sets, say $\{U_1, \dots, U_n\}$. We know that $(U_1 \cup \cdots \cup U_{n-1}) \cap U_n$ has a good cover with $n-1$ sets, namely, $\{U_1 \cap U_n, \dots, U_{n-1} \cap U_n\}$. Thus, our induction hypothesis gives that Poincare duality holds for $U_1 \cup \cdots \cup U_{n-1}$, U_n, and $(U_1 \cup \cdots \cup U_{n-1}) \cap U_n$. Thus, as was mentioned at the beginning of the proof this gives that it holds for $U_1 \cup \cdots \cup U_n$ as well. Hence, we have the result by induction for any manifold of finite type. □

Corollary 3.7.25. *Let M be a connected oriented m-manifold of finite type. Then we have*

$$\mathrm{H}^m_{\mathrm{dR},c}(M) \cong \mathbb{R}.$$

In particular, if M is compact, oriented, and connected then

$$\mathrm{H}^m_{\mathrm{dR}}(M) \cong \mathbb{R}.$$

Proof. We have by Poincare duality that $(\mathrm{H}^m_{\mathrm{dR},c}(M))^\vee \cong \mathrm{H}^0_{\mathrm{dR}}(M) \cong \mathbb{R}$ since we are assuming M is connected. However, we know that a finite dimensional $\mathbb{R}$-vector space V satisfies $V \cong V^\vee$, and so we have the first statement. If M happens to be compact, then $\mathrm{H}^m_{\mathrm{dR},c}(M) \cong \mathrm{H}^m_{\mathrm{dR}}(M)$, which gives the second statement. □

Note that this result justifies the statement that was used in calculating the cohomology of the torus T, namely that $\mathrm{H}^2_{\mathrm{dR}}(T) \cong \mathbb{R}$.

Let M and N be compact connected oriented m-manifolds and $f : M \to N$ a smooth map. We define the degree of f, denoted $\deg(f)$, via the diagram

$$\begin{array}{ccc} \mathrm{H}^m_{\mathrm{dR}}(N) & \xrightarrow{\mathrm{H}^m_{\mathrm{dR}}(f)} & \mathrm{H}^m_{\mathrm{dR}}(M) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \mathbb{R} & \xrightarrow{\deg(f)} & \mathbb{R}. \end{array}$$

In other words, $\deg(f)$ is defined to be the real number so that

$$\int_M f^*(\omega) = \deg(f) \int_N \omega.$$

Lemma 3.7.26. *The degree of f depends only on the homotopy class of $f : M \to N$.*

Proof. This is clear as the map $\mathrm{H}^k_{\mathrm{dR}}(f)$ depends only on the homotopy class of f. $\square$

Exercise 3.7.27. Show directly from the definition that the n^{th} power map on S^1 has degree n.

We conclude this chapter with a proof of the fundamental theorem of algebra.

Theorem 3.7.28. *(Fundamental Theorem of Algebra) Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with complex coefficients. If $n \geq 1$ then f has a root in $\mathbb{C}$.*

Proof. Suppose that f has no roots. For $r \geq 0$, define

$$\begin{aligned} g_r : S^1 &\longrightarrow S^1 \\ z &\mapsto \frac{f(rz)}{|f(rz)|}. \end{aligned}$$

Note that this is a smooth map and is well-defined by our assumption. Given $r, s \geq 0$, we have that g_r is homotopic to g_s by setting

$$F_t(z) = \frac{f((1-t)rz + tsz)}{|f((1-t)rz + tsz)|}.$$

Thus, we have that for all $r \geq 0$ that $\deg(g_r) = \deg(g_0)$. However, $g_0(z) = 1$ and so $\deg(g_r) = 0$.

Let $r \geq 0$ and consider now the map $G : S^1 \times [0,1] \to S^1$ defined by

$$G_t(z) = \frac{(rz)^n + t(f(rz) - (rz)^n)}{|(rz)^n + t(f(rz) - (rz)^n)|}.$$

This is well-defined as long as $(rz)^n + t(f(rz) - (rz)^n) \neq 0$. Observe that we have

$$\begin{aligned}|(rz)^n + t(f(rz) - (rz)^n)| &\geq |rz|^n - |t(a_{n-1}(rz)^{n-1} + \cdots + a_1(rz) + a_0)| \\ &\geq r^n - |a_{n-1}(rz)^{n-1} + \cdots + a_1(rz) + a_0| \\ &\geq r^n - |a_{n-1}|r^{n-1} - \cdots - a_1 r - a_0.\end{aligned}$$

We know that the limit of $r^n - |a_{n-1}|r^{n-1} - \cdots - |a_1|r - |a_0|$ as $r \to \infty$ diverges to ∞, so certainly there exists a large r so that $|(rz)^n + t(f(rz) - (rz)^n)| \neq 0$. Thus, choosing such a r the map $G_t(z)$ is well-defined and gives a homotopy between g_r and the map $z \mapsto z^n$. Thus, $\deg(g_r) = n$. This contradicts the fact that $n > 0$. □

Though a priori we only have that $\deg(f)$ is a real number, it turns out that if M and N are compact m-manifolds, M connected, and f smooth then $\deg(f)$ is in fact an integer. Let $f : M \to N$ be smooth with M compact, connected, and oriented and N compact and oriented. Let $y \in N$ be a regular value of f and $x \in f^{-1}(y)$. The *local index* is defined by

$$\mathrm{Ind}(f;x) = \begin{cases} 1 & \text{if } D_x f : T_x M \to T_y N \text{ preserves orientation} \\ -1 & \text{otherwise.} \end{cases}$$

Exercise 3.7.29. Let $y \in N$ be a regular value for the smooth map $f : M \to N$ between m-manifolds with M compact. Show that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_n$. Moreover, show that there exist disjoint open neighborhoods V_i of x_i in M and an open neighborhood U of y in N so that $f^{-1}(U) = \cup_{i=1}^n V_i$ and f maps each V_i diffeomorphically onto U. The Inverse Function Theorem may be of some help here.

Theorem 3.7.30. *With M, N, and f as above, for every regular value $y \in N$ we have*

$$\deg(f) = \sum_{x \in f^{-1}(y)} \mathrm{Ind}(f;x).$$

In particular, we have that $\deg(f) \in \mathbb{Z}$.

Proof. Let $x_1, \ldots, x_n$ be the elements in $f^{-1}(y)$. Let U and $V_1, \ldots, V_n$ be as in Exercise 3.7.29. We may assume that U is connected, and so each V_i is necessarily connected as well. We have that the diffeomorphism $f|_{V_i} : V_i \to U$ is either positively or negatively oriented. This is determined by whether $\mathrm{Ind}(f;x_i)$ is 1 or -1. Let $\omega \in \Omega^m(N)$ with $\mathrm{supp}(\omega) \subset U$ and $\int_N \omega = 1$. We necessarily have that $\mathrm{supp}(f^*(\omega)) \subset f^{-1}(U) = \cup_{i=1}^n V_i$. Thus, we can write

$$f^*(\omega) = \sum_{i=1}^n \omega_i$$

with $\omega_i \in \Omega^m(M)$ and $\operatorname{supp}(\omega_i) \subset V_i$. Note that $\omega_i|_{V_i} = (f|_{V_i})^*(\omega|_U)$. Then we have

$$\begin{aligned}
\deg(f) &= \deg(f) \int_N \omega \\
&= \int_M f^*(\omega) \\
&= \sum_{i=1}^{n} \int_M \omega_i \\
&= \sum_{i=1}^{n} \int_{V_i} (f|_{V_i})^*(\omega|_U) \\
&= \sum_{i=1}^{n} \operatorname{Ind}(f; x_i) \int_U \omega|_U \\
&= \sum_{i=1}^{n} \operatorname{Ind}(f; x_i)
\end{aligned}$$

where we have used that $\operatorname{supp}(\omega) \subset U$ and so $\int_U \omega|_U = \int_N \omega = 1$. □

Exercise 3.7.31. Use Theorem 3.7.30 to give an alternative computation of the degree of the n^{th} power map in Exercise 3.7.27.

Chapter 4

Singular Homology and Cohomology

In this chapter we provide the basics of singular homology and cohomology. We begin with a section on simplicial homology as it is a bit easier of an introduction to the material. For the most part we follow the presentation of [5].

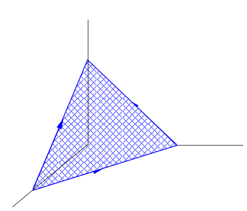
4.1 Simplicial Homology

In this section we give definitions of the simplicial homology groups as well as some basic examples. The simplicial homology groups are easier to compute with for simple examples, so this allows us to get our hands on some computations almost immediately. However, showing basic properties such as the fact that if X and Y are homotopic then they have the same homology groups requires us to work in the setting of singular homology. Before we can define the homology groups, we need to construct the relevant chain complex.

Our first step is to define a n-simplex. The standard n-simplex is defined by

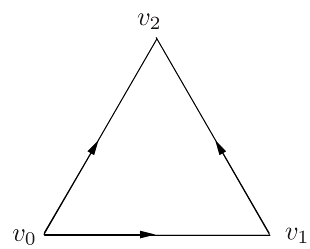
$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \text{ for } i = 0, \ldots, n \right\}.$$

For example, $\Delta^2 \subset \mathbb{R}^3$ is given by

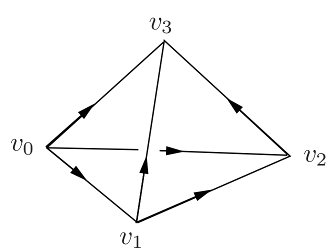


Note that there are arrows around the edges of Δ^2. It will be important for us to keep track of an ordering. A general *n-simplex* is the smallest convex set in $\mathbb{R}^{n+1}$ containing $n+1$ points $v_0, \dots, v_n$ so that $v_n - v_0, \dots, v_1 - v_0$ are linearly independent. The points v_i are the *vertices* of the simplex. We denote the simplex by $[v_0, \dots, v_n]$. Note that by writing the simplex in this way we are including an ordering of the vertices. This determines an orientation on the boundary edges: $[v_i, v_j]$ is positively oriented if $j > i$. Note that specifying such an ordering gives a canonical linear homomorphism from Δ^n to $[v_0, \dots, v_n]$ by sending $(t_0, \dots, t_n)$ to $\sum_i t_i v_i$. For the point $p = \sum_i t_i v_i \in [v_0, \dots, v_n]$, we call $(t_0, \dots, t_n)$ the *barycentric coordinates* of p.

Example 4.1.1. A 0-simplex is simply a point. The 1-simplex $[v_0, v_1]$ is the line between v_0 and v_1 oriented from v_0 to v_1. The 2-simplex $[v_0, v_1, v_2]$ is the triangle



The 3-simplex $[v_0, v_1, v_2, v_3]$ is the tetrahedron



Observe that given a n-simplex $[v_0, \dots, v_n]$, if we remove a vertex we are left with a $(n-1)$-simplex. We give this the orientation induced by the orientation of the original n-simplex. Such a $(n-1)$-simplex is called a *face of* $[v_0, \dots, v_n]$. For instance, given the 2-simplex $[v_0, v_1, v_2]$, we have faces $[v_0, v_1]$, $[v_0, v_2]$, and $[v_1, v_2]$. The union of all the faces is referred to as the *boundary* of $[v_0, \dots, v_n]$ and denoted $\partial[v_0, \dots, v_n]$. The *interior* of the simplex $[v_0, \dots, v_n]$ is given by $\mathrm{Int}([v_0, \dots, v_n]) = [v_0, \dots, v_n] - \partial[v_0, \dots, v_n]$.

Definition 4.1.2. Let X be a topological space. A Δ*-complex structure* on X is a collection of maps $\sigma_i : \Delta^n \to X$ with n depending on i so that

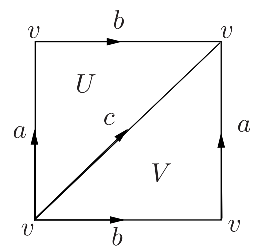
1. The restriction $\sigma_i|_{\mathrm{Int}(\Delta^n)}$ is injective and each point of X is in the image of exactly one such restriction.

2. Each restriction of σ_i to a face of Δ^n is one of the maps $\sigma_j : \Delta^{n-1} \to X$. (Note here we identify a face of Δ^n with Δ^{n-1} via the canonical linear homomorphism between them that preserves the orientation.)

3. A set $U \subset X$ is open if and only if $\sigma_i^{-1}(U)$ is open in Δ^n for each σ_i.

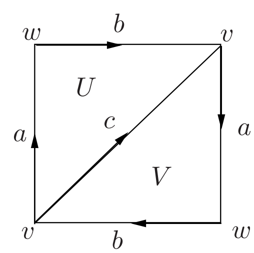
We call X with a Δ-complex structure a Δ-*complex.*

Example 4.1.3. We can decompose the torus into two triangles, three edges, and one vertex as in the following picture.



This gives T as a Δ-complex with six σ_i's.

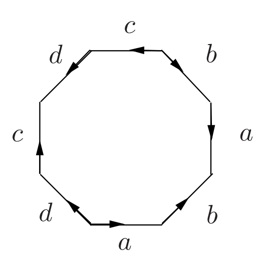
Example 4.1.4. We can decompose $\mathbb{RP}^2$ into two triangles, 3 edges, and 2 vertices as in the following picture.



This gives $\mathbb{RP}^2$ as a Δ-complex with seven σ_i's.

Exercise 4.1.5. Decompose the Klein bottle into simplices and show it is a Δ-complex as above.

Exercise 4.1.6. The torus with two holes can be formed as a quotient space via the following picture.



Decompose this into simplices and show the torus with two holes is a Δ-complex. (Hint: There should be six triangles.)

We can now define the simplicial homology groups. Let X be a Δ-complex. Let G be an abelian group. It is typical to focus on $G = \mathbb{Z}$ here, but there is really no need to specify G.

Definition 4.1.7. The *n-chains* of X are elements of the free abelian group $\Delta_n(X;G)$ generated over G by the maps $\sigma_i : \Delta^n \to X$. In other words, a n-chain is a formal sum $\sum_i g_i\sigma_i$.

In order to form homology groups, we need to put the $\Delta_n(X;G)$ into a chain complex. Define the *boundary homomorphism*

$$d_n : \Delta_n(X;G) \to \Delta_{n-1}(X;G)$$

by setting

$$d_n(\sigma_i) = \sum_{j=0}^{n}(-1)^j \sigma_i|_{[v_0,\ldots,\widehat{v}_j,\ldots,v_n]}$$

and then extending linearly to all other elements in $\Delta_n(X;G)$. First, observe that each element $\sigma_i|_{[v_0,\ldots,\widehat{v}_j,\ldots,v_n]}$ lies in $\Delta_{n-1}(X;G)$, so the map is well-defined. The negative signs are inserted in order to keep track of orientations.

Example 4.1.8. We look at the easiest cases.

1. Recall the 1-simplex $[v_0, v_1]$. In this case $d_1([v_0, v_1]) = [v_1] - [v_0]$.
2. Recall the 2-simplex $[v_0, v_1, v_2]$. In this case we have $d_2([v_0, v_1, v_2]) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$.

Exercise 4.1.9. Write out the map d_3 on the 3-simplex $[v_0, v_1, v_2, v_3]$.

Lemma 4.1.10. *The composition*

$$\Delta_n(X;G) \xrightarrow{d_n} \Delta_{n-1}(X;G) \xrightarrow{d_{n-1}} \Delta_{n-2}(X;G)$$

is exactly 0.

Proof. We have

$$\begin{aligned} d_{n-1} \circ d_n(\sigma) &= d_{n-1}\left(\sum_{i=0}^{n}(-1)^i \sigma|_{[v_0,\ldots,\widehat{v}_i,\ldots,v_n]}\right) \\ &= \sum_{j<i}(-1)^j(-1)^i \sigma|_{[v_0,\ldots,\widehat{v}_j,\ldots,\widehat{v}_i,\ldots,v_n]} \\ &= \ +\sum_{j>i}(-1)^j(-1)^i \sigma|_{[v_0,\ldots,\widehat{v}_i,\ldots,\widehat{v}_j,\ldots,v_n]}. \end{aligned}$$

Now observe that the second sum is the negative of the first sum, as you should check, and so they cancel out. □

Thus, we have a chain complex

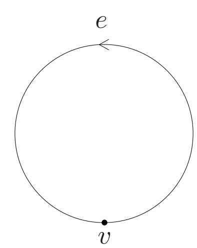
$$\cdots \longrightarrow \Delta_n(X;G) \xrightarrow{d_n} \Delta_{n-1}(X;G) \xrightarrow{d_{n-1}} \Delta_{n-2}(X;G) \longrightarrow \cdots \longrightarrow \Delta_1(X;G) \xrightarrow{d_1} \Delta_0(X;G) \xrightarrow{d_0} 0.$$

In particular, we have $\operatorname{im}(d_{n+1}) \subset \ker(d_n)$ and so we can form simplicial homology groups by setting

$$\mathrm{H}_n^{\Delta}(X;G) = \ker(d_n)/\operatorname{im}(d_{n+1}).$$

The elements of $\ker(d_n)$ are referred to as *n-cycles* and the elements of $\operatorname{im}(d_{n+1})$ as *n-boundaries*. Note that is it customary that if $G = \mathbb{Z}$, we simply write $\mathrm{H}_n^{\Delta}(X)$ for $\mathrm{H}_n^{\Delta}(X;\mathbb{Z})$.

Example 4.1.11. Let $X = S^1$. This is a Δ-complex with one vertex v and one edge e as pictured.



This gives that $\Delta_0(X;G) = Gv \cong G$ and $\Delta_1(X;G) = Ge \cong G$. Note that since there are no n-simplices for $n \geq 2$, we have $\Delta_n(X;G) = 0$ for all $n \geq 2$. The boundary map $d_1 : \Delta_1(X;G) \to \Delta_0(X;G)$ is given by $d_1(e) = v - v = 0$. Thus, we have that $\mathrm{H}_1^{\Delta}(X;G) = \ker(d_1)/\operatorname{im}(d_2) \cong G$ and $\mathrm{H}_0^{\Delta}(X;G) = \ker(d_0)/\operatorname{im}(d_1) \cong G$. In particular, we see that if we take $G = \mathbb{Z}$ we get

$$\mathrm{H}_k^{\Delta}(X) \cong \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.1.12. Let $X = T$. As we saw above, this is a Δ-complex with two 2-simplices U and V, three 1-simplices a, b, and c, and one 0-simplex v. Immediately from this we see that $\Delta_n(X;G) = 0$ for all $n \geq 3$. Let $G = \mathbb{Z}$. Observe that $d_1(a) = d_1(b) = d_1(c) = v - v = 0$, so the map d_1 is exactly 0. We know that $\Delta_0(X)$ is generated by v, and so we have

$$\begin{aligned} \mathrm{H}_0^{\Delta}(X) &= \ker(d_0)/\operatorname{im}(d_1) \\ &= \ker(d_0) \\ &= \mathbb{Z}v \\ &\cong \mathbb{Z}. \end{aligned}$$

We have that $d_2(U) = a + b - c = d_2(V)$. Thus, we see that $\operatorname{im}(d_2) = \mathbb{Z}(a + b - c) \cong \mathbb{Z}$. As a basis for $\Delta_1(X)$ we take $\{a, b, a + b - c\}$ so that we easily see that

$$\begin{aligned}\mathrm{H}_1^\Delta(X) &= \ker(d_1)/\operatorname{im}(d_2)\\ &= (\mathbb{Z}a + \mathbb{Z}b + \mathbb{Z}(a+b-c))/\mathbb{Z}(a+b-c)\\ &\cong \mathbb{Z}a \oplus \mathbb{Z}b\\ &\cong \mathbb{Z}\oplus\mathbb{Z}.\end{aligned}$$

Finally, since $\Delta_3(X) = 0$, we know that $\mathrm{H}_2^\Delta(X) = \ker(d_2)$. Note that $d_2(mU + nV) = (m+n)(a+b-c)$. This is equal to zero precisely when $m = -n$. Thus, we have that $\ker(d_2) = \mathbb{Z}(U - V) \cong \mathbb{Z}$. Thus, $\mathrm{H}_2^\Delta(X) \cong \mathbb{Z}$.

To summarize, we have

$$\mathrm{H}_k^\Delta(X) \cong \begin{cases} \mathbb{Z} & k = 0, 2\\ \mathbb{Z}\oplus\mathbb{Z} & k = 1\\ 0 & \text{otherwise.}\end{cases}$$

Exercise 4.1.13. Compute the homology groups $\mathrm{H}_k^\Delta(T; \mathbb{Z}/2\mathbb{Z})$.

Exercise 4.1.14. Show that the simplicial homology groups of $\mathbb{RP}^2$ are given by

$$\mathrm{H}_k^\Delta(\mathbb{RP}^2) \cong \begin{cases} \mathbb{Z} & k = 0\\ \mathbb{Z}/2\mathbb{Z} & k = 1\\ 0 & \text{otherwise.}\end{cases}$$

One should note here that for each of the examples, there was a choice of Δ-complex for the space X. It is natural to ask if the homology groups depend upon this choice. Moreover, if X and Y are homeomorphic, are the homology groups isomorphic? What if X and Y are only homotopic? These are all important questions, but it turns out it is easier to work with singular homology to answer such questions. We will study singular homology and then show that the singular homology groups agree with the simplicial homology groups for any Δ-complex X.

4.2 Definitions and Basic Properties of Singular Homology

In this section we will define and prove many of the basic properties of singular homology. We will also show how the singular homology groups agree with those calculated in the previous section.

Let X be a topological space. A *singular n-simplex of X* is a continuous map $\sigma : \Delta^n \to X$. Note that we do not require this to be a nice embedding at all, only that the map is continuous. Given an abelian group G, we let $C_n(X; G)$ denote the free abelian group over G generated by singular n-simplices of X, i.e. $C_n(X; G)$ consists of elements of the form $\sum_i g_i\sigma_i$. This is huge group as we do not put requirements such as used to define a Δ-complex on $C_n(X; G)$. We refer to elements of $C_n(X; G)$ as singular n-chains (or simply n-chains.)

We define the boundary maps $d_n : C_n(X;G) \to C_{n-1}(X;G)$ as in § 4.1, namely,

$$d_n(\sigma) = \sum_{i=1}^{n} (-1)^i \sigma|_{[v_0,\ldots,\widehat{v}_i,\ldots,v_n]}.$$

In this we are identifying Δ^{n-1} with $[v_0, \ldots, \widehat{v}_i, \ldots, v_n]$ and so $\sigma|_{[v_0,\ldots,\widehat{v}_i,\ldots,v_n]}$ is a $(n-1)$-simplex. Again we see that the boundary maps form a chain complex and so we can define singular homology groups by

$$\mathrm{H}_i(X;G) = \ker(d_n)/\operatorname{im}(d_{n+1}).$$

Proposition 4.2.1. *Let $\{X_i\}_{i\in I}$ be the path-components of X. Then we have*

$$\mathrm{H}_i(X;G) \cong \bigoplus_{i\in I} \mathrm{H}_i(X_i;G).$$

Proof. First, observe that since Δ^n is path-connected for all n and any singular n-simplex is continuous, we have that $\sigma(\Delta^n)$ must be path-connected as well. Thus, we have that $C_n(X;G) = \bigoplus_{i\in I} C_n(X_i;G)$. Furthermore, the definition of the boundary map shows that it respects this decomposition. Thus, we have that $\ker(d_n)$ and $\operatorname{im}(d_{n+1})$ also decompose into direct sums, which gives the result. □

Proposition 4.2.2. *We have that $\mathrm{H}_0(X;G)$ splits into a direct sum of copies of G, one for each path component of X.*

Proof. In light of Proposition 4.2.1, it is enough to show that if X is path-connected then $\mathrm{H}_0(X;G) \cong G$.

We know that d_0 is the zero map, so $\mathrm{H}_0(X;G) = C_0(X;G)/\operatorname{im}(d_1)$. Define

$$\varepsilon : C_0(X;G) \to G$$

by setting

$$\varepsilon\left(\sum_i g_i\sigma_i\right) = \sum_i g_i.$$

This map is clearly surjective and a homomorphism, so it remains to show that $\ker(\varepsilon) = \operatorname{im}(d_1)$.

Let $\sigma : \Delta^1 \to X$ be a singular 1-simplex. We have

$$\begin{aligned}\varepsilon(d_1(\sigma)) &= \varepsilon(\sigma|_{[v_1]} - \sigma|_{[v_0]})\\ &= 1 - 1\\ &= 0.\end{aligned}$$

This gives that $\operatorname{im}(d_1) \subset \ker(\varepsilon)$. Suppose now that $\varepsilon\left(\sum_i g_i\sigma_i\right) = 0$, i.e., $\sum_i g_i = 0$. We know that the σ_i's are singular 0-simplices, so they are points of X. Let x_0 be a point in X. Choose a path $\gamma_i : I \to X$ from x_0 to $\sigma_i(v_0)$, which is possible since X is assumed to be path-connected. Let σ_0 be the singular 0-simplex with image x_0. Then we have that γ_i is a singular 1-simplex, $\gamma_i : [v_0, v_1] \to X$. We have $d_1(\gamma_i) = \sigma_i - \sigma_0$. Thus,

$$\begin{aligned} d_1\left(\sum_i g_i\gamma_i\right) &= \sum_i g_i\sigma_i - \sum_i g_i\sigma_0 \\ &= \sum_i g_i\sigma_i \end{aligned}$$

where we have used that $\sum_i g_i\sigma_0 = \sigma_0 \sum_i g_i = 0$. Thus, we see that if $\sum_i g_i\sigma_i \in \ker(\varepsilon)$, then it is a boundary and so we have $\ker(\varepsilon) = \operatorname{im}(d_1)$ as claimed. □

In the particularly easy case that X is a single point, we can calculate all the homology groups.

Proposition 4.2.3. *If X is a single point then*

$$\mathrm{H}_k(X; G) \cong \begin{cases} 0 & k > 0 \\ G & k = 0. \end{cases}$$

Proof. We already know the result in the case $k = 0$. Observe that for each k we have a unique singular n-simplex σ_n since X is a single point. In particular, we have that $C_n(X; G) \cong G$ for $n \geq 0$. We have $d_n(\sigma_n) = \sum_{i=0}^{n}(-1)^i\sigma_{n-1}$. Thus, we get that $d_n = 0$ if n is odd and σ_{n-1} if n is even, $n \geq 0$. Thus, our chain complex becomes

$$\cdots \longrightarrow G \xrightarrow{\simeq} G \xrightarrow{0} G \xrightarrow{\simeq} G \xrightarrow{0} G \longrightarrow 0.$$

This gives the result. Note here that we have used that the map $g\sigma_n \mapsto g\sigma_{n-1}$ from $C_n(X; G)$ to $C_{n-1}(X; G)$ is an isomorphism to get the chain complex. □

Exercise 4.2.4. It is often desirable to have all the homology groups of a point vanish. One can accomplish this by using *reduced homology*. To define the reduced homology groups, use the chain complex

$$\cdots \longrightarrow C_2(X; G) \xrightarrow{d_2} C_1(X; G) \xrightarrow{d_1} C_0(X; G) \xrightarrow{\varepsilon} G \longrightarrow 0.$$

Check that this gives a chain complex. The resulting homology groups are referred to as the reduced homology and denoted $\widetilde{\mathrm{H}}_k(X; G)$. Show that for a space X one has $\widetilde{\mathrm{H}}_k(X; G) \cong \mathrm{H}_k(X; G)$ for $k > 0$ and $\mathrm{H}_0(X; G) \cong \widetilde{\mathrm{H}}_0(X; G) \oplus G$.

We would now like to show that homotopic spaces X and Y have isomorphic homology groups. As the set-up mirrors what we have done before for cohomology in Chapter 3, we leave many of the verifications to exercises. Let $f : X \to Y$ be a map between topological spaces. There is an induced map $f_* : C_n(X; G) \to C_n(Y; G)$ defined by $f_*(\sigma) = f \circ \sigma$ and then extending linearly.

Exercise 4.2.5. Check that the induced maps f_* satisfy $f_* d_n = d_n f_*$.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1}(X;G) & \xrightarrow{d_{n+1}} & C_n(X;G) & \xrightarrow{d_n} & C_{n-1}(X;G) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ \cdots & \longrightarrow & C_{n+1}(Y;G) & \xrightarrow{d_{n+1}} & C_n(Y;G) & \xrightarrow{d_n} & C_{n-1}(Y;G) & \longrightarrow & \cdots . \end{array}$$

Thus, we have that the collection $\{f_*\}$ gives a chain map, i.e, the following diagram commutes:

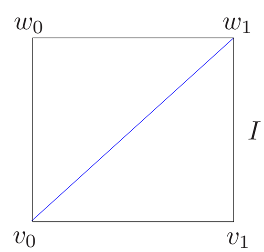
Exercise 4.2.6. Show that the maps f_* descend to maps on the homology groups.

Exercise 4.2.7. Show that if $f : X \to Y$ and $g : Y \to Z$ are maps, then $(f \circ g)_* = f_* \circ g_*$.

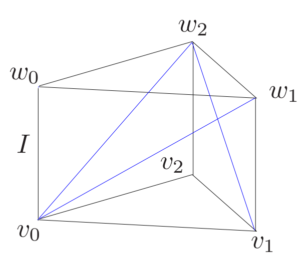
Exercise 4.2.8. Show that id is the identity map, then $\mathrm{id}_* = \mathrm{id}$. Be sure you understand what id means on each side of the equation!

Theorem 4.2.9. *If maps $f, g : X \to Y$ are homotopic then the induced maps on cohomology are equal.*

Proof. Our first step is to decompose $\Delta^n \times I$ into $(n+1)$-simplices. For example, if we consider the case of a 1-simplex we have



So in this case we see that we can break $\Delta^1 \times I$ into the 2-simplices $[v_0, v_1, w_1]$ and $[v_0, w_0, w_1]$. The case of a 2-simplex can be given by



In this case we see that we can break $\Delta^2 \times I$ into the 3-simplices $[v_0, v_1, v_2, w_2]$, $[v_0, v_1, w_1, w_2]$, and $[v_0, w_0, w_1, w_2]$. More generally, let $[v_0, \ldots, v_n] = \Delta^n \times \{0\}$

and $[w_0, \ldots, w_n] = \Delta^n \times \{1\}$ with w_i chosen so that w_i and v_i have the same image under the projection $\Delta^n \times I \to \Delta^n$. Our goal is to show that the union of the $(n+1)$-simplices $[v_0, \ldots, v_i, w_i, \ldots, w_n]$ give $\Delta^n \times I$. It is easy to see this is the case in the two examples we gave above.

For i between 0 and $n-1$ define a map $\phi_i : \Delta^n \to I$ by $\phi_i(t_0, \ldots, t_n) = t_{i+1} + \cdots + t_n$ where this is defined in terms of barycentric coordinates. Note that we define ϕ_n to be 0 and ϕ_{-1} to be 1. Observe that the graph of ϕ_i is precisely the n-simplex $[v_0, \ldots, v_i, w_{i+1}, \ldots, w_n]$. For instance, in our example above with the 1-simplex we have $\phi_0(t_0, t_1) = t_1$, which shows that $[v_0, v_1]$ maps onto $[v_0, w_1]$ and $\phi_1(t_0, t_1) = 0$ so $[v_0, v_1]$ maps onto $[v_0, v_1]$. Note that $\phi_i(t_0, \ldots, t_n) \leq \phi_{i-1}(t_0, \ldots, t_n)$ and so the graph of ϕ_i lies below the graph of ϕ_{i-1}. The region between these graphs is precisely the $(n+1)$-simplex $[v_0, \ldots, v_i, w_i, \ldots, w_n]$. We obtain that $[v_0, \ldots, v_i, w_i, \ldots, w_n]$ is a true $(n+1)$-simplex since w_i does not lie on the graph of ϕ_i. We have that

$$0 = \phi_n(t_0, \ldots, t_n) \leq \phi_{n-1}(t_0, \ldots, t_n) \leq \cdots \leq \phi_0(t_0, \ldots, t_n) \leq \phi_{-1}(t_0, \ldots, t_n) = 1.$$

This shows that we get all of $\Delta^n \times I$.

We can now define a chain homotopy between f_* and g_*. Let σ be a singular n-simplex and define $\sigma \times \mathrm{id} : \Delta^n \times I \to X \times I$. Let $F : X \times I \to Y$ be a homotopy between f and g. Define $P : C_n(X; G) \to C_{n+1}(Y; G)$ by setting

$$P_n(\sigma) = \sum_{i=1}^{n} (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}.$$

We will show that

$$d_{n+1} P_n - g_* + f_* = P_{n+1} d_n$$

which gives that P_n is the chain homotopy we seek. Observe we have

$$\begin{aligned} d_{n+1} P_n(\sigma) &= d_{n+1} \left(\sum_{i=1}^{n} (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]} \right) \\ &= \sum_{j \leq i} (-1)^j (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0, \ldots, \widehat{v}_j, \ldots, v_i, w_i, \ldots, w_n]} \\ &+ \sum_{j \geq i} (-1)^{j+1} (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0, \ldots, v_i, w_i, \ldots, \widehat{w}_j, \ldots, w_n]}. \end{aligned}$$

Observe that in the case that $i = j$, the terms in the two sums cancel except for the terms $F \circ (\sigma \times \mathrm{id})|_{[\widehat{v}_0, w_0, \ldots, w_n]}$ and $-F \circ (\sigma \times \mathrm{id})|_{[v_0, \ldots, v_n, \widehat{w}_n]}$. However, we have that $F \circ (\sigma \times \mathrm{id})|_{[\widehat{v}_0, w_0, \ldots, w_n]} = g \circ \sigma = g_*(\sigma)$ and $-F \circ (\sigma \times \mathrm{id})|_{[v_0, \ldots, v_n, \widehat{w}_n]} = -f \circ \sigma = f_*(\sigma)$. Thus, we have that

$$\begin{aligned} d_{n+1} P_n(\sigma) - g_*(\sigma) + f_*(\sigma) &= \sum_{j < i} (-1)^j (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0, \ldots, \widehat{v}_j, \ldots, v_i, w_i, \ldots, w_n]} \\ &+ \sum_{j > i} (-1)^{j+1} (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0, \ldots, v_i, w_i, \ldots, \widehat{w}_j, \ldots, w_n]}. \end{aligned}$$

However, we have

$$\begin{aligned} P_{n+1}(d_n\sigma) &= \sum_{i<j}(-1)^j(-1)^i F\circ(\sigma\times \mathrm{id})|_{[v_0,\dots,v_i,w_i,\dots,\widehat{v}_j,\dots,w_n]} \\ &\quad + \sum_{i>j}(-1)^j(-1)^{i-1} F\circ(\sigma\times \mathrm{id})|_{[v_0,\dots,\widehat{v}_j,\dots,v_i,w_i,\dots,w_n]}, \end{aligned}$$

which is precisely $d_{n+1}P_n - g_* + f_*$ as desired.

Thus, we have a chain homotopy between f_* and g_*. As before, we see that if $[\sigma]\in \mathrm{H}_n(X;G)$, then we have

$$\begin{aligned} f_*([\sigma]) - g_*([\sigma]) &= d_{n+1}(P_n([\sigma])) + P_{n+1}(d_n([\sigma])) \\ &= d_{n+1}(P_n([\sigma])) \\ &= 0. \end{aligned}$$

Thus, we see the maps agree on homology as claimed. □

Corollary 4.2.10. *If X and Y are homotopy equivalent, then $\mathrm{H}_n(X;G)\cong \mathrm{H}_n(Y;G)$ for all n.*

Example 4.2.11. We can combine Corollary 4.2.10 along with Proposition 4.2.3 to see that

$$\mathrm{H}_k(\mathbb{R}^m;G)\cong \begin{cases} 0 & k\neq 0 \\ G & k=0. \end{cases}$$

since Euclidean space is contractible to a point by straight-line homotopy.

Let X be a topological space and A a subspace of X. We would like to be able to relate the groups $\mathrm{H}_n(X;G)$, $\mathrm{H}_n(A;G)$, and $\mathrm{H}_n(X/A;G)$ where X/A is the quotient space as defined in Example 2.9.5 of § 2.9. In general it is very difficult to compute homology groups straight from the definition, so we would like to be able to relate the homology groups of X to subspaces to help us actually compute the homology groups. At first glance one might conjecture that $\mathrm{H}_n(X/A;G)\cong \mathrm{H}_n(X;G)/\mathrm{H}_n(A;G)$. However, this is not the case in general. Actually, it is good that this does not happen because if it did, our theory would be useless. For instance, we can consider X as a subspace of the cone of X: $CX = (X\times I)/(X\times\{0\})$. The cone of X is contractible and so has trivial homology groups, which would give that all the homology groups of X as trivial as well since X embeds into its cone. In order to get a handle on how the homology groups of X and A are related, we define the relative homology groups.

Let A be a subspace of X. Set $C_n(X,A;G) = C_n(X;G)/C_n(A;G)$. It is clear from the definition that d_n restricts to a map from $C_n(A;G)$ to $C_{n-1}(A;G)$, and so we have a map $d_n : C_n(X,A;G)\to C_{n-1}(X,A;G)$. We again have that $d_{n-1}\circ d_n = 0$ and so we have a chain complex

$$\cdots \longrightarrow C_n(X,A;G) \xrightarrow{d_n} C_{n-1}(X,A;G) \xrightarrow{d_{n-1}} C_{n-2}(X,A;G) \longrightarrow \cdots$$

which allows us to define the relative homology groups as

$$\mathrm{H}_n(X, A; G) = \mathrm{H}_n(C_*(X, A; G)).$$

Observe that elements of $\mathrm{H}_n(X, A; G)$ are represented by n-chains $\sigma \in C_n(X; G)$ so that $d_{n-1}(\sigma) \in C_{n-1}(A; G)$. Furthermore, a relative cycle $\sigma \in C_n(X, A; G)$ is trivial in $\mathrm{H}_n(X, A; G)$ if and only if it is a relative boundary, i.e., $\sigma = d_{n+1}\alpha + \beta$ for some $\alpha \in C_{n+1}(X; G)$ and some $\beta \in C_n(A; G)$. This allows us to view the group $\mathrm{H}_n(X, A; G)$ as really the homology of X modulo the homology of A.

Exercise 4.2.12. Let $i : A \to X$ be the inclusion map and $j : X \to X/A$ the projection map. Show that for each $n \geq 0$ we have an exact sequence

$$0 \longrightarrow C_n(A; G) \xrightarrow{i_*} C_n(X; G) \xrightarrow{j_*} C_n(X, A; G) \longrightarrow 0.$$

As in the case of cohomology, we would like to have a long exact sequence of homology. Observe that we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n+1}(A; G) & \xrightarrow{i_*} & C_{n+1}(X; G) & \xrightarrow{j_*} & C_{n+1}(X, A; G) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_{n+1}} & & \downarrow{\scriptstyle d_{n+1}} & & \downarrow{\scriptstyle d_{n+1}} & & \\
0 & \longrightarrow & C_n(A; G) & \xrightarrow{i_*} & C_n(X; G) & \xrightarrow{j_*} & C_n(X, A; G) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \\
0 & \longrightarrow & C_{n-1}(A; G) & \xrightarrow{i_*} & C_{n-1}(X; G) & \xrightarrow{j_*} & C_{n-1}(X, A; G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

We have already seen that the maps i_* and j_* descend to maps on cohomology, so it remains to define a connecting homomorphism

$$\partial_n : \mathrm{H}_n(X, A; G) \to \mathrm{H}_{n-1}(A; G)$$

so that we have the following long exact sequence in homology

$$\cdots \xrightarrow{\partial_{n+1}} \mathrm{H}_n(A; G) \xrightarrow{i_*} \mathrm{H}_n(X; G) \xrightarrow{j_*} \mathrm{H}_n(X, A; G) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A; G) \xrightarrow{i_*} \cdots.$$

Let $[\sigma] \in \mathrm{H}_n(X, A; G)$ and let $\sigma \in C_n(X, A; G)$ be a representative. We know that $j_* : C_n(X; G) \to C_n(X, A; G)$ is surjective, so there exists a $\tau \in C_n(X; G)$ so that $j_*(\tau) = \sigma$. We have that $d_n(\tau) \in C_{n-1}(X; G)$ and the fact that the diagram above commutes, we have $j_*(d_n(\tau)) = d_n(j_*(\tau)) = d_n(\sigma) = 0$ because $\sigma \in \ker(d_n)$ by assumption of it being a cycle. Thus, using the exactness of the $(n-1)$-row we see that there exists $u \in C_{n-1}(A; G)$ so that $i_*(u) = d_n(\tau)$. Define $\partial_n([\sigma]) = [u]$.

Exercise 4.2.13. Check that ∂_n is well-defined. Namely, show that u is a cycle and that the definition of ∂_n does not depend on any of the choices made.

Exercise 4.2.14. Show that this definition of ∂_n yields a long exact sequence of homology groups as given above.

Theorem 4.2.15. *Given a topological space X and a subspace A, one has the following long exact sequence of homology groups*

$$\cdots \xrightarrow{\partial_{n+1}} \mathrm{H}_n(A; G) \xrightarrow{i_*} \mathrm{H}_n(X; G) \xrightarrow{j_*} \mathrm{H}_n(X, A; G) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A; G) \xrightarrow{i_*} \cdots .$$

Exercise 4.2.16. Show that for any $x_0 \in X$ one has $\mathrm{H}_k(X, \{x_0\}) \cong \widetilde{\mathrm{H}}_k(X)$ for all $k \geq 0$.

Exercise 4.2.17. Let A be a nonempty space of X. Show that the boundary operator $\partial_1 : \mathrm{H}_1(X, A; G) \to \mathrm{H}_0(A; G)$ sends $\mathrm{H}_1(X, A; G)$ into the subgroup $\widetilde{\mathrm{H}}_0(A; G)$ of $\mathrm{H}_0(A; G)$ and show the following sequence is exact:

$$\cdots \xrightarrow{j_*} \mathrm{H}_1(X, A; G) \xrightarrow{\partial_1} \widetilde{\mathrm{H}}_0(A; G) \xrightarrow{i_*} \widetilde{\mathrm{H}}_0(X; G) \xrightarrow{j_*} \mathrm{H}_0(X, A; G) \longrightarrow 0.$$

Let $f : X \to Y$ be a continuous map and let $A \subset X$ and $B \subset Y$ be subspaces. Moreover, suppose that $f(A) \subset B$. We denote this by writing $f : (X, A) \to (Y, B)$. We have already seen that f induces a map on chains $f_* : C_n(X; G) \to C_n(Y; G)$ by sending σ to $f \circ \sigma$. Observe that if $\sigma \in C_n(A; G)$, then the assumption that $f(A) \subset B$ gives that $f_*(\sigma) = f \circ \sigma \in C_n(B; G)$. Thus, f_* maps $C_n(A; G)$ into $C_n(B; G)$ and so f_* gives a map on the relative chains

$$f_* : C_n(X, A; G) \to C_n(Y, B; G).$$

Furthermore, one can easily check that this descends to a map on the relative homology groups

$$f_* : \mathrm{H}_n(X, A; G) \to \mathrm{H}_n(Y, B; G).$$

Proposition 4.2.18. *If two maps $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs (X, A) and (Y, B), then*

$$f_* = g_* : \mathrm{H}_n(X, A; G) \to \mathrm{H}_n(Y, B; G).$$

Proof. One uses the same argument as in the proof of Theorem 4.2.9. The same map P works here as well. ☐

Exercise 4.2.19. Let $B \subset A \subset X$ be subspaces. Show that one has an exact sequence

$$0 \longrightarrow C_n(A, B; G) \longrightarrow C_n(X, B; G) \longrightarrow C_n(X, A; G) \longrightarrow 0.$$

Use this to show that one has a long exact sequence in homology

$$\cdots \longrightarrow \mathrm{H}_n(A,B;G) \longrightarrow \mathrm{H}_n(X,B;G) \longrightarrow \mathrm{H}_n(X,A;G) \longrightarrow \mathrm{H}_{n-1}(A,B;G) \longrightarrow \cdots.$$

We recover the long exact sequence in Theorem 4.2.15 by setting B to be a single point.

The following theorem shows that given subspaces $Z \subset A \subset X$, we can "excise" the subset Z and the relative homology groups are not changed. It is known as the Excision Theorem.

Theorem 4.2.20. *(Excision Theorem) Let $Z \subset A \subset X$ with $\mathrm{Cl}(Z) \subset \mathrm{Int}(A)$. Then natural inclusion $(X-Z, A-Z) \hookrightarrow (X,A)$ induces isomorphisms*

$$\mathrm{H}_k(X-Z, A-Z;G) \cong \mathrm{H}_k(X,A;G)$$

for all k.

We will not prove this theorem as it is fairly long and involved. It essentially amounts to showing that one can compute homology groups by using simplices that are small. For instance, if X is a metric space, for any $\epsilon > 0$ one can insist all the simplices lie in cubes of size ϵ. One can find a proof of this theorem in Chapter 2 of [5].

Corollary 4.2.21. *Let A, B be subspaces of X so that $X \subset \mathrm{Int}(A) \cup \mathrm{Int}(B)$. The inclusion $(B, A\cap B) \hookrightarrow (X,A)$ induces isomorphisms*

$$\mathrm{H}_k(B, A\cap B;G) \cong \mathrm{H}_k(X,A;G).$$

Proof. We reduce this to Theorem 4.2.20. Set $Z = X - B$. Then we have $A \cap B = A - Z$. The condition that $X \subset \mathrm{Int}(A) \cup \mathrm{Int}(B)$ implies that $\mathrm{Cl}(Z) \subset \mathrm{Int}(A)$. □

We can now relate the relative homology groups $\mathrm{H}_n(X,A;G)$ to the reduced homology groups $\widetilde{\mathrm{H}}_n(X/A;G)$, at least in the case that A is closed, nonempty and is a deformation retract of an open neighborhood in X. We call such a pair (X,A) a *good pair.*

Proposition 4.2.22. *Let (X,A) be a good pair with V an open neighborhood of A that deformation retracts to A. Then the quotient map j induces isomorphisms $\mathrm{H}_n(X,A;G) \cong \widetilde{\mathrm{H}}_n(X/A;G)$ for all n.*

Proof. Observe that we have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{H}_n(X,A;G) & \longrightarrow & \mathrm{H}_n(X,V;G) & \longleftarrow & \mathrm{H}_n(X-A,V-A;G) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}_n(X/A,A/A;G) & \longrightarrow & \mathrm{H}_n(X/A,V/A;G) & \longleftarrow & \mathrm{H}_n(X/A-A/A,V/A-A/A;G) \end{array}$$

where the horizontal maps are induced from inclusions and the vertical ones from projection. Since A is a deformation retract of V, we have that $\mathrm{H}_n(V,A;G) \cong \mathrm{H}_n(A,A;G) = 0$. Thus, using Exercise 4.2.19 we see that $\mathrm{H}_n(X,A;G) \cong \mathrm{H}_n(X,V;G)$. Furthermore, we see that $\mathrm{H}_n(X/A,A/A;G) \cong \mathrm{H}_n(X/A,V/A;G)$ by the exact same argument since V/A retracts to A/A. Thus, both of the left horizontal arrows are isomorphisms. We obtain that the horizontal right two arrows are isomorphisms from Theorem 4.2.20.

Observe that the projection map $j : X \to X/A$ is a homeomorphism when we restrict to $X - A$, namely, $j : X - A \to X/A - A/A$ is a homeomorphism. Thus, the right vertical arrow is an isomorphism. One can now use the commutativity of the diagram to see that the left-most vertical arrow is an isomorphism as well. Thus, we have that

$$\mathrm{H}_n(X,A;G) \cong \mathrm{H}_n(X/A,A/A;G).$$

However, we know that A/A is a single point so Exercise 4.2.16 gives that $\mathrm{H}_n(X/A,A/A;G) \cong \widetilde{\mathrm{H}}_n(X/A;G)$ as desired. □

Corollary 4.2.23. *We have*

$$\widetilde{\mathrm{H}}_k(S^n;G) \cong \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

Proof. Let $X = D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $A = S^{n-1}$ so that $A = \partial X$. Observe that $X/A \cong S^n$. (Be sure you understand why!) The fact that D^n is contractible gives $\widetilde{\mathrm{H}}_k(D^n;G) = 0$ for all k. We apply the long exact sequence given in Exercise 4.2.17 to obtain

$$0 \longrightarrow \mathrm{H}_k(D^n, S^{n-1};G) \longrightarrow \mathrm{H}_{k-1}(S^{n-1};G) \longrightarrow 0,$$

for $k > 1$ and

$$0 \longrightarrow \mathrm{H}_1(D^n, S^{n-1};G) \longrightarrow \widetilde{\mathrm{H}}_0(S^{n-1};G) \longrightarrow 0.$$

Thus we have that

$$\mathrm{H}_k(D^n, S^{n-1};G) \cong \widetilde{\mathrm{H}}_{k-1}(S^{n-1};G)$$

for all k where we have used the homology and reduced homology groups agree for $k > 0$. We have that (D^n, S^{n-1}) is a good pair, so Proposition 4.2.22 gives that

$$\mathrm{H}_k(D^n, S^{n-1};G) \cong \widetilde{\mathrm{H}}_k(S^n;G).$$

Combining these results we see that

$$\widetilde{\mathrm{H}}_k(S^n; G) \cong \widetilde{\mathrm{H}}_{k-1}(S^{n-1}; G).$$

Observing that S^0 consists of two points, we obtain the result by using Propositions 4.2.1 and 4.2.3 and induction. □

Exercise 4.2.24. Use this to give alternative proofs of Lemma 3.4.28 and Theorem 3.4.27.

Example 4.2.25. We can construct explicit generators of the groups $\mathrm{H}_n(D^n, \partial D^n)$ and $\widetilde{\mathrm{H}}_n(S^n)$ that will be useful to know. First we deal with $\mathrm{H}_n(D^n, \partial D^n)$. Observe that we can replace the pair by $(\Delta^n, \partial\Delta^n)$ since they are homotopy equivalent. We claim that the identity map $\mathrm{id} : \Delta^n \to \Delta^n$ when viewed as a singular n-simplex is a cycle generating $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$. We proceed by induction. The case $n = 0$ is clear. Let $\Lambda \subset \Delta^n$ be the union of all but one of the $(n-1)$ dimensional faces of Δ^n. Observe that we have an isomorphism

$$\mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{\simeq} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)$$

that arises from considering the triple in Exercise 4.2.19 as $(\Delta^n, \partial\Delta^n, \Lambda)$ and using that $\mathrm{H}_n(\Delta^n, \Lambda) = 0$ since Δ^n deformation retracts onto Λ and so (Δ^n, Λ) is homotopy equivalent to (Λ, Λ). We also have an isomorphism

$$\mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \xrightarrow{\simeq} \mathrm{H}_{n-1}(\partial\Delta^n).$$

To obtain this isomorphism, observe that the inclusion $\Delta^{n-1} \hookrightarrow \partial\Delta^n$ of the face not contained in Λ induces a homeomorphism of quotients $\Delta^{n-1}/\partial\Delta^{n-1} \cong \partial\Delta^n/\Lambda$. One then uses that the pairs being considered are good pairs along with Proposition 4.2.22 to obtain the isomorphism. We now have the result by induction since the cycle id is sent under the first isomorphism to $\partial\,\mathrm{id}$, which equals $\pm\,\mathrm{id}$ in $C_{n-1}(\partial\Delta^n, \Lambda)$.

We can now regard S^n as two n-simplices Δ^n_1 and Δ^n_2 with the boundaries identified in the obvious way. We can then view the difference $\Delta^n_1 - \Delta^n_2$ as a singular n-chain that is a cycle. We claim this generates $\widetilde{\mathrm{H}}_n(S^n)$ for $n > 0$. Using the long exact sequence for the pair (S^n, Δ^n_2) we obtain an isomorphism

$$\widetilde{\mathrm{H}}_n(S^n) \xrightarrow{\simeq} \mathrm{H}_n(S^n, \Delta^n_2).$$

We can also use the same argument given above with quotients to obtain an isomorphism

$$\mathrm{H}_n(\Delta^n_1, \partial\Delta^n_1) \xrightarrow{\simeq} \mathrm{H}_n(S^n, \Delta^n_2).$$

Under these isomorphisms, the cycle $\Delta^n_1 - \Delta^n_2$ in the group $\widetilde{\mathrm{H}}_n(S^n)$ corresponds to the cycle Δ^n_1 in the group $\mathrm{H}_n(\Delta^n_1, \partial\Delta^n_1)$, which represents a generator of this group. Thus, we have that $\Delta^n_1 - \Delta^n_2$ is a generator of the group $\widetilde{\mathrm{H}}_n(S^n)$.

When studying topology, there is always the issue of picking the "right" spaces to study. For instance, if we try and be too general it will be difficult to prove any interesting theorems. Of course, we would like to be as general as is reasonably possible to encompass as many spaces as we can. In studying singular homology, the "right" spaces are generally CW-complexes. A CW-complex is built up inductively. We call a set a *n-cell* if it is homeomorphic to $U^n = \{x \in \mathbb{R}^n : |x| < 1\}$. Let Y be a Hausdorff space and X a closed subspace of Y. Suppose that $Y - X$ consists of a disjoint union of n-cells $\{e_i^n\}_{i \in I}$. Furthermore, assume that for each $i \in I$ there is a continuous map $f_i : D^n \to \mathrm{Cl}(e^n)$ so that f_i maps U^n homeomorphically onto e^n and $f_i(S^{n-1}) \subset X$. If I is a finite set, we say Y is obtained from X by attaching n-cells. If I is not finite, we require that Y has the *weak topology determined by the maps f_i and the inclusion $X \hookrightarrow Y$.* The weak topology condition means that $A \subset Y$ is closed if and only if $A \cap X$ is closed and the sets $f_i^{-1}(A)$ are closed for all $i \in I$.

Definition 4.2.26. A structure of a *CW-complex* is prescribed on a Hausdorff space X by an ascending chain of closed subspaces

$$X^0 \subset X^1 \subset \cdots$$

that satisfy:

1. X^0 has the discrete topology
2. For $n > 0$, X^n is obtained from X^{n-1} by attaching a collection of n-cells as described above.
3. X is the union of X^i for $i \geq 0$.
4. The space X and the subspaces X^i all have the weak topology: A subset A is closed if and only if $A \cap \mathrm{Cl}(e^n)$ is closed for all n-cells e^n for $n = 0, 1, \ldots$.

We call the subset X^0 of X the *vertices*. The subset X^n is the *n-skeleton* of X. We say X is *finite* or *infinite* according to whether the number of cells is finite or infinite. If $X = X^n$ we say X is *finite dimensional* and call n the *dimension* of X.

Definition 4.2.27. A subset A of a CW-complex X is called a *subcomplex* if A is the union of cells of X, and if for any cell e^n we have that $e^n \subset A$ implies that $\mathrm{Cl}(e^n) \subset A$.

In fact, if A is a subcomplex of X then the sets $A^n = A \cap X^n$ define a CW-complex structure on A. It is a fact that if $A \subset X$ is a subcomplex of a CW-complex X, then (X, A) is a good pair. One can see Appendix A of [5] for a proof of this fact and many others about the basic topology of CW-complexes.

Definition 4.2.28. A continuous map $f : X \to Y$ of CW-complexes is called *cellular* if $f(X^n) \subset Y^n$ for $n \geq 0$.

Example 4.2.29. Let $X = S^n$. Then X is a CW-complex with one e^0 and one e^n. For instance, the case $n = 2$ is given by identifying the boundary of the open unit disc U^2 to a point, which gives the sphere. In particular, for $0 \le k < n$, X^k consists of a single point and $X^n = S^n$ and the map that attaches e^n to X^{n-1} is by mapping the boundary of D^n to a point.

Example 4.2.30. Let X be a 1-dimensional CW-complex. Then X consists of vertices and edges, 0-cells and 1-cells. This is a graph.

Example 4.2.31. The Δ-complexes given in § 4.1 are CW-complexes. Note that for $n \ge 1$, $\mathrm{Int}(\Delta^n)$ is homeomorphic to U^n. Thus, we have that the torus, $\mathbb{RP}^2$, and the Klein bottle are all CW-complexes.

Example 4.2.32. Consider $\mathbb{RP}^n$. Observe that $\mathbb{RP}^0$ is just a point. We also have that $\mathbb{RP}^1$ is the circle. For instance, one can see this via the map

$$[x : y] \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right).$$

We can use these base skeletons to build $\mathbb{RP}^n$. We proceed by induction. Suppose that we have constructed $\mathbb{RP}^{n-1}$. We can build $\mathbb{RP}^n$ from $\mathbb{RP}^{n-1}$ by adjoining a single n-cell. We define the map

$$f_n : D^n \to \mathbb{RP}^n$$

by setting

$$f_n(x_1, \ldots, x_n) = [x_1 : \cdots : x_n : \sqrt{1 - |x|^2}]$$

where $x = (x_1, \ldots, x_n)$. It now straight-forward to check that f_n maps $D^n - S^{n-1}$ homeomorphically onto $\mathbb{RP}^n - \mathbb{RP}^{n-1}$ and maps S^{n-1} onto $\mathbb{RP}^{n-1}$, though not homeomorphically. Thus, we have that $\mathbb{RP}^n$ is a CW-complex with one cell of each dimension, i.e., $\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n$.

Exercise 4.2.33. Show that $\mathbb{CP}^n$ is a CW-complex with decomposition $\mathbb{CP}^n = e^0 \cup e^2 \cup e^4 \cup \cdots \cup e^{2n}$.

We have the excision property for subcomplexes of CW-complexes as well.

Corollary 4.2.34. *If the CW-complex X is the union of subcomplexes A and B, then the inclusion map $(B, A \cap B) \to (X, A)$ induces isomorphisms*

$$\mathrm{H}_n(B, A \cap B; G) \cong \mathrm{H}_n(X, A; G)$$

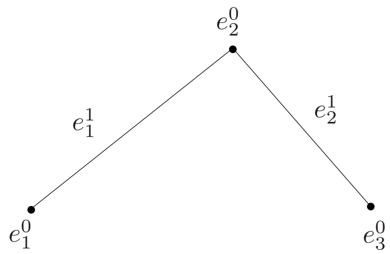
for all n.

Proof. We know that all CW-pairs are good pairs, and so we can use Proposition 4.2.22 to pass to the quotient spaces $B/A \cap B$ and X/A. These spaces are homeomorphic since $X = A \cup B$, and so we have the result. □

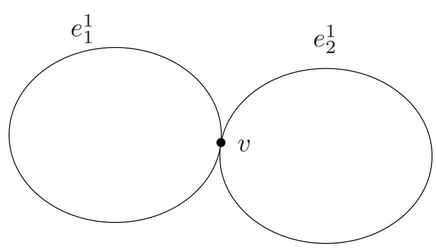
The *wedge sum* is a useful operation when studying CW-complexes. Let X and Y be topological spaces and let $x_0 \in X$, $y_0 \in Y$. We define the *wedge sum* $X \vee Y$ of X and Y with respect to x_0 and y_0 to be the quotient of the disjoint union of X and Y obtained by identifying x_0 and y_0. More generally, we can define a wedge sum of an arbitrary collection of spaces X_i with respect to points $x_i \in X_i$ for $i \in I$ by forming the quotient space of the disjoint union of the X_i by identifying the x_i to a single point. We denote this space by $\vee_{i\in I} X_i$.

Example 4.2.35. The wedge sum $S^1 \vee S^1$ is a figure eight.

Example 4.2.36. Consider the set $X^0 = \{e_1^0, e_2^0, e_3^0\}$. Attach two 1-cells e_1^1 and e_2^1 to form X^1 as in the following picture:



Consider the quotient space X^1/X^0. Note we identify all the vertices to one point, say v. We have the picture:



Note that X^1/X^0 is homeomorphic to $S^1 \vee S^1$.

More generally, if X^n is a n-skeleton formed from X^{n-1} by attaching n-cells, then X^n/X^{n-1} is homeomorphic to the wedge sum of copies of S^n, one copy for each n-cell attached.

We will return to CW-complexes in § 4.4. We finish this section by proving that the simplicial and singular homology groups agree for a Δ-complex X. This will give us the singular homology groups of the examples calculated in § 4.1. Let X be a Δ-complex and A a subcomplex, i.e., A is a Δ-complex formed via the union of simplicies in X. We can define relative simplicial homology groups by setting $\Delta_k(X, A; G) = \Delta_k(X; G)/\Delta_n(A; G)$. As in the singular case this gives a chain complex and so we define the relative simplicial homology groups via that chain complex. We obtain a long exact sequence of simplicial homology groups as in Theorem 4.2.15. There is a natural map $\Delta_k(X, A; G) \to C_k(X, A; G)$. Note we include the case $A = \emptyset$ to recover the natural map $\Delta_k(X; G) \to C_k(X; G)$. This natural map induces a map on homology $\mathrm{H}_k^{\Delta}(X, A; G) \to \mathrm{H}_k(X, A; G)$. Note that a Δ-complex X is certainly a CW-complex as well.

Theorem 4.2.37. *The homomorphisms $\mathrm{H}_k^{\Delta}(X, A; G) \to \mathrm{H}_k(X, A; G)$ are isomorphisms for all k and all Δ-complex pairs (X, A).*

Proof. We begin with the case that X is finite dimensional and A is empty. We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
\mathrm{H}_{n+1}^{\Delta}(X^k, X^{k11}) & \longrightarrow & \mathrm{H}_n^{\Delta}(X^{k-1}) & \longrightarrow & \mathrm{H}_n^{\Delta}(X^k) & \longrightarrow & \mathrm{H}_n^{\Delta}(X^k, X^{k11}) & \longrightarrow & \mathrm{H}_{n-1}^{\Delta}(X^{k-1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}_{n+1}(X^k, X^{k-1}) & \longrightarrow & \mathrm{H}_n(X^{k-1}) & \longrightarrow & \mathrm{H}_n(X^k) & \longrightarrow & \mathrm{H}_n(X^k, X^{k-1}) & \longrightarrow & \mathrm{H}_{n-1}(X^{k-1}).
\end{array}$$

Note that we drop the G from the notation here to save space, but the result follows with coefficients as well.

We have that $\Delta_n(X^k, X^{k-1}; G)$ is zero for $n \neq k$ and is a free abelian group generated by the k-simplices when $k = n$. Thus, we have that $\mathrm{H}_n^{\Delta}(X^k, X^{k-1}; G)$ is zero for $n \neq k$ and is free abelian generated by the k-cycles for $n = k$. Consider the map $\Psi : \coprod_i (\Delta_i^k, \partial\Delta_i^k) \longrightarrow (X^k, X^{k-1})$ given by the maps $\Delta^k \to X$ for each k-simplex. We have that Ψ gives a homeomorphism of the spaces $\coprod_i \Delta_i^k / \coprod_i \partial\Delta_i^k$ with X^k/X^{k-1}, and so induces an isomorphism of singular homology groups. This shows that $\mathrm{H}_n(X^k, X^{k-1}; G)$ is zero for $n \neq k$. For $n = k$ we have that it is free abelian with basis the relative cycles given by characteristic maps of the k-simplices of X since $\mathrm{H}_k(\Delta^k, \partial\Delta^k; G)$ is generated by the identity map $\Delta^k \longrightarrow \Delta^k$ (check this). Thus, we have the map $\mathrm{H}_k^{\Delta}(X^k, X^{k-1}; G) \longrightarrow \mathrm{H}_k(X^k, X^{k-1}; G)$ is an isomorphism. This gives that the first and fourth vertical maps in the commutative diagram above are isomorphisms. We may assume the second and fifth vertical maps are isomorphisms by induction, so the Five Lemma gives that the middle one is an isomorphism as well. This gives the result in this case.

Suppose now that X is infinite dimensional. One has that a compact set in X can only meet finitely many open simplices of X. One can see Appendix A of [5] for a proof of this fact. We will encounter it again when we continue our study of CW-complexes. In the case of a Δ-complex it is even easier and should be proved as an exercise. Let $[z] \in \mathrm{H}_n(X; G)$ with z a representative in $C_n(X; G)$. By definition this is a linear combination of finitely many singular simplices with compact images, so it must be contained in X^k for some k. Since we already have that $\mathrm{H}_n^{\Delta}(X^k; G) \to \mathrm{H}_n(X^k; G)$ is an isomorphism for all n, we have that z is homologous to a simplicial cycle. Thus, we have that the map $\mathrm{H}_n^{\Delta}(X; G) \to \mathrm{H}_n(X; G)$ is surjective. Now suppose that $[\sigma] \in \mathrm{H}_n^{\Delta}(X; G)$ is represented by $\sigma \in \Delta^n(X; G)$ and it is the boundary of a singular chain in X. We know this chain must have compact image, and so is contained in X^k for some

k. Thus, σ represents an element in the kernel of $\mathrm{H}_n^{\Delta}(X^k;G) \to \mathrm{H}_n(X^k;G)$, which is trivial. Thus, σ must be a simplicial boundary and we are done.

One can do the case when $A \neq \emptyset$ by the same method as above if one replaces the commutative diagram with the corresponding commutative diagram of relative homology and simplicial homology groups. □

An efficient way for computing the homology groups of a CW complex is via cellular homology, which we introduce after a few basic definitions and facts.

The following corollary of Proposition 4.2.22 will also be necessary in computing the homology of CW complexes.

Corollary 4.2.38. *Let X_k be a collection of topological spaces and $\vee_k X_k$ the wedge sum. The inclusion maps $i_k : X_k \longrightarrow \vee_k X_k$ induce an isomorphism*

$$\oplus_k (i_k)_* : \bigoplus_k \widetilde{\mathrm{H}}_n(X_k) \longrightarrow \widetilde{\mathrm{H}}_n(\vee_k X_k)$$

assuming that the wedge sum is formed with respect to base points x_k so that $(X_k, \{x_k\})$ is a good pair.

Proof. Consider the pair $(\coprod_k X_i, \coprod_k \{x_k\})$. We have that

$$\begin{aligned} \mathrm{H}_n\left(\coprod_k X_k, \coprod_k \{x_k\}\right) &\cong \bigoplus_k \mathrm{H}_n(X_k, \{x_k\}) \\ &\cong \bigoplus_k \widetilde{\mathrm{H}}_n(X_k). \end{aligned}$$

On the other hand, Proposition 4.2.22 gives that

$$\mathrm{H}_n\left(\coprod_k X_k, \coprod_k \{x_k\}\right) \cong \widetilde{\mathrm{H}}_n(\vee_k X_k).$$

Combining these gives the result. □

Lemma 4.2.39. *Let X be a CW complex.*

1. *$\mathrm{H}_n(X^k, X^{k-1}) = 0$ for $n \neq k$ and is free abelian for $n = k$ with a basis in one to one correspondence with the n-cells of X.*

2. *$\mathrm{H}_n(X^k) = 0$ for $n > k$. In particular, if X is finite dimensional then $\mathrm{H}_n(X) = 0$ for $n > \dim X$.*

3. *The inclusion map $i : X^k \to X$ induces an isomorphism $i_* : \mathrm{H}_n(X^k) \to \mathrm{H}_n(X)$ if $n < k$.*

Proof. 1. Observe that (X^k, X^{k-1}) is a good pair and X^k/X^{k-1} is a wedge sum of k-spheres. We now use Corollary 4.2.38 combined with our previous calculations of the homology of spheres to get the result.

2. Consider the long exact sequence arising from the pair (X^k, X^{k-1}). In this exact sequence we have

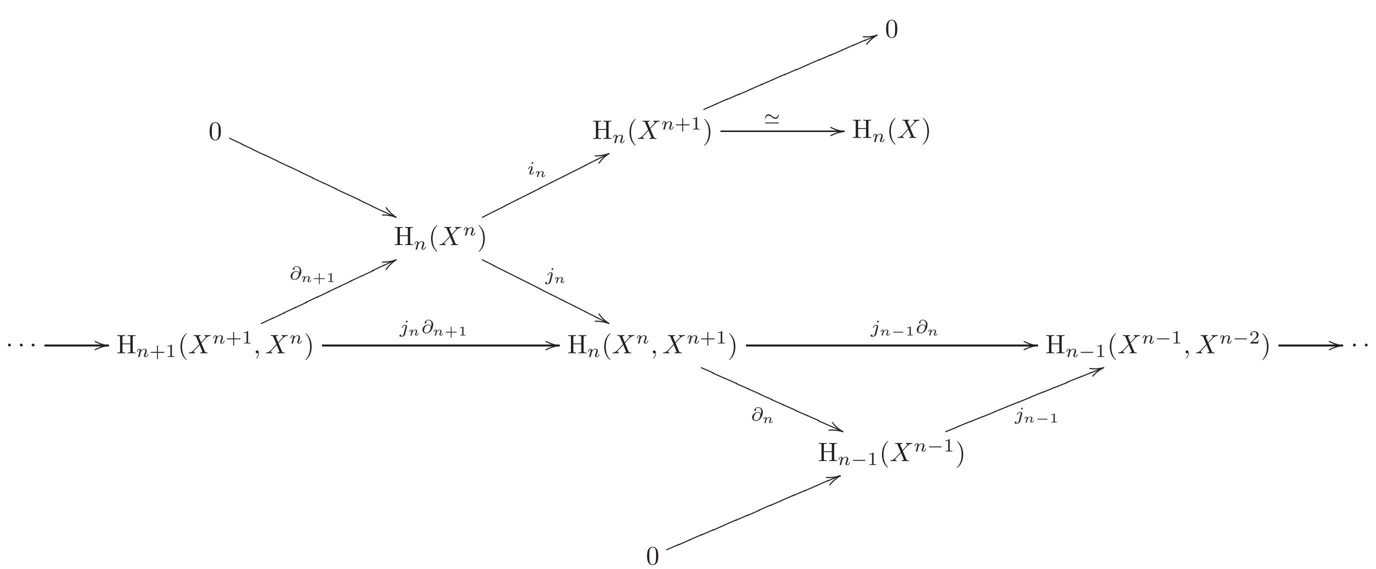
$$\mathrm{H}_{n+1}(X^k, X^{k-1}) \longrightarrow \mathrm{H}_n(X^{k-1}) \longrightarrow \mathrm{H}_n(X^k) \longrightarrow \mathrm{H}_n(X^k, X^{k-1}).$$

If $n \neq k, k-1$, then the outer two groups are 0 and so we have $\mathrm{H}_n(X^{k-1}) \cong \mathrm{H}_n(X^k)$ for all $n \neq k, k-1$. Thus, if $n > k$ then we have $\mathrm{H}_n(X^k) \cong \mathrm{H}_n(X^{k-1}) \cong \cdots \cong \mathrm{H}_n(X^1) \cong \mathrm{H}_n(X^0) = 0$. This proves the second part.

3. Note that we can use what we have just shown in the case that X is finite dimensional to conclude that if $n < k$ then $\mathrm{H}_n(X^k) \cong \mathrm{H}_n(X^{k+1}) \cong \cdots \cong \mathrm{H}_n(X^{k+m})$ for all $m \geq 0$, which gives the third result if X is finite dimensional. In the case that X is not finite dimensional, we use the same argument we've used before to observe that any singular n-chain in X has compact image so sits inside X^m for some m. Thus, a n-cycle in X is homologous to a cycle in X^k if $k > n$ by the finite dimensional case. Thus, the induced map $i_* : \mathrm{H}_n(X^k) \to \mathrm{H}_n(X)$ is surjective. Similarly, if a n-cycle in X^k bounds a chain in X, this chain lies in X^m for some m with $m \geq k$, so by the finite dimensional case the cycle bounds a chain in X^k if $k > n$.

□

We can now build the cellular chain complex which can be very useful in computing homology groups of CW complexes. Consider the following diagram:



Observe that the composition of two of the maps in the horizontal part of the diagram consists of at two successive maps in one of the diagonal exact sequences, so must be zero. Thus, the horizontal row gives a chain complex. This complex is known as the *cellular chain complex*. We denote the homology groups of this chain by $\mathrm{H}_n^{\mathrm{CW}}(X)$.

Theorem 4.2.40. *One has* $\mathrm{H}_n^{\mathrm{CW}}(X) \cong \mathrm{H}_n(X)$ *for all* n.

Proof. To ease the notation write d_n^{CW} for $j_{n-1} \circ \partial_n$. Observe that we have

$$\mathrm{H}_n(X) \cong \mathrm{H}_n(X^n)/\ker(i_n) \cong \mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}).$$

The fact that j_n is injective gives that $\operatorname{im}(\partial_{n+1})$ maps isomorphically onto $\operatorname{im}(j_n\partial_{n+1}) = \operatorname{im}(d_{n+1}^{\mathrm{CW}})$. Furthermore, we see that $\mathrm{H}_n(X^n)$ maps isomorphically onto $\operatorname{im}(j_n) = \ker(\partial_n)$. We have that j_{n-1} is injective, so $\ker(\partial_n) = \ker(d_n^{\mathrm{CW}})$. Thus, the map j_n induces an isomorphism between the quotient $\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1})$ and $\ker(d_n^{\mathrm{CW}})/\operatorname{im}(d_{n+1}^{\mathrm{CW}})$, which gives the result. □

Corollary 4.2.41. *If X is a CW complex with k n-cells, then $\mathrm{H}_n(X)$ is generated by at most k elements. In particular, if there are no n-cells in X then $\mathrm{H}_n(X) = 0$.*

Proof. We know that $\mathrm{H}_n(X^n, X^{n-1})$ is a free abelian group generated by the n-cells. Thus, we must have that the kernel of d_n^{CW} is generated by at most these k n-cells. Thus, $\mathrm{H}_n^{\mathrm{CW}}(X) \cong \mathrm{H}_n(X)$ is generated by at most k elements. □

Corollary 4.2.42. *Let X be a CW complex that has no two of its cells in adjacent dimensions. Then $\mathrm{H}_n(X)$ is free abelian with basis in one to one correspondence with the n-cells of X. In particular, this gives*

$$\mathrm{H}_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & \text{for } k = 0, 2, 4, \cdots, 2n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is clear because in this case all of the maps d_n^{CW} must be 0. □

Another nice application we have is to the Euler characteristic of a finite CW complex. Define the Euler characteristic of a finite CW complex X to be

$$\chi(X) = \sum_n (-1)^n c_n$$

where c_n is the number of n-cells of X. Note that this is a generalization of the familiar formula for 2-complexes that reads $\chi(X)$ is the number of vertices - number of edges + number of faces. In fact, we can now see that $\chi(X)$ can be defined completely in terms of the homology of X and so is a homotopy invariant of the space.

Theorem 4.2.43. *For X a finite CW complex one has*

$$\chi(X) = \sum_n (-1)^n \operatorname{rank}(\mathrm{H}_n(X)).$$

Proof. This result is a purely algebraic result. Let

$$0 \longrightarrow C_k \xrightarrow{d_k} C_{k-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \longrightarrow 0$$

be a descending chain complex of finitely generated abelian groups. Defining Z_n, B_n, and $H_n(C_*)$ as always, we have short exact sequences

$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0$$

and

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow \mathrm{H}_n(C_*) \longrightarrow 0.$$

We now apply the fact that if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of finitely generated abelian groups, then $\mathrm{rank}(B) = \mathrm{rank}(A) + \mathrm{rank}(C)$ to obtain that

$$\begin{aligned}\mathrm{rank}(C_n) &= \mathrm{rank}(Z_n) + \mathrm{rank}(B_{n-1}) \\ &= \mathrm{rank}(B_n) + \mathrm{rank}(\mathrm{H}_n(C_*)) + \mathrm{rank}(B_{n-1}).\end{aligned}$$

If we multiply this equation by $(-1)^n$ and sum over all n we obtain

$$\sum_n (-1)^n \,\mathrm{rank}(C_n) = \sum_n (-1)^n \,\mathrm{rank}(\mathrm{H}_n(C_*)).$$

Now just apply this purely algebraic result to the complex $C_n = \mathrm{H}_n(X^n, X^{n-1})$ to get the result. □

4.3 The Universal Coefficient Theorem

In § 4.2 we saw that working with homology with coefficients in an arbitrary abelian group did not add any difficulties in proving theorems. It can be the case that using arbitrary coefficients can make working examples a little trickier, but essentially it did not appear anything new was gained or lost by switching from G to $\mathbb{Z}$ or vice versa. In this section we justify that statement by relating the homology groups $\mathrm{H}_k(X, A; G)$ to $\mathrm{H}_k(X, A) \otimes G$. Note when a subscript is not added to the tensor product it is understood to be a tensor product over $\mathbb{Z}$. One of the main reasons for interest in such a result for us is to prepare us for the same result when we introduce cohomology. In that context a universal coefficient theorem will be important when we compare different cohomology theories.

Let X be a topological space and G an abelian group. Recall that $C_n(X; G)$ is the free abelian group over G generated by the chains $\sigma : \Delta^n \to X$. In particular, given a chain $\tau \in C_n(X; G)$, we can write $\tau = \sum_i g_i \sigma_i$ for $\sigma_i : \Delta^n \to X$ singular n-simplices. From this it is clear that we have

$$C_n(X; G) \cong C_n(X) \otimes G$$

where the map is given by $\sum_i g_i\sigma_i$ maps to $\sum_i \sigma_i \otimes g_i$. From this we see that we can write our chain complex

$$\cdots \longrightarrow C_n(X;G) \xrightarrow{d_n} C_{n-1}(X;G) \xrightarrow{d_{n-1}} C_{n-2}(X;G) \longrightarrow \cdots$$

as

$$\cdots \longrightarrow C_n(X) \otimes G \xrightarrow{d_n\otimes 1} C_{n-1}(X) \otimes G \xrightarrow{d_{n-1}\otimes 1} C_{n-2}(X) \otimes G \longrightarrow \cdots .$$

Thus, the homology groups we defined relative to $C_n(X;G)$ can also be given as the homology groups of this chain complex, namely,

$$\mathrm{H}_n(X;G) = \mathrm{H}_n(C_*(X) \otimes G).$$

The group G is referred to as the *coefficients* of the homology group. We can view the relative homology groups in this way as well, in particular, $C_n(X,A;G) \cong C_n(X,A) \otimes G$.

Before we relate $\mathrm{H}_n(X,A;G)$ to $\mathrm{H}_n(X,A) \otimes G$, we consider the induced homomorphisms $\mathrm{H}_n(X,A;G) \to \mathrm{H}_n(X,A;H)$ when we have a homomorphism between the groups G and H. Let $\phi : G \to H$ be a group homomorphism. Clearly, we have an induced homomorphism on chains given by

$$1 \otimes \phi : C_n(X,A) \otimes G \longrightarrow C_n(X,A) \otimes H.$$

These maps descend to maps on homology, though to avoid confusion we add some details here. (Note that we do not have that $\mathrm{H}_n(X,A;G) \cong \mathrm{H}_n(X,A)\otimes G$!) Let $[\tau] \in \mathrm{H}_n(X,A;G)$ and let $\tau = \sum_i \sigma_i \otimes g_i \in C_n(X,A)\otimes G$ be a representative of $[\tau]$. We have that $(1\otimes\phi)(\tau) = \sum_i \sigma_i \otimes \phi(g_i) \in C_n(X,A) \otimes H$. It is therefore natural to define

$$\begin{aligned} \phi_\sharp : \mathrm{H}_n(X,A;G) &\to \mathrm{H}_n(X,A;H) \\ [\tau] &\mapsto [(1\otimes\phi)(\tau)]. \end{aligned}$$

There are two things to check. The first is that the image of this map is actually a cycle, and the second is that it does not depend on the choice of representative for $[\tau]$. To see the image is a cycle, observe that

$$\begin{aligned} (d_n \otimes 1)\left(\sum_i \sigma_i \otimes \phi(g_i)\right) &= \sum_i d_n(\sigma_i) \otimes \phi(g_i) \\ &= (1\otimes\phi)\left(\sum_i d_n(\sigma_i) \otimes g_i\right) \\ &= (1\otimes\phi)(0) \\ &= 0 \end{aligned}$$

where we have used that $\sum_i d_n(\sigma_i) \otimes g_i = 0$ because τ is a cycle and that ϕ is a homomorphism so takes 0 to 0. Thus, we have that the image lies in $\mathrm{H}_n(X,A;H)$.

Exercise 4.3.1. Check that if one chooses different representatives of $[\tau]$ that they are mapped to the same thing in $\mathrm{H}_n(X, A; H)$.

For this induced map to be useful it is important that the map is "natural", i.e., it commutes with the induced maps we have already defined. Namely, given a continuous map $f : (X, A) \to (Y, B)$ and a group homomorphism $\phi : G \to H$, it is important that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(X, A; G) & \xrightarrow{f_*} & \mathrm{H}_n(Y, B; G) \\ \downarrow{\scriptstyle \phi_\sharp} & & \downarrow{\scriptstyle \phi_\sharp} \\ \mathrm{H}_n(X, A; H) & \xrightarrow{f_*} & \mathrm{H}_n(Y, B; H). \end{array}$$

However, this follows immediately from the definitions. It is also important that the induced map $\phi_\sharp$ behaves well with respect to the boundary homomorphism ∂_n. In particular, we have that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(X, A; G) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A; G) \\ \downarrow{\scriptstyle \phi_\sharp} & & \downarrow{\scriptstyle \phi_\sharp} \\ \mathrm{H}_n(X, A; H) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A; H). \end{array}$$

Again, one only needs to write down the definitions of the maps to see that this is true.

This induced map has many important uses in homology theory. For example, let R be a ring and suppose that G is a R-module. We write the action of R on G as left multiplication. Thus, for each $r \in R$ we have an endomorphism of G given by $g \mapsto rg$. Using what we have just shown, each $r \in R$ induces a map $r_\sharp$ of $\mathrm{H}_n(X, A; G)$.

Exercise 4.3.2. Show that the induced maps $r_\sharp$ give the group $\mathrm{H}_n(X, A; G)$ the structure of a R-module. Furthermore, show that the commutative diagrams above show that f_* and ∂_n are homomorphisms of R-modules.

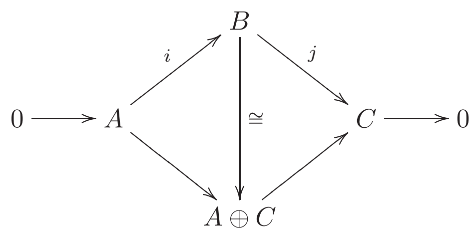
A particularly nice case is when R is a field and G is a vector space over R. In this case one has that $\mathrm{H}_n(X, A; G)$ is a vector space over R as well and the maps f_* and ∂_n are linear transformations. What makes this situation particularly nice is that one can then use all the tools of linear algebra to study the homology groups.

We now return to studying the relation between $\mathrm{H}_n(X, A; G)$ and $\mathrm{H}_n(X, A) \otimes G$. In order to do this we introduce some definitions and results from general homological algebra.

Definition 4.3.3. Let A, B, and C be abelian groups with

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

an exact sequence. We say the exact sequence *splits* if there is an isomorphism $B \cong A \oplus C$ that makes the following diagram commute
where the maps $A \to A \oplus C$ is the natural inclusion map and the map $A \oplus C \to C$



is the natural projection map.

Exercise 4.3.4. Show that if A, B, and C are free abelian groups then every short exact sequence is split. In fact, given a short exact sequence, show it is enough for C to be free for the short exact sequence to split.

Lemma 4.3.5. *Let A, B, and C be abelian groups and*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

a split short exact sequence. Given any abelian group G, the sequence

$$0 \longrightarrow A \otimes G \xrightarrow{i \otimes 1} B \otimes G \xrightarrow{j \otimes 1} C \otimes G \longrightarrow 0$$

is a split exact sequence.

Proof. First, observe that the tensor product operation is right exact, namely, we get that

$$A \otimes G \xrightarrow{i \otimes 1} B \otimes G \xrightarrow{j \otimes 1} C \otimes G \longrightarrow 0$$

is exact for any exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0.$$

To get the injectivity of the first map, observe that

$$(A \oplus C) \otimes G \cong (A \otimes G) \oplus (C \otimes G)$$

and so clearly the inclusion map

$$A \otimes G \to (A \otimes G) \oplus (C \otimes G)$$

is injective. This gives the injectivity since the sequence is split. It is also clear now that the sequence

$$0 \longrightarrow A \otimes G \xrightarrow{i \otimes 1} B \otimes G \xrightarrow{j \otimes 1} C \otimes G \longrightarrow 0$$

is split. □

Let $\{C_n, d_n\}$ be a descending chain complex of free abelian groups, i.e., a chain complex given as

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

with each C_n a free abelian group. Set $Z_n = \ker(d_n) \subset C_n$ and $B_n = \operatorname{im}(d_{n+1}) \subset C_n$. We can define the homology group of C_n as

$$\mathrm{H}_n(C_*) = Z_n/B_n.$$

We have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{d_n} & B_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_{n-1}} & & \\
0 & \longrightarrow & Z_{n-1} & \longrightarrow & C_{n-1} & \xrightarrow{d_{n-1}} & B_{n-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where each horizontal row is exact. Observe that each row is split because B_n is free as it is a subgroup of the free abelian group C_n. Let G be any abelian group. Lemma 4.3.5 combined with the above commutative diagram gives us an exact sequence of chain complexes

$$0 \longrightarrow Z_* \otimes G \longrightarrow C_* \otimes G \longrightarrow B_{*-1} \otimes G \longrightarrow 0.$$

This gives a long exact sequence of homology groups

$$\cdots \longrightarrow \mathrm{H}_{n+1}(B_* \otimes G) \longrightarrow \mathrm{H}_n(Z_* \otimes G) \longrightarrow \mathrm{H}_n(C_* \otimes G) \longrightarrow \mathrm{H}_n(B_* \otimes G) \longrightarrow \cdots .$$

However, we know that $d_n \otimes 1$ maps all elements of $Z_n \otimes G$ and $B_n \otimes G$ to 0, so $\mathrm{H}_n(Z_* \otimes G) = Z_n \otimes G$ and $\mathrm{H}_n(B_* \otimes G) = B_{n-1} \otimes G$. Thus, the long exact sequence becomes

$$\cdots \longrightarrow B_n \otimes G \longrightarrow Z_n \otimes G \longrightarrow \mathrm{H}_n(C_* \otimes G) \longrightarrow B_{n-1} \otimes G \longrightarrow \cdots . \tag{4.1}$$

The connecting homomorphism $\partial_{n+1} : \mathrm{H}_{n+1}(B_* \otimes G) \to \mathrm{H}_n(Z_* \otimes G)$ is the map $i_n \otimes 1 : B_n \otimes G \to Z_n \otimes G$ where i_n is the inclusion map. One should check this as an exercise. We break up the long exact sequence (4.1) into short exact sequences

$$0 \longrightarrow \operatorname{coker}(i_n \otimes 1) \longrightarrow \mathrm{H}_n(C_* \otimes G) \longrightarrow \ker(i_{n-1} \otimes 1) \longrightarrow 0. \tag{4.2}$$

Consider the short exact sequence defining $\mathrm{H}_n(C_*)$, namely,

$$0 \longrightarrow B_n \xrightarrow{i_n} Z_n \xrightarrow{j_n} \mathrm{H}_n(C_*) \longrightarrow 0.$$

As was stated above, it is a fact from commutative algebra that tensoring with G is right exact so we have the exact sequence

$$B_n \otimes G \xrightarrow{i_n \otimes 1} Z_n \otimes G \xrightarrow{j_n \otimes 1} \mathrm{H}_n(C_*) \otimes G \longrightarrow 0.$$

In particular, we see that

$$\begin{aligned} \mathrm{H}_n(C_*) \otimes G &\cong (Z_n \otimes G)/\ker(j_n \otimes 1) \\ &\cong (Z_n \otimes G)/\operatorname{im}(i_n \otimes 1) \\ &= \operatorname{coker}(i_n \otimes 1). \end{aligned}$$

Thus, we can rewrite the short exact sequence (4.2) as

$$0 \longrightarrow \mathrm{H}_n(C_*) \otimes G \longrightarrow \mathrm{H}_n(C_* \otimes G) \longrightarrow \ker(i_{n-1} \otimes 1) \longrightarrow 0.$$

It remains to study $\ker(i_{n-1} \otimes 1)$. This requires a little more background and work.

Definition 4.3.6. Let H be an abelian group. A *free resolution* $F_* = \{F_i, f_i\}$ of H is an exact sequence

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0$$

where each F_i is a free abelian group.

We know that if we tensor the free resolution of H with an abelian group G the resulting sequence is not necessarily exact. We define

$$\mathrm{H}_n(F_* \otimes G) = \ker(f_n \otimes 1)/\operatorname{im}(f_{n+1} \otimes 1).$$

This group only depends upon G and H and not the free resolution F_* used as we will see in Lemma 4.3.8. First, we need the following result on chain maps of free resolutions.

Lemma 4.3.7. *Let E_* a free resolution of an abelian group G and F_* be a free resolution of an abelian group H. Then every homomorphism $\phi : G \to H$ extends to a chain map from E_* to F_*, i.e., the following diagram commutes*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & E_2 & \xrightarrow{e_2} & E_1 & \xrightarrow{e_1} & E_0 & \xrightarrow{e_0} & G & \longrightarrow 0 \\
 & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi} & \\
\cdots \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow 0.
\end{array}$$

Moreover, any two such chain maps extending ϕ are chain homotopic.

Proof. We construct the maps ϕ_i inductively. It is enough to define ϕ_i on a basis of E_i since E_i is free. Let $x \in E_0$ be a basis element. Since f_0 is surjective, there exists a $y \in F_0$ so that $f_0(y) = \phi(e_0(x))$. Define $\phi_1(x) = y$. This defines ϕ_0. We would like to define ϕ_1 in the same manner. Let x now be a basis element in E_1. We want to find $y \in F_1$ so that $f_1(y) = \phi_0(e_1(x))$. We see that such a y will exist if $\phi_0(e_1(x)) \in \operatorname{im}(f_1) = \ker(f_0)$. However, we have that $\phi_0(e_1(x)) \in \ker(f_0)$ because $\phi_0(e_1(X)) = \phi(e_0(e_1(x))) = \phi(0) = 0$. Thus, we can define ϕ_1 as desired. The rest of the maps are defined in the same manner.

Now suppose there is another chain map $\phi' : E_* \to F_*$ extending ϕ. Consider the maps $\psi_i : E_i \to F_i$ defined by $\psi_i = \phi_i - \phi'_i$. This is a chain map extending the zero map from G to H. It is enough to show that ψ_i is chain homotopic to 0, i.e., to construct maps $T_i : E_i \to F_{i+1}$ so that $\psi_i = f_{i+1}T_i + T_{i-1}e_i$. We construct the maps T_i inductively much as the maps ϕ_i were constructed. For $i = 0$, set $T_{-1} : G \to F_0$ to be the zero map. The relation we need in this case is $\psi_0 = f_1T_0$. Define T_0 as follows. Let $x \in E_0$ a basis element. Note that there is a $y \in F_1$ so that $f_1(y) = \psi_0(x)$ because $\operatorname{im}(f_1) = \ker(f_0)$ and $f_0(\psi_0(x)) = \psi(e_0(x)) = 0$ since ψ is the zero map. Thus, we can define $T_0(x) = y$ and this gives the desired relation. Now we show the inductive step. We need to define T_i so that it takes a basis element $x \in E_i$ to a basis element $y \in F_{i+1}$so that $f_{i+1}(y) = \psi_i(x) - T_{i-1}e_i(x)$. This is possible if $\psi_i(x) - T_{i-1}e_i(x)$ lies in $\operatorname{im}(f_{i+1}) = \ker(f_i)$, i.e, if $f_i(\psi_i - T_{i-1}e_i) = 0$. However, using the relations $f_i\psi_i = \psi_{i-1}e_i$ and $\psi_{i-1} = f_iT_{i-1} + T_{i-2}e_{i-1}$ (which holds by our induction hypothesis), we have

$$\begin{aligned}
f_i(\psi_i - T_{i-1}e_i) &= f_i\psi_i - f_iT_{i-1}e_i \\
&= \psi_{i-1}e_i - f_iT_{i-1}e_i \\
&= (\psi_{i-1} - f_iT_{i-1})e_i \\
&= T_{i-1}e_{i-1}e_i \\
&= 0.
\end{aligned}$$

Thus, we have the result. □

Lemma 4.3.8. *Let E_* and F_* be two free resolutions of H. For each n there is a canonical isomorphism*

$$\mathrm{H}_n(E_* \otimes G) \cong \mathrm{H}_n(F_* \otimes G).$$

Proof. Let E_* and F_* be two free resolutions of H. We tensor these free resolutions with G to obtain chain complexes $E_* \otimes G$ and $F_* \otimes G$ with maps $\phi_n \otimes 1$

giving the chain map between them. These maps descend to maps on homology

$$\phi_* : \mathrm{H}_n(E_* \otimes G) \to \mathrm{H}_n(F_* \otimes G).$$

Note that the maps on homology are independent of the chain maps ϕ_* as if one has a different sent of chain maps, we have that they are chain homotopic, which in homology means they agree.

Suppose that we have a composition $H_1 \xrightarrow{\phi} H_2 \xrightarrow{\psi} H_3$ with free resolutions F_*, F'_*, and F''_* respectively. The induced maps on homology satisfy $(\psi\phi)_* = \psi_*\phi_*$. One can see this by choosing the chain map $F_* \to F''_*$ to be the composition of chain maps $F_* \to F'_* \to F''_*$. Now if ϕ is an isomorphism and ψ the inverse of ϕ, then we have $\psi_*\phi_* = (\psi\phi)_* = 1_* = 1$ and likewise for $\phi_*\psi_*$. Now take ϕ to be the identity with two different free resolutions. Then we obtain a canonical isomorphism

$$1_* : \mathrm{H}_n(E_* \otimes G) \to \mathrm{H}_n(F_* \otimes G).$$

□

Since the group $\mathrm{H}_n(F_* \otimes G)$ depends only on H and G, we denote it as $\mathrm{Tor}_n(H, G)$. One always has a free resolution given by

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow H \longrightarrow 0$$

for any abelian group G. In particular, choose a set of generators for H and let F_0 be the free abelian group with basis in one to one correspondence with the generators of H. Then we have a surjective map $F_0 \to H$. Let F_1 be the kernel of this map. We have that F_1 is free abelian since it is a subgroup of F_0, which gives the resolution. This free resolution shows that $\mathrm{Tor}_n(H, G) = 0$ for $n > 1$. Thus, we write $\mathrm{Tor}(H, G)$ for $\mathrm{Tor}_1(H, G)$ as it is the group of interest. In fact, $\mathrm{Tor}(H, G)$ measures the common torsion of G and H, which is where the "Tor" comes from.

Note that in this current definition, we have that $\mathrm{Tor}_0(H, G) = 0$ because tensoring is right exact. This is not entirely what we would like, so instead we set $\mathrm{Tor}_n(H, G)$ to be the homology groups of the sequence

$$\cdots \longrightarrow F_2 \otimes G \longrightarrow F_1 \otimes G \longrightarrow F_0 \otimes G \longrightarrow 0,$$

where we remove the $H \otimes G$ term. This does not effect the groups $\mathrm{Tor}_n(H, G)$ for $n \geq 1$, but it sets $\mathrm{Tor}_0(H, G) = H \otimes G$, which is more suitable for our theory.

We have the following theorem, known as the universal coefficient theorem for homology.

Theorem 4.3.9. *Let C_* be a chain complex of free abelian groups. Then for all $n \geq 0$ there are short exact sequences*

$$0 \longrightarrow \mathrm{H}_n(C_*) \otimes G \longrightarrow \mathrm{H}_n(C_* \otimes G) \longrightarrow \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G) \longrightarrow 0$$

that are split. Moreover, the maps in the short exact sequence are natural with respect to chain maps and coefficient homomorphisms, i.e., given a chain map $C_* \to D_*$, the chain map induces a map between the short exact sequences with commuting squares and given a homomorphism $\phi : G \to H$ between abelian groups, ϕ induces a chain map between the short exact sequences with commuting squares.

Proof. Note that we have already proven that we have such an exact sequence, it is simply the short exact sequence (4.2) combined with the fact that $\text{coker}(i_n \otimes 1) \cong \text{H}_n(C_*) \otimes G$ and $\ker(i_{n-1} \otimes 1) \cong \text{Tor}(\text{H}_{n-1}(C_*), G)$. Checking that the maps are natural is just a matter of running through the definitions.

It remains to check that the short exact sequences split. Recall that we have already seen that the short exact sequence

$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0$$

splits. In particular, this implies there is a projection map $p : C_n \to Z_n$ that restricts to the identity on Z_n. Furthermore, we can use p to give an extension of the quotient map $Z_n \to \text{H}_n(C_*)$ to a map $C_n \to \text{H}_n(C_*)$. Thus, as we vary n we obtain a chain map from the chain C_* to the chain of homology groups $\text{H}_*(C_*)$ where we regard $\text{H}_*(C_*)$ as a chain with trivial boundary maps. Tensoring this with G we obtain a chain map $C_* \otimes G \to \text{H}_*(C_*) \otimes G$. We now take homology groups and using that $\text{H}_*(C_*)$ is a chain with trivial boundary maps we obtain the induced homomorphisms

$$\text{H}_n(C_*; G) \to \text{H}_n(C) \otimes G.$$

These homomorphisms give the splitting when combined with the following exercise and the fact that these homomorphisms are trivial on cycles by the definition of p. ☐

Exercise 4.3.10. Let A, B, and C be abelian groups and

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

be an exact sequence.

1. Show that the exact sequence is split if and only if there is a homomorphism $p : B \to A$ so that $p \circ i : A \to A$ is the identity map.
2. Show that the exact sequence is split if and only if there is a homomorphism $s : C \to B$ so that $j \circ s : C \to C$ is the identity map.

Corollary 4.3.11. *Let X be a topological space, A a subspace of X, and G an abelian group. Then for each $n \geq 0$ there is a split exact sequence*

$$0 \longrightarrow \text{H}_n(X, A) \otimes G \longrightarrow \text{H}_n(X, A; G) \longrightarrow \text{Tor}(\text{H}_{n-1}(X, A), G) \longrightarrow 0.$$

Moreover, these sequences are natural with respect to maps $(X, A) \to (Y, B)$ and coefficient homomorphisms $\phi : G \to H$.

For this corollary to be useful, it is important that we can actually calculate the groups $\mathrm{Tor}(\mathrm{H}_{n-1}(X,A),G)$ for abelian groups G. We have the following calculations.

Proposition 4.3.12. *Let A, B, C, D, and A_i for $i \in I$ be abelian groups.*

1. $\mathrm{Tor}\left(\bigoplus_i A_i, B\right) \cong \bigoplus_i \mathrm{Tor}(A_i, B)$.
2. $\mathrm{Tor}\left(\mathbb{Z}/n\mathbb{Z}, A\right) \cong \ker(A \xrightarrow{n} A)$.
3. $\mathrm{Tor}(A,B) = 0$ *if A or B is torsion-free.*
4. *For each short exact sequence*

$$0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0,$$

there is naturally associated an exact sequence

$$0 \longrightarrow \mathrm{Tor}(A,B) \longrightarrow \mathrm{Tor}(A,C) \longrightarrow \mathrm{Tor}(A,D) \longrightarrow A\otimes B \longrightarrow A\otimes C \longrightarrow A\otimes D \longrightarrow 0.$$

5. $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$.
6. $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(T(A),B)$ *where $T(A)$ is the torsion subgroup of A.*

Proof.

1. We can choose a free resolution of $\bigoplus_i A_i$ to be a direct sum of free resolutions of the A_i, which gives this result immediately.
2. Consider the free resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}/n\mathbb{Z}$. If we tensor this with A we obtain

$$0 \longrightarrow A \xrightarrow{n} A \longrightarrow \mathbb{Z}/n\mathbb{Z} \otimes A \longrightarrow 0.$$

Thus, we have that $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, A) = \ker(A \xrightarrow{n} A)$ as claimed.

3. We prove the third statement here in the case that A or B is free and then return to prove the general case momentarily. Assuming A is free, it has a free resolution with $F_i = 0$ for $i \geq 0$ and so $\mathrm{Tor}(A,B) = 0$ for all B. If B is free, then tensoring a free resolution of A with B preserves exactness and so $\mathrm{Tor}(A,B) = 0$ in this case as well.
4. Let

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

be a free resolution of A. We tensor with the given short exact sequence to obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_1 \otimes B & \longrightarrow & F_1 \otimes C & \longrightarrow & F_1 \otimes D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0 \otimes B & \longrightarrow & F_0 \otimes C & \longrightarrow & F_0 \otimes D & \longrightarrow & 0.
\end{array}$$

We have that the rows are exact because tensoring an exact sequence with a free group preserves exactness. We can now extend this diagram to the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 \otimes B & \longrightarrow & F_1 \otimes C & \longrightarrow & F_1 \otimes D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0 \otimes B & \longrightarrow & F_0 \otimes C & \longrightarrow & F_0 \otimes D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

This gives a short exact sequence of chain complexes. The associated long exact sequence is the desired sequence.

5. We now apply the exact sequence just shown to the free resolution

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow B \longrightarrow 0.$$

We know that $\mathrm{Tor}(A, F_1)$ and $\mathrm{Tor}(A, F_0)$ both vanish because F_1 and F_0 are free, so the sequence reduces to

$$0 \longrightarrow \mathrm{Tor}(A, B) \longrightarrow A \otimes F_1 \longrightarrow A \otimes F_0 \longrightarrow A \otimes B \longrightarrow 0.$$

We can combine this with the definition of $\mathrm{Tor}(B, A)$ to obtain the diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{Tor}(A, B) & \longrightarrow & A \otimes F_1 & \longrightarrow & A \otimes F_0 & \longrightarrow & A \otimes B & \longrightarrow & 0 \\
 & & & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
0 & \longrightarrow & \mathrm{Tor}(B, A) & \longrightarrow & F_1 \otimes A & \longrightarrow & F_0 \otimes A & \longrightarrow & B \otimes A & \longrightarrow & 0.
\end{array}$$

Since all of the squares commute, we have that the induced map from $\mathrm{Tor}(A, B)$ to $\mathrm{Tor}(B, A)$ is an isomorphism by the five lemma.

We also are now able to prove the rest of 3 in the torsion free case. Let

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

be a free resolution of A. Suppose that B is torsion free and $\sum x_i \otimes b_i$ lies in the kernel of $f_1 \otimes 1 : F_1 \otimes B \to F_0 \otimes B$. Then we have that $\sum f_1(x_i) \otimes b_i$ can be reduced to 0 after a finite number of applications of the defining relations for a tensor product. Only a finite number of elements of B are involved in this process. These lie in a finitely generated subgroup $B_0 \subset B$, so $\sum x_i \otimes b_i$ lies in the kernel of $f_1 \otimes 1 : F_1 \otimes B_0 \to F_1 \otimes B_0$. This kernel must be 0 because $\mathrm{Tor}(A, B_0) = 0$ as B_0 is finitely generated and torsion free, hence free. Thus, we have $\mathrm{Tor}(A, B) = 0$ as claimed.

6. To obtain this last statement we just apply the six term exact sequence already shown to the short exact sequence

$$0 \longrightarrow T(A) \longrightarrow A \longrightarrow A/T(A) \longrightarrow 0$$

and use the fact that $A/T(A)$ is torsion free.

□

Exercise 4.3.13. Show that $\mathrm{Tor}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z}$. Use this to show that for finitely generated A and B, $\mathrm{Tor}(A, B)$ is the tensor product of the torsion subgroups of A and B and so is the common torsion of the groups.

As vector spaces are often easier to work with than modules, it is often easier to calculate the homology of a space with coefficients in a field. For example, it is usually easier to work with coefficients in $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ than with coefficients in $\mathbb{Z}$. We often lose information in doing this, but sometimes what remains is still enough to work with.

Corollary 4.3.14. *We have* $\mathrm{H}_n(X; \mathbb{Q}) \cong \mathrm{H}_n(X) \otimes \mathbb{Q}$. *Thus, if* $\mathrm{H}_n(X)$ *is finitely generated, the dimension of* $\mathrm{H}_n(X; \mathbb{Q})$ *as a* $\mathbb{Q}$*-vector space over* $\mathbb{Q}$ *equals the rank of* $\mathrm{H}_n(X)$.

Proof. This follows immediately from Corollary 4.3.11 and the fact that $\mathbb{Q}$ is torsion free. □

Corollary 4.3.15. *Let* p *be a prime and assume that* $\mathrm{H}_n(X)$ *and* $\mathrm{H}_{n-1}(X)$ *are finitely generated. Then we have that* $\mathrm{H}_n(X; \mathbb{Z}/p\mathbb{Z})$ *consists of*

1. *a* $\mathbb{Z}/p\mathbb{Z}$ *summand for each* $\mathbb{Z}$ *summand of* $\mathrm{H}_n(X)$*;*

2. *a* $\mathbb{Z}/p\mathbb{Z}$ *summand for each* $\mathbb{Z}/p^k\mathbb{Z}$ *summand of* $\mathrm{H}_n(X)$ *with* $k \geq 1$*;*

3. *a* $\mathbb{Z}/p\mathbb{Z}$ *summand for each* $\mathbb{Z}/p^k\mathbb{Z}$ *summand of* $\mathrm{H}_{n-1}(X)$ *with* $k \geq 1$.

Proof. This amounts to just writing down the exact sequence in Corollary 4.3.11.

□

We also have the following local to global type result for checking the vanishing of a homology group.

Proposition 4.3.16. *We have* $\widetilde{\mathrm{H}}_n(X) = 0$ *if and only if* $\widetilde{\mathrm{H}}_n(X; \mathbb{Q}) = 0$ *and* $\widetilde{\mathrm{H}}_n(X; \mathbb{Z}/p\mathbb{Z}) = 0$ *for all* n *and all primes* p.

Proof. We know that if $\widetilde{\mathrm{H}}_n(X) = 0$ for all n then the homology groups $\widetilde{\mathrm{H}}_n(X;\mathbb{Q})$ and $\widetilde{\mathrm{H}}_n(X;\mathbb{Z}/p\mathbb{Z})$ vanish for all n and all primes p by Corollary 4.3.11. For the other direction, it is enough to show that if A is an abelian group and $A\otimes\mathbb{Q} = 0$ and $\mathrm{Tor}(A,\mathbb{Z}/p\mathbb{Z}) = 0$ for all primes p, then $A = 0$. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0.$$

The six term exact sequence of Proposition 4.3.12 gives

$$0 \longrightarrow \mathrm{Tor}(A,\mathbb{Z}/p\mathbb{Z}) \longrightarrow A \xrightarrow{p} A \longrightarrow A\otimes\mathbb{Z}/p\mathbb{Z} \longrightarrow 0.$$

If $\mathrm{Tor}(A,\mathbb{Z}/p\mathbb{Z}) = 0$ for all p, we see that the map on A given by multiplication by p is injective for all p. This shows that A must be torsion free.

Similarly, the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

yields the exact sequence

$$0 \longrightarrow \mathrm{Tor}(A,\mathbb{Q}/\mathbb{Z}) \longrightarrow A \longrightarrow A\otimes\mathbb{Q} \longrightarrow A\otimes\mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Since A is torsion free, we must have $\mathrm{Tor}(A,\mathbb{Q}/\mathbb{Z}) = 0$ by Proposition 4.3.12, and so we have that the map $A \to A\otimes\mathbb{Q}$ is injective, hence $A = 0$.

□

4.4 Singular Cohomology

In defining relative homology with coefficients in a group G of a space X with respect to a subspace A there were three steps that were taken. The first was that we formed a chain complex $C_* = \{C_n(X,A), d_n\}$. The second step was to tensor this chain complex with the group G to obtain a new chain complex $C_*(X,A;G) = C_*(X,A)\otimes G$. Finally, we took the homology groups of this chain complex. In order to define cohomology groups, we replace tensoring with G in the second step by applying the functor $\mathrm{Hom}(\,,G)$ instead.

Let X be a topological space and G an abelian group. Define $C^n(X,A;G) = \mathrm{Hom}(C_n(X,A),G)$. The elements of $C^n(X,A;G)$ are referred to as the *singular n-cochains with coefficients in G*. Note that $\phi \in C^n(X,A;G)$ is a map that sends each chain $\sigma : \Delta^n \to X$ to a point in G. The *coboundary maps* $d^n : C^n(X,A;G) \to C^{n+1}(X,A;G)$ are the dual maps to the boundary maps d_{n+1}, i.e., they are the composition $d^n(\phi) = \phi\circ d_{n+1}$. In particular, given $\phi \in C^n(X,A;G)$, we define

$$d^n(\phi)(\sigma) = \sum_i (-1)^i \phi(\sigma|_{[v_0,\ldots,\widehat{v}_i,\ldots,v_{n+1}]})$$

for $\sigma \in C_{n+1}(X,A;G)$. We immediately get that $d^{n+1}\circ d^n = 0$ because they are the dual maps to the maps d_*. Thus, we obtain a chain complex

$$0 \longrightarrow C^0(X,A;G) \longrightarrow \cdots \longrightarrow C^n(X,A;G) \xrightarrow{d^n} C^{n+1}(X,A;G) \longrightarrow \cdots.$$

From this we can form the singular cohomology groups

$$\mathrm{H}^n(X, A; G) = \mathrm{H}^n(C^*(X, A; G)).$$

Exercise 4.4.1. Show that $\mathrm{H}^0(X, A; G) \cong \mathrm{Hom}(\mathrm{H}_0(X, A), G)$.

Let $f : (X, A) \to (Y, B)$ be a continuous map. Recall we have an induced map

$$f_* : C_n(X, A; G) \to C_n(Y, B; G)$$

given by composition, namely, for $\sigma : \Delta^n \to X$ a chain in $C_n(X, A; G)$, we define $f_*(\sigma)$ by $f \circ \sigma : \Delta^n \to Y$. From this we obtain an induced map

$$f^* : C^n(Y, B; G) \to C^n(X, A; G)$$

by setting $f^* = f_*^\vee$, i.e., $f^*(\phi) = \phi \circ f_*$. Thus, we obtain an induced map f^* on the cohomology groups.

Consider a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

In general we know that the functor $\mathrm{Hom}(\ , G)$ is not an exact functor and applying this to the short exact sequence yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}(C, G) \longrightarrow \mathrm{Hom}(B, G) \longrightarrow \mathrm{Hom}(A, G).$$

However, if the original short exact sequence happens to be split then we will obtain a short exact sequence

$$0 \longrightarrow \mathrm{Hom}(C, G) \longrightarrow \mathrm{Hom}(B, G) \longrightarrow \mathrm{Hom}(A, G) \longrightarrow 0.$$

Recalling that

$$0 \longrightarrow C_n(A) \longrightarrow C_n(X) \longrightarrow C_n(X, A) \longrightarrow 0$$

is a split exact sequence, we obtain a split exact sequence of cochain complexes

$$0 \longrightarrow C^n(X, A; G) \longrightarrow C^n(X; G) \longrightarrow C^n(A; G) \longrightarrow 0.$$

Thus, we obtain a long exact sequence of cohomology groups as well

$$\cdots \longrightarrow \mathrm{H}^{n-1}(A; G) \xrightarrow{\partial^{n-1}} \mathrm{H}^n(X, A; G) \xrightarrow{j^*} \mathrm{H}^n(X; G) \xrightarrow{i^*} \mathrm{H}^n(A; G) \xrightarrow{\partial^n} \mathrm{H}^{n+1}(X, A; G) \longrightarrow \cdots.$$

We define the reduced cohomology groups by using the augmented chain used to define the reduced homology groups. As in the case of homology, $\widetilde{\mathrm{H}}^n(X, A; G) \cong \mathrm{H}^n(X, A; G)$ if $n > 0$ and in the case $n = 0$ we have a split short exact sequence

$$0 \longrightarrow G \longrightarrow \mathrm{H}^0(X, A; G) \longrightarrow \widetilde{\mathrm{H}}^0(X, A; G) \longrightarrow 0.$$

Our next step is a universal coefficient theorem in the case of cohomology. This will relate the cohomology groups of X with coefficients in G to the homology groups. This will be useful right from the start of the theory so we do not delay in proving it. We proceed generally here as we did in § 4.3.

Consider a descending chain complex $C_* = \{C_n, d_n\}$ of free abelian groups. Set $C^n = \mathrm{Hom}(C_n, G)$. As above, we obtain a cochain complex $C^* = \{C^n, d^n\}$ and we can form the cohomology groups $\mathrm{H}^n(C^*; G)$ of this cochain complex. We define a product between $\mathrm{H}_n(C_*)$ and $\mathrm{H}^n(C^*; G)$. Let $[x] \in \mathrm{H}_n(C_*)$ and $[\phi] \in \mathrm{H}^n(C^*; G)$ with $x \in C_n$ a representative of $[x]$ and $\phi \in \mathrm{Hom}(C_n, G)$ a representative of $[\phi]$. Define

$$\langle[\phi], [x]\rangle = \phi(x) \in G.$$

Exercise 4.4.2. Show that the product $\langle[\phi], [x]\rangle$ is independent of the choice of representatives. Furthermore, show that it is additive in each variable separately, i.e.,

$$\begin{aligned}\langle[\phi_1 + \phi_2], [x]\rangle &= \langle[\phi_1], [x]\rangle + \langle[\phi_2], [x]\rangle \\ \langle[\phi], [x_1 + x_2]\rangle &= \langle[\phi], [x_1]\rangle + \langle[\phi], [x_2]\rangle.\end{aligned}$$

This product allows us to define a homomorphism

$$\alpha : \mathrm{H}^n(C^*; G) \to \mathrm{Hom}(\mathrm{H}_n(C_*), G)$$

by setting

$$(\alpha([\phi]))(x) = \langle[\phi], [x]\rangle$$

for $[\phi] \in \mathrm{H}^n(C^*; G)$ and $[x] \in \mathrm{H}_n(C_*)$. This map is a natural map. In particular, we have the following results.

Exercise 4.4.3. Let $f : C_* \to D_*$ be a chain map. Show that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n(D^*; G) & \xrightarrow{\alpha} & \mathrm{Hom}(\mathrm{H}_n(D_*), G) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{H}^n(C^*; G) & \xrightarrow{\alpha} & \mathrm{Hom}(\mathrm{H}_n(C_*), G). \end{array}$$

Exercise 4.4.4. Let

$$0 \longrightarrow C_* \longrightarrow D_* \longrightarrow E_* \longrightarrow 0$$

be a split short exact sequence of chain complexes. Show that the following chain complex is also exact

$$0 \longrightarrow \mathrm{Hom}(E_*, G) \longrightarrow \mathrm{Hom}(D_*, G) \longrightarrow \mathrm{Hom}(C_*, G) \longrightarrow 0$$

and the following diagram commutes

$$\begin{CD}
\mathrm{H}^n(C^*;G) @>\alpha>> \mathrm{Hom}(\mathrm{H}_n(C_*),G)\\
@V{\partial^n}VV @VV{\partial_{n+1}^{\vee}}V\\
\mathrm{H}^{n+1}(E^*;G) @>\alpha>> \mathrm{Hom}(\mathrm{H}_{n+1}(E_*),G).
\end{CD}$$

Exercise 4.4.5. Let $\phi : G \to H$ be a homomorphism of abelian groups. Show there is a natural induced map

$$\phi^\sharp : \mathrm{H}^n(C^*;G) \longrightarrow \mathrm{H}^n(C^*;H).$$

Moreover, show the following diagram commutes

$$\begin{CD}
\mathrm{H}^n(C^*;G) @>\alpha>> \mathrm{Hom}(\mathrm{H}_n(C_*),G)\\
@V{\phi^\sharp}VV @VV{\phi}V\\
\mathrm{H}^n(C^*,H) @>\alpha>> \mathrm{Hom}(\mathrm{H}_n(C_*),H).
\end{CD}$$

As we needed some homological algebra to state the universal coefficient theorem for homology, we need some additional homological algebra in this case as well. The dual notion to a free abelian group is that of a divisible group.

Definition 4.4.6. An abelian group G is said to be *divisible* if given any $g \in G$ and any nonzero integer n, there exists $h \in G$ so that $nh = g$.

The typical example of a divisible group is $\mathbb{Q}$.

Exercise 4.4.7. Show that quotients of divisible groups are divisible and direct sums of divisible groups are divisible.

We have used that any abelian group is isomorphic to a quotient of a free abelian group when studying free resolutions in § 4.3. We have the corresponding result for divisible groups.

Proposition 4.4.8. *Any group is isomorphic to a subgroup of a divisible group.*

Proof. Let G be an abelian group. We know there is a free abelian group F so that $G \cong F/R$ for some subgroup R of F. We can consider F as a subgroup of a divisible group. For instance, let $\{x_i\}$ be a basis for F. The rational vector space D with basis $\{x_i\}$ is divisible and F is a subgroup of this group. Then G is isomorphic to a subgroup of the divisible group D/R. □

Let H be an abelian group and consider a free resolution of H:

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0.$$

We can take the dual of this long exact sequence with respect to an abelian group G to obtain a chain complex

$$0 \longrightarrow \mathrm{Hom}(H, G) \xrightarrow{f_0^\vee} \mathrm{Hom}(F_0, G) \xrightarrow{f_1^\vee} \mathrm{Hom}(F_1, G) \longrightarrow \cdots .$$

The cohomology groups of this chain are denoted by $\mathrm{Ext}(H, G)$, i.e.,

$$\mathrm{Ext}^n(H, G) = \mathrm{H}^n(\mathrm{Hom}(F_*, G)).$$

One can show, much as was done with the Tor groups, that the group $\mathrm{Ext}(H, G)$ is independent of the choice of free resolution used. Furthermore, using that we can always write a free resolution of H as

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow H \longrightarrow 0,$$

we have again that $\mathrm{Ext}^n(H, G) = 0$ for $n > 1$. Thus, write $\mathrm{Ext}(H, G) = \mathrm{Ext}^1(H, G)$.

Exercise 4.4.9. Show that $\mathrm{Ext}^0(H, G) \cong \mathrm{Hom}(H, G)$.

Proposition 4.4.10. *Let A, B, C, and D be abelian groups.*

1. *If A is free then* $\mathrm{Ext}(A, B) = 0$.
2. *If B is divisible then* $\mathrm{Ext}(A, B) = 0$.
3. $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, B) \cong B/nB$.
4. *Given an exact sequence*

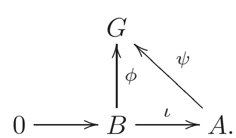
$$0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0,$$

we have an exact sequence

$$0 \to \mathrm{Hom}(D, A) \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A) \to \mathrm{Ext}(D, A) \to \mathrm{Ext}(C, A) \to \mathrm{Ext}(B, A) \to 0.$$

Proof. The proof of this proposition follows the same lines as those used to prove Proposition 4.3.12 and so is left as an exercise. □

Definition 4.4.11. An abelian group G is called *injective* if given any injection $\iota : B \to A$ and any homomorphism $\phi : B \to G$, there exists a homomorphism $\psi : A \to G$ so that the following diagram commutes:



Proposition 4.4.12. *An abelian group is injective if and only if it is divisible.*

Proof. First, suppose that G is injective. Let $g \in G$ and $n \in \mathbb{Z}$. Define a homomorphism $\phi : n\mathbb{Z} \to G$ by sending $n \mapsto g$. We have that $n\mathbb{Z} \hookrightarrow \mathbb{Z}$, and since G is injective there is an extension ψ of ϕ from $n\mathbb{Z}$ to $\mathbb{Z}$. Thus, using the commutativity of the diagram we have $g = \phi(n) = n\phi(1)$, and so G is divisible.

Suppose now that G is divisible. Let A, B, ι, and ϕ be as in the definition given for an injective group. We may assume that B is a subgroup of A and ι is the inclusion map. Consider the set of pairs (G_i, ι_i) where G_i is a subgroup of A that contains B and $\iota_i : G_i \to G$ is a homomorphism so that $\iota_i|_B = \phi$. This set is nonempty as the pair (B, ϕ) satisfies the hypotheses. Write $(G_i, \iota_i) < (G_j, \iota_j)$ if $G_i \subset G_j$ and $\iota_j|_{g_i} = \iota_i$. We can apply Zorn's lemma to conclude there exists a maximal pair, $(G_{\max}, \iota_{\max})$. We claim $G_{\max} = A$. If not, there exists $a \in A - G_{\max}$. Since G is divisible, we can extend $\iota_{\max}$ to the subgroup of A generated by $G_{\max}$ and a. This contradicts the maximality of $G_{\max}$. □

Proposition 4.4.13. *If G is a divisible group then the homomorphism*

$$\alpha : \mathrm{H}^n(C^*; G) \to \mathrm{Hom}(\mathrm{H}_n(C_*), G)$$

is an isomorphism for any chain complex C_*.

Proof. We have already shown such a homomorphism exists, it only remains to show it is bijective. We sketch this, leaving the details to the interested reader. Let $\phi \in \mathrm{Hom}(\mathrm{H}_n(C_*), G)$. Then we can view ϕ as a homomorphism $\phi : Z_n \to G$ that vanishes on B_n. The fact that Z_n injects into C_n and that G is divisible allows us to lift ϕ to a homomorphism $\psi : C_n \to G$. Thus, we have a map $\psi \in \mathrm{Hom}(C_n, G)$ that extends the map ϕ. To prove surjectivity, it only remains to check that ψ is a cocycle, which we leave as an exercise. We leave injectivity as an exercise as well. □

We now have the following universal coefficient theorem for cohomology groups.

Theorem 4.4.14. *Let C_* be a descending chain complex of free abelian groups and let G be an arbitrary abelian group. There exists a split exact sequence*

$$0 \longrightarrow \mathrm{Ext}(\mathrm{H}_{n-1}(C_*), G) \xrightarrow{\beta} \mathrm{H}^n(C^*; G) \xrightarrow{\alpha} \mathrm{Hom}(\mathrm{H}_n(C_*), G) \longrightarrow 0.$$

Note that the map β is natural with respect to coefficient homomorphisms and chain maps. The splitting is natural with respect to coefficient homomorphisms, but not with respect to chain maps.

Applying this to the case of interest where $C_* = C_*(X, A)$, we have the following.

Corollary 4.4.15. *For any pair (X, A) and any abelian group G, there exists a split exact sequence*

$$0 \longrightarrow \mathrm{Ext}(\mathrm{H}_{n-1}(X, A), G) \xrightarrow{\beta} \mathrm{H}^n(X, A; G) \xrightarrow{\alpha} \mathrm{Hom}(\mathrm{H}_n(X, A), G) \longrightarrow 0.$$

The homomorphisms α and β are natural with respect to homomorphisms induced by continuous maps of pairs and coefficient homomorphisms. The splitting can be chosen to be natural with respect to coefficient homomorphisms, but not with respect to homomorphisms induced by continuous maps.

As cohomology is obtained by dualizing the construction for homology, we can recover many of the results from homology for cohomology. For example, we have the excision property.

Theorem 4.4.16. *Let $Z \subset A \subset X$ with $\mathrm{Cl}(Z) \subset \mathrm{Int}(A)$. The inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces an isomorphism*

$$\mathrm{H}^n(X, A; G) \xrightarrow{\simeq} \mathrm{H}^n(X - Z, A - Z; G).$$

Proof. Let (X, A) be a pair and $Z \subset A$. Recall that for each n we have a split short exact sequence

$$0 \longrightarrow C_n(X - Z, A - Z) \longrightarrow C_n(X, A) \longrightarrow C_n(X, A)/C_n(X - Z, A - Z) \longrightarrow 0.$$

We pass to the long exact sequence in homology groups to see that

$$\mathrm{H}_n(X - Z, A - Z) \xrightarrow{\simeq} \mathrm{H}_n(X, A)$$

if and only if

$$\mathrm{H}_n(C_*(X, A)/C_*(X - Z, A - Z)) = 0$$

for all n. This is precisely what is proven when proving the excision theorem for homology groups.

Instead of passing to the long exact sequence in homology, apply the functor $\mathrm{Hom}(\,, G)$ to the split exact sequence above to obtain the following exact sequence

$$0 \to \mathrm{Hom}(C_n(X, A)/C_n(X - Z, A - Z), G) \to C^n(X, A; G) \to C^n(X - Z, A - Z; G) \to 0.$$

If we now take cohomology groups, we see that

$$\mathrm{H}^n(X, A; G) \cong \mathrm{H}^n(X - Z, A - Z; G)$$

if and only if $\mathrm{H}^n(\mathrm{Hom}(C_*(X, A)/C_*(X - Z, A - Z), G)) = 0$ for all n. However, we can apply Corollary 4.4.15 along with the fact that we know $\mathrm{H}_n(C_*(X, A)/C_*(X - Z, A - Z)) = 0$ for all n by excision for homology to conclude that we must have $\mathrm{H}^n(\mathrm{Hom}(C_*(X, A)/C_*(X - Z, A - Z), G)) = 0$ for all n. ☐

Chapter 5

Sheaves and Čech Cohomology

In this chapter we will introduce sheaves and Čech cohomology and see how de Rham and singular cohomology are special cases of Čech cohomology. We will also introduce algebraic sheaves, Serre's GAGA theorems, and give a brief introduction to the theory of algebraic curves. We will end with a short expository section on the Hodge conjecture. Most of the material in this chapter can be found in [4], [7], or [12].

NOTE: When I have time I will go back and introduction sheaf cohomology theory generally using "fine" sheaves and then insert the proofs of the comparison isomorphisms between singular, de Rham, and Čech cohomologies on a smooth manifold.

5.1 Sheaves

We begin by introducing presheaves as they are the natural precursor to sheaves and are a bit easier to grasp and work with. In our settings most all the presheaves we work with will also be sheaves, but in general this is not the case so it is important to understand the distinction. Throughout this section X stands for a topological space.

Definition 5.1.1. Let X be a topological space. A *presheaf of groups* $\mathcal{F}$ on X is a collection of groups $\{\mathcal{F}(U)\}_{U\in\mathcal{T}_X}$ along with a collection of group homomorphisms $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ for each $V \subset U$ satisfying

1. $\mathcal{F}(\emptyset)$ is the trivial group with one element,
2. $\rho^U_U = \text{id}$ on $\mathcal{F}(U)$,
3. if $W \subset V \subset U$, then $\rho^U_W = \rho^V_W \circ \rho^U_V$.

The maps ρ_V^U are referred to as the *restriction maps.* The elements of $\mathcal{F}(U)$ are referred to as *the sections of* $\mathcal{F}$ *over* U*.* The elements in $\mathcal{F}(X)$ are referred to as the *global sections.* Global sections are often denoted as $\Gamma(X, \mathcal{F})$ as well.

One can define a presheaf of rings in the same manner, one just requires the $\mathcal{F}(U)$ to be rings and the restriction maps to be ring homomorphisms. Clearly any presheaf of rings is also a presheaf of groups just by restricting to the group operation.

Example 5.1.2. Let X be a real manifold and set $C_X^\infty(U)$ to be the set of functions $f : U \to \mathbb{R}$ that are C^∞. This presheaf is known as the presheaf of C^∞ functions and is a presheaf of rings. We denote it by C_X^∞.

Example 5.1.3. Let X be a complex manifold and set $\mathcal{O}_X(U)$ to be the set of holomorphic functions $f : U \to \mathbb{C}$. This presheaf of rings is known as the sheaf of holomorphic functions and is denoted $\mathcal{O}_X$.

Example 5.1.4. Let X be a complex manifold and set $\mathcal{O}_X^*(U)$ to be the set of nowhere vanishing holomorphic functions $f : U \to \mathbb{C}^\times$. This is a presheaf of groups with the operation being multiplication of functions and is denoted $\mathcal{O}_X^*$.

Example 5.1.5. Let X be a complex manifold and set $\mathcal{M}_X(U)$ to be the set of meromorphic functions $f : U \to \mathbb{C}$. This is a presheaf of rings known as the the sheaf of meromorphic functions and is denoted $\mathcal{M}_X$.

Example 5.1.6. Let X be a complex manifold and set $\mathcal{M}_X^{(n)}(U)$ to be the set of meromorphic n-forms on X. This is a presheaf of rings known as the sheaf of meromorphic n-forms and is denoted $\mathcal{M}_X^{(n)}$.

Example 5.1.7. Let X be a real manifold and set $\Omega_X^n(U)$ to be the set of C^∞ differential n-forms on U as studied in Chapter 3. This is a presheaf of groups known as the sheaf of C^∞ differential n-forms on X and denoted Ω_X^n.

Example 5.1.8. Let X be a complex manifold and set $\Omega_{X,\mathrm{hol}}^n(U)$ to be the set of holomorphic differential n-forms on U. The difference between these and the ones studied in Chapter 3 is that we require the forms to be holomorphic and not just C^∞. This gives a presheaf of groups known as the holomorphic differential n-forms on X. We can also define a presheaf $\overline{\Omega}_{X,\mathrm{hol}}^n$ by setting $\overline{\Omega}_{X,\mathrm{hol}}^n(U)$ to be the complex conjugates of the forms in $\Omega_{X,\mathrm{hol}}^n(U)$.

Example 5.1.9. Recall that in § 3.3 we saw that if we are given a complex manifold X of dimension n we can consider it as a real manifold of dimension $2n$. (Technically we just saw this for $X = \mathbb{C}^n$, but it is clear it generalizes to any complex manifold.) In this case, we can take $dz_1, \ldots, dz_n, d\overline{z}_1, \ldots, d\overline{z}_n$ as a basis for the space of C^∞ differential 1-forms. We write $\Omega_X^{p,q}$ for the presheaf of C^∞ differential $(p+q)$-forms that are generated by p of the forms dz_i and q of the forms $d\overline{z}_j$. For example, $\Omega_X^{p,q}(U)$ consists of sums of the form

$$\omega = \sum_{I,J} f_{I,J} dz_I \wedge d\overline{z}_J$$

where $f_{I,J}$ is a C^∞ function, I runs over sets of the form $(i_1, \ldots, i_p)$ and J runs over sets of the form $(j_1, \ldots, j_q)$.

Example 5.1.10. Let G be a group and X any topological space. Define a presheaf G^X on X by setting $G^X(U)$ to be the set of all functions $f : U \to G$. We place no restrictions at all on the functions. The group structure on $G^X(U)$ is point-wise multiplication of functions. This presheaf is known as the constant presheaf G^X.

Example 5.1.11. Let X be a topological space and for each $x \in X$ assign a group G_x. Define $\mathcal{S}$ by setting

$$\mathcal{S}(U) = \prod_{x \in U} G_x.$$

This is a presheaf on X. One particular case of this of interest is called the *skyscraper sheaf*. We obtain this by assigning a single group at a single point x and then the trivial group to all other points. We denote this presheaf by G_x. Thus, one has

$$G_x(U) = \begin{cases} \{0\} & \text{if } x \notin U \\ G & \text{if } x \in U. \end{cases}$$

In particular, we will use the presheaf $\mathbb{C}_x$ which is $\mathbb{C}$ around the point x and 0 everywhere else.

Example 5.1.12. Let X be a Riemann surface. A *divisor on* X is function $D : X \to \mathbb{Z}$ whose support is a discrete subset of X. The divisors on X form a group under pointwise addition denoted by $\mathrm{Div}(X)$. We write $D \geq 0$ if $D(x) \geq 0$ for all x. Write $D > 0$ if $D \geq 0$ and $D \neq 0$. For divisors D_1 and D_2 we write $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$. One generally denotes a divisor as

$$D = \sum_{x \in X} D(x) \cdot x.$$

One has a presheaf of divisors by setting $\mathrm{Div}(U)$ to be the divisors on U.

Let $f : X \to \mathbb{C}$ be a function that is meromorphic at $x \in X$. By choosing local coordinates around x, we can expand f in a Laurent expansion so that near x one has

$$f(z) = \sum_n c_n (z - x)^n.$$

We define the *order of* f *at* x by

$$\mathrm{ord}_x(f) = \min\{n : c_n \neq 0\}.$$

We define the *divisor associated to* f by setting

$$\mathrm{div}(f) = \sum_{x \in X} \mathrm{ord}_x(f) \cdot x.$$

Let D be a divisor on X. Let $\mathcal{O}_X[D](U)$ be the set of meromorphic functions on U that satisfy the condition

$$\text{ord}_x(f) \geq -D(x)$$

for all $x \in U$. This is a presheaf of groups on X. Note that $\mathcal{O}_X[D](U)$ are the functions with poles no worse than D on U. The global sections of this sheaf are often denoted by $L(D)$. For instance, this shows up in the classical statement of the Riemann-Roch theorem.

Example 5.1.13. Let X be a Riemann surface and D a divisor on X. We can consider the sheaf $\Omega_{X,\text{hol}}[D]$ where $\Omega_{X,\text{hol}}[D](U)$ consists of 1-forms on U that have poles bounded by D.

Example 5.1.14. Let X be a Riemann surface and D a divisor on X. We can consider the sheaf of meromorphic 1-forms with poles bounded by D, i.e., the sheaf $\mathcal{M}_X^{(1)}[D]$ given by where the sections $\mathcal{M}_X^{(1)}[D](U)$ are 1-forms $\omega \in \mathcal{M}_X^{(1)}(U)$ that satisfy $\text{div}(\omega)(x) \geq -D(x)$ for all $x \in U$. The global sections of this sheaf are denoted in the classical literature as $L^{(1)}(D)$.

Example 5.1.15. Let X be a Riemann surface and D a divisor on X. In this case there is a particular skyscraper sheaf we will be interested in. Given any $x \in X$, one can choose local coordinates around x. Assign to each x the group of Laurent polynomials in the local coordinates that whose top term has degree strictly less than $-D(x)$. In other words, this is the group of Laurent tails truncated at $-D(x)$. We denote the skyscraper sheaf with these groups at each point by $\mathcal{T}_X[D]$.

Given two divisors D_1 and D_2 with $D_1 \leq D_2$, we can form a skyscraper sheaf $\mathcal{T}_X[D_1/D_2]$ as follows. For each point x, associate the group of Laurent polynomials that have top term with degree strictly less than $-D_1(x)$ and lowest term has degree at least $-D_2(x)$.

Exercise 5.1.16. Show that $\mathcal{O}_X[0]$ is the presheaf of holomorphic functions on X.

Note that all of the presheaves given, and presheaves in general, arise from the fact that if $\mathcal{F}(U)$ is defined to be functions with some property, it is the case that when one restricts to a smaller domain the property remains valid. For a presheaf to be a sheaf, we require that this can be reversed. Namely, if U is an open set with an open covering $\{U_i\}$ and the property holds for all U_i, then it must hold on U as well.

Definition 5.1.17. Let $\mathcal{F}$ be a presheaf on X. We say $\mathcal{F}$ is a *sheaf* on X if for any open set U and any open covering $\{U_i\}$ of U, whenever there are elements $s_i \in \mathcal{F}(U_i)$ that agree on overlaps, namely, one has

$$\rho^{U_i}_{U_i \cap U_j}(s_i) = \rho^{U_j}_{U_i \cap U_j}(s_j)$$

for every i, j, then there exists a unique $s \in \mathcal{F}(U)$ so that

$$\rho_{U_i}^U(s) = s_i$$

for each i.

Exercise 5.1.18. Let U be an open set with open covering $\{U_i\}_{i \in I}$. If $\mathcal{F}$ is a sheaf and there is a section $s \in \mathcal{F}(U)$ so that $\rho_{U_i}^U(s) = 0$ for all $i \in I$, show that $s = 0$.

Definition 5.1.19. Let $\mathcal{F}$ be a sheaf. We say a sheaf $\mathcal{G}$ is a *subsheaf* of $\mathcal{F}$ if $\mathcal{G}(U)$ is a subgroup of $\mathcal{F}(U)$ for every U and the restriction maps of the sheaf $\mathcal{G}$ are induced by those of $\mathcal{F}$.

It is straightforward to check that the presheaves defined above are all actually sheaves except the constant sheaf. This is not a sheaf as being constant is not a local property for a function. For example, if X is the disjoint union of two open sets U and V, a function can be constant on U and constant on V without being globally constant if it happens to take different values on U and V. This shows in particular that a constant presheaf on a space X is never a sheaf unless the group is trivial or the space enjoys the property that any two open sets have to intersect. However, we can associate a sheaf to the constant sheaf by setting $\underline{G}(U)$ to be the set of $f : U \to G$ that are locally constant, i.e., for any point $x \in U$ there is an open neighborhood V of x with $V \subset U$ and $f|_V$ a constant function. The locally constant sheaves we will encounter most often are $\underline{\mathbb{Z}}$, $\underline{\mathbb{R}}$ and $\underline{\mathbb{C}}$.

Example 5.1.20. Let X be a compact Riemann surface, i.e., X is a compact complex manifold of dimension 1. Then $\mathcal{O}_X(X) = \mathbb{C}$ since the only holomorphic functions a compact Riemann surface are constant.

Exercise 5.1.21. Let X be a connected topological space. Show that $\underline{G}(X) = G$ for any group G.

Exercise 5.1.22. Let $\mathcal{F}$ be a sheaf on a space X and let Y be an open subset of X. Show that the *restriction sheaf* $\mathcal{F}|_Y$ defined by

$$\mathcal{F}|_Y(U) = \mathcal{F}(U)$$

for any U open in Y defines a sheaf on Y.

Exercise 5.1.23. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a space X. Define the direct sum $\mathcal{F} \oplus \mathcal{G}$ by

$$\mathcal{F} \oplus \mathcal{G}(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$$

for an open set $U \subset X$. Define the restriction maps for $\mathcal{F} \oplus \mathcal{G}$ using the restriction maps for $\mathcal{F}$ and $\mathcal{G}$. Show that $\mathcal{F} \oplus \mathcal{G}$ is a sheaf on X.

In order for sheaves to be useful, we will need a notion of maps between sheaves. This is a fairly straightforward thing to define as we want any morphism to respect the structures we already have.

Definition 5.1.24. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a space X. A *sheaf homomorphism* from $\mathcal{F}$ to $\mathcal{G}$ is a collection of homomorphisms

$$\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

for each open set U so that the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ {\scriptstyle \rho^U_V}\downarrow & & \downarrow{\scriptstyle \rho^U_V} \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \end{array}$$

for $V \subset U$ any open set.

One can generally obtain inclusion maps of sheaves whenever $\mathcal{F}(U) \subset \mathcal{G}(U)$ for all open sets U. When we have an inclusion map of sheaves $\mathcal{F} \hookrightarrow \mathcal{G}$ we write $\mathcal{F} \subset \mathcal{G}$. In terms of the examples above, we have the following inclusions.

Example 5.1.25. Regardless of the space X, we have the following inclusions of constant sheaves:

$$\underline{\mathbb{Z}} \subset \underline{\mathbb{R}} \subset \underline{\mathbb{C}}.$$

Example 5.1.26. For X a real manifold, we have

$$\underline{\mathbb{R}} \subset C^\infty_X.$$

Example 5.1.27. For X a complex manifold, we have

$$\underline{\mathbb{C}} \subset \mathcal{O}_X \subset \mathcal{M}_X.$$

We also have

$$\mathcal{O}_X \subset C^\infty_X.$$

Example 5.1.28. For X a Riemann surface, if $D_1 \leq D_2$ are divisors on X, then

$$\mathcal{O}_X[D_1] \subset \mathcal{O}_X[D_2].$$

Furthermore, we also have

$$\mathcal{I}_X[D_2] \subset \mathcal{I}_X[D_1]$$

and

$$\mathcal{I}_X[D_1/D_2] \subset \mathcal{I}_X[D_1].$$

Example 5.1.29. For X a real manifold, the differentiation maps d^n are sheaf maps from Ω^n_X to Ω^{n+1}_X.

Example 5.1.30. Let X be a Riemann surface and $x \in X$ a point. We can define a sheaf map by evaluation

$$\mathrm{eval}_x : C_X^\infty \to \mathbb{C}_x$$

which on any open set U containing x sends the C^∞ function f defined on U to be the constant $f(x)$. On an open set not containing x the sheaf map is the zero map.

Example 5.1.31. Let X be a Riemann surface and D a divisor on X. Let $f \in \mathcal{O}_X[D](U)$. Then the Laurent series for f near $x \in U$ has terms with degrees at least $-D(x)$, i.e., we can write for z near x

$$f(z) = \sum_{n \geq -D(x)} c_n (z-x)^n.$$

For $x \in U$ we can define a map

$$\mathrm{eval}_x : \mathcal{O}_X[D](U) \to \mathbb{C}_x(U)$$

by sending f to $c_{-D(x)}$. If $x \notin U$, we define eval_x to be the zero map. This gives a sheaf homomorphism from $\mathcal{O}_X[D]$ to $\mathbb{C}_x$.

Example 5.1.32. Let X be a Riemann surface and U an open set in X. Let f be a holomorphic function on U. Then $g(z) = e^{2\pi i f(z)}$ is a holomorphic function on U that is nonvanishing on U. This gives a map

$$\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X^*(U)$$

for each $U \subset X$ open. This commutes with restriction so gives the exponential map $\exp(2\pi i -)$ of sheaves:

$$\exp(2\pi i -) : \mathcal{O}_X \longrightarrow \mathcal{O}_X^*.$$

Exercise 5.1.33. Let X be a Riemann surface and fix a divisor D on X. Let $f \in \mathcal{M}_X(X)$. Show that multiplication by f gives a sheaf homomorphism

$$\mathcal{O}_X[D] \longrightarrow \mathcal{O}_X[D - \mathrm{div}(f)].$$

Exercise 5.1.34. Let X be a Riemann surface and fix a divisor D on X. Let ω be a nonzero global meromorphic 1-form on X. Show that the multiplication by ω map gives a sheaf isomorphism

$$\mathcal{O}_X[D] \to \Omega^1_{X,\mathrm{hol}}[D - \mathrm{div}(\omega)].$$

Example 5.1.35. We also have several truncation maps. Let D be a divisor on a Riemann surface X. One clearly obtains a truncation map

$$\alpha_D : \mathcal{M}_X \to \mathcal{T}_X[D]$$

by truncating any meromorphic function.

Similarly, if $D_1 \le D_2$ are divisors, given a Laurent tail divisor in $\mathcal{T}_X[D_1]$, we can truncate this at $-D_2(x)$ for each $x \in X$ and obtain a sheaf homomorphism

$$t_{D_2}^{D_1} : \mathcal{T}_X[D_1] \to \mathcal{T}_X[D_2].$$

Finally, given a meromorphic function with poles bounded by D_2, we may truncate the Laurent series at $-D_1$ and obtain a sheaf map

$$\alpha_{D_1/D_2} : \mathcal{O}_X[D_2] \to \mathcal{T}_X[D_1/D_2].$$

Let X be a topological space and $\mathcal{F}$, $\mathcal{G}$ sheaves on X. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a sheaf homomorphism. We can define a *kernel presheaf* $\ker(\phi)$ by setting

$$\ker(\phi)(U) = \ker(\phi_U).$$

Exercise 5.1.36. Check that $\ker(\phi)$ is a presheaf.

Proposition 5.1.37. *The presheaf* $\ker(\phi)$ *is a sheaf on* X*.*

Proof. Let U be an open set with open covering $\{U_i\}$. Let $s_i \in \ker(\phi)(U_i)$ be sections so that the s_i agree on overlaps. The fact that $\mathcal{F}$ is a sheaf and $\ker(\phi)(U_i) \subset \mathcal{F}(U_i)$ gives that there exists $s \in \mathcal{F}(U)$ so that $\rho_{U_i}^U(s) = s_i$. It remains to show that $s \in \ker(\phi)(U)$, i.e., $\phi_U(s) = 0$.

Let $t_i = \rho_{U_i}^U(\phi_U(s))$. Since ϕ is a sheaf homomorphism it commutes with restriction maps, so we have

$$\begin{aligned} t_i &= \rho_{U_i}^U(\phi_U(s)) \\ &= \phi_{U_i}(\rho_{U_i}^U(s)) \\ &= \phi_{U_i}(s_i) \\ &= 0 \end{aligned}$$

since each $s_i \in \ker(\phi_{U_i})$. Now we use the fact that $\mathcal{G}$ is a sheaf to see that $\phi_U(s)$ must be 0 as well. ☐

One can define presheaves $U \mapsto \operatorname{im}(\phi_U)$ and $U \mapsto \operatorname{coker}(\phi_U)$ as well. However, unlike the kernel sheaf, it turns out that these are not sheaves! One can associate a sheaf to them, as we will see later.

The definition for injective and surjective for sheaf homomorphisms is not the one that first comes to mind, but is the correct one when one remembers that properties of sheaves should reflect locally properties. Namely, we do not require that ϕ_U be injective (resp. surjective) for each U, only that for small enough U we have ϕ_U is injective (resp. surjective.) In particular, we define injective and surjective as follows.

Definition 5.1.38. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a sheaf homomorphism. We say ϕ is *injective* if for every $x \in X$ and every open set U containing x, there is an open subset $V \subset U$ containing x so that ϕ_V is injective. We say ϕ is *surjective* if for every $x \in X$ and every open set U containing x and every section $f \in \mathcal{G}(U)$, there is an open $V \subset U$ containing x so that $\rho_V^U(f)$ is in the image of ϕ_V.

Example 5.1.39. Let $X = \mathbb{C} - \{0\} = \mathbb{C}^\times$. Let $g(z) = 1/z \in \mathcal{O}_X^*(X)$. It is well known from complex analysis there is no function f so that $e^{2\pi i f(z)} = 1/z$ for all $z \in X$. Thus, we see the exponential map is not surjective for the set $U = X$. However, at any point $x \in \mathbb{C}^\times$ there is a branch of the logarithm defined in a neighborhood V of x and so $f(z) = -\frac{\ln(z)}{2\pi i}$ maps under the exponential map to $1/z$. Thus, as a sheaf map the exponential map is a surjective map.

This type of issue does not arise when talking about injectivity.

Proposition 5.1.40. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a sheaf homomorphism. The following are equivalent:*

1. *ϕ is injective;*
2. *ϕ_U is injective for every open set $U \subset X$;*
3. *The kernel sheaf for ϕ is the identically 0 sheaf.*

Proof. It is clear that the second and third statements are equivalent and that the second statement implies the first. Thus, it only remains to show that if ϕ is injective then ϕ_U is injective for every open set $U \subset X$. Let U be an open set and let $s \in \mathcal{F}(U)$ so that $\phi_U(s) = 0$. Using the fact that $\mathcal{F}$ is a sheaf, it is enough to show that $\rho_V^U(s) = 0$ in $\mathcal{F}(V)$ for each subset V in an open covering of U.

Let $x \in U$. Since ϕ is injective, there exists an open set $V_x \subset U$ containing x so that ϕ_{V_x} is injective. Let $s_x = \rho_{V_x}^U(s)$. Since the V_x cover U, it is enough to show that s_x is 0 for each x. However, we have

$$\begin{aligned}\phi_{V_x}(s_x) &= \phi_{V_x}(\rho_{V_x}^U(s)) \\ &= \rho_{V_x}^U(\phi_U(s)) \\ &= \rho_{V_x}^U(0) \\ &= 0.\end{aligned}$$

However, since ϕ_{V_x} is injective this gives the result. $\square$

Another way to deal with injectivity and surjectivity of sheaf homomorphisms is to work with stalks, which we introduce now.

Definition 5.1.41. Let I be a nonempty set with a partial order $\leq$. For each $i \in I$, let G_i be an additive abelian group. Suppose for every pair $i, j \in I$ with $i \leq j$ there is a map $\rho_{ij} : A_i \to A_j$ so that

1. $\rho_{jk} \circ \rho_{ij} = \rho_{ik}$ whenever $i \leq j \leq k$ and
2. $\rho_{ii} = 1$ for all $i \in I$.

Let H be the disjoint union of all the G_i. Define an equivalence relation $\sim$ on H by setting $g \sim h$ if and only if there exists a k with $i, j \leq k$ and $\rho_{ik}(g) = \rho_{jk}(h)$ for $g \in G_i$, $h \in G_j$. The set of equivalence classes is called the *direct limit of the G_i* and is denoted $\varinjlim_i G_i$.

Definition 5.1.42. Let $\mathcal{F}$ be a presheaf on a space X. Let $x \in X$. Define the *stalk* $\mathcal{F}_x$ *of* $\mathcal{F}$ *at* x to be the direct limit of the groups $\mathcal{F}(U)$ for all open sets U containing x via the restriction maps ρ.

One should view the stalk $\mathcal{F}_x$ as zooming in to see what is happening at the point x. An element of $\mathcal{F}_x$ is represented by a pair $\langle U, s\rangle$ where U is an open neighborhood of x and s is an element of $\mathcal{F}(U)$. We denote such an equivalence class by s_x. Two such pairs $\langle U, s\rangle$ and $\langle V, t\rangle$ define the same element in $\mathcal{F}_x$ if there is an open neighborhood $W \subset U \cap V$ so that $s|_W = t|_W$. This generalizes the classical notion of *germs of functions* from complex analysis. One should think of this as kind of like a Taylor series, you just use local information (derivatives) to get information about the function in the form of a power series.

Example 5.1.43. Let X be a complex manifold of dimension n. Let $x \in X$. Then $\mathcal{O}_{X,x}$ is the ring of convergent power series in n variables.

Exercise 5.1.44. Show that the stalks of a locally constant sheaf $\underline{G}$ are all isomorphic to the group G.

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a sheaf homomorphism. Thus, for open sets $U \subset V$ we have a commuting diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \Big\downarrow \rho^U_V & & \Big\downarrow \rho^U_V \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \end{array}$$

This shows that for any $x \in X$ the map ϕ induces a map of stalks

$$\phi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x.$$

We set $\operatorname{supp}(\phi)$ to be the set of $x \in X$ so that ϕ_x is not the zero map.

Theorem 5.1.45. *The sheaf homomorphism $\phi : \mathcal{F} \to \mathcal{G}$ is injective if and only if $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$. Similarly, ϕ is surjective if and only if ϕ_x is surjective for every $x \in X$.*

Proof. First, suppose that ϕ is injective so that $\ker(\phi) = 0$. Thus, we have that each map ϕ_U is injective. Upon passing to the direct limit we see that ϕ_x must be injective for each $x \in X$ as well. Conversely, suppose that ϕ_x is injective for each $x \in X$. Let U be an open set and let $s \in \mathcal{F}(U)$ be such that $\phi_U(s) = 0$. Thus, for every $x \in U$ we have that $\phi_U(s)_x = 0$ in the stalk $\mathcal{G}_x$. Since we have that ϕ_x is injective, we must have $s_x = 0$ in the stalk $\mathcal{F}_x$ for each $x \in U$. The statement that $s_x = 0$ means that there exists an open neighborhood $W_x \subset U$ of x so that $\rho^U_{W_x}(s) = 0$. The fact that U is covered by such neighborhoods along with the fact that $\ker(\phi)$ is a sheaf gives that $s = 0$ and so ϕ_U is injective. Since U was arbitrary, this gives that ϕ is injective.

Suppose now that ϕ is surjective. Let $x \in X$ and let $t_x \in \mathcal{G}_x$. Let t_x be represented by the pair $\langle U, t\rangle$. Since ϕ is surjective, there is an open neighborhood $V \subset U$ of x so that ϕ_V is surjective. Thus, there is a $s \in \mathcal{F}(V)$ so that $\phi_V(s) = t$. In particular, we have that ϕ_x maps $\langle V, s\rangle$ to $\langle V, t\rangle$, which clearly represents t_x. Thus, ϕ_x is surjective as well. Conversely, suppose that ϕ_x is surjective for each $x \in X$. Let $t \in \mathcal{G}(U)$ and let t_x be the image of t in $\mathcal{G}_x$. The fact that ϕ_x is surjective implies there exists a $s_x \in \mathcal{F}_x$ so that $\phi_x(s_x) = t_x$. Let $\langle V, s\rangle$ represent s_x and set $W = U \cap V$. Then $\langle W, s\rangle$ represents s_x and we have $\phi_W(s) = t \in \mathcal{G}(W)$. Thus, we have that ϕ is surjective since we have found an element in $\mathcal{F}(W)$ mapping to $t \in \mathcal{G}(W)$ for a small enough open set W. $\square$

Definition 5.1.46. We say a sequence of sheaf homomorphisms

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \stackrel{\phi}{\longrightarrow} \mathcal{G} \longrightarrow 0$$

is a *short exact sequence of sheaves* if the sheaf map ϕ is surjective and $\mathcal{K} = \ker(\phi)$.

We have the following corollary immediately from Theorem 5.1.45.

Corollary 5.1.47. *A sequence of sheaf homomorphisms*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \stackrel{\phi}{\longrightarrow} \mathcal{G} \longrightarrow 0$$

is a short exact sequence of sheaves if and only if the sequence of stalks

$$0 \longrightarrow \mathcal{K}_x \longrightarrow \mathcal{F}_x \stackrel{\phi_x}{\longrightarrow} \mathcal{G}_x \longrightarrow 0$$

is a short exact sequence of abelian groups for each $x \in X$.

Example 5.1.48. Let X be a real manifold. The sequence

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow C^\infty_X \stackrel{d}{\longrightarrow} \Omega^1_X \longrightarrow 0$$

is a short exact sequence of sheaves. We have seen that for any open set U the space $\Omega^1_X(U)$ is generated by dx_i's, and so clearly d is a surjective map of sheaves. The kernel is precisely the locally constant functions, which gives the exact sequence.

Example 5.1.49. Let X be a complex manifold. The sequence

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \mathcal{O}_X \stackrel{d}{\longrightarrow} \Omega^1_{X,\mathrm{hol}} \longrightarrow 0$$

is a short exact sequence of sheaves for the same reasons as the last example was short exact.

Example 5.1.50. Let X be a Riemann surface. The sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \stackrel{\exp(2\pi i -)}{\longrightarrow} \mathcal{O}^\times_X \longrightarrow 0$$

is a short exact sequence of sheaves.

Exercise 5.1.51. Let X be a Riemann surface and let D be a divisor on X. Show that the sequence

$$0 \longrightarrow \mathcal{O}_X[D-x] \longrightarrow \mathcal{O}_X[D] \stackrel{\text{eval}_x}{\longrightarrow} \mathbb{C}_x \longrightarrow 0$$

is a short exact sequence of sheaves.

Show that the sequence

$$0 \longrightarrow \mathcal{O}_X[D] \longrightarrow \mathcal{M}_X \stackrel{\alpha_D}{\longrightarrow} \mathcal{T}_X[D] \longrightarrow 0$$

is a short exact sequence of sheaves.

If $D_1 \leq D_2$ are divisors, then the sequence

$$0 \longrightarrow \mathcal{O}_X[D_1] \longrightarrow \mathcal{O}_X[D_2] \stackrel{\alpha_{D_1/D_2}}{\longrightarrow} \mathcal{T}_X[D_1/D_2] \longrightarrow 0$$

is a short exact sequence of sheaves.

Finally, show that the sequence

$$0 \longrightarrow \mathcal{T}_X[D_1/D_2] \longrightarrow \mathcal{T}_X[D_1] \longrightarrow \mathcal{T}_X[D_2] \longrightarrow 0$$

is a short exact sequence of sheaves where the map from $\mathcal{T}_X[D_1]$ to $\mathcal{T}_X[D_2]$ is the truncation map.

Definition 5.1.52. A sheaf homomorphism $\phi : \mathcal{F} \to \mathcal{G}$ is an *isomorphism* if it has an inverse, i.e., if there exists a sheaf homomorphism $\psi : \mathcal{G} \to \mathcal{F}$ so that $\phi \circ \psi = \text{id}$ and $\psi \circ \phi = \text{id}$.

Proposition 5.1.53. *A sheaf homomorphism $\phi : \mathcal{F} \to \mathcal{G}$ is a sheaf isomorphism if and only if it is injective and surjective.*

Proof. First, suppose that ϕ has an inverse ψ. By definition one has that ψ_U is the inverse of ψ_U for each U, and so each ϕ_U must be bijective. Thus, ϕ is bijective.

Now suppose that ϕ is bijective. If we show that each ϕ_U is an isomorphism, then we can define ψ to be the collection of inverse maps and we will be done. Let U be an open set. Since ϕ is injective we know that each ϕ_U is also injective by Proposition 5.1.40. So it only remains to show that ϕ_U is surjective.

Let $g \in \mathcal{G}(U)$. Since ϕ is surjective, for each $x \in X$ there is an open set U_x so that ϕ_{U_x} is surjective. Thus, there exists $f_x \in \mathcal{F}(U_x)$ so that $\phi_{U_x}(f_x) = \rho^U_{U_x}(g)$. We claim that the sections f_x agree on overlaps. Let $W = U_x \cap U_y$ for $x \neq y$. Observe that we have

$$\begin{aligned}
\phi_W(\rho^{U_x}_W(f_x)) &= \rho^{V_x}_W(\phi_{V_x}(f_x)) \\
&= \rho^{V_x}_W(\rho^U_{V_x}(g)) \\
&= \rho^U_W(g).
\end{aligned}$$

Similarly, we obtain that $\phi_W(\rho^{U_y}_W(f_y)) = \rho^U_W(g)$ and so

$$\phi_W(\rho^{U_x}_W(f_x)) = \phi_W(\rho^{U_y}_W(f_y)).$$

However, we can now use that ϕ_W is injective to conclude that

$$\rho_W^{U_x}(f_x) = \rho_W^{U_y}(f_y)$$

for all $x, y \in X$. Thus, the sections agree on overlaps and so we can use the fact that $\mathcal{F}$ is a sheaf to glue them together into a section $f \in \mathcal{F}(U)$.

It only remains to show that $\phi_U(f) = g$. We can again use that we have sheafs to reduce to checking this locally on each U_x. But this is clear because we have

$$\begin{aligned}\rho_{U_x}^U(\phi_U(f)) &= \phi_{U_x}(\rho_{U_x}^U(f)) \\ &= \phi_{U_x}(f_x) \\ &= g_x \\ &= \rho_{U_x}^U(g).\end{aligned}$$

Thus, $\phi_U(f)$ and g agree on each U_x and so must be equal by the fact that $\mathcal{G}$ is a sheaf. □

Exercise 5.1.54. Show $\phi : \mathcal{F} \to \mathcal{G}$ is a sheaf isomorphism if and only if $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for all $x \in X$.

Exercise 5.1.55. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups on X. For any open $U \subset X$ show that the set $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ of morphisms of restricted sheaves has the structure of an abelian group. Show that the presheaf $U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf. It is referred to as the *sheaf of local morphisms* of $\mathcal{F}$ into $\mathcal{G}$, or "sheaf homs" for short. It is denoted by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

As we saw with the constant presheaf, there are times one has a presheaf of interest that is not a sheaf. It turns out that one can always associate a sheaf to a presheaf.

Definition 5.1.56. Let $\mathcal{F}$ be a presheaf on a space X. For any open set U in X, define $\widetilde{\mathcal{F}}(U)$ to be the set of functions $s : U \to \coprod_{x \in U} \mathcal{F}_x$ so that

1. for each $x \in U$, $s(x) \in \mathcal{F}_x$,
2. for each $x \in U$, there is a neighborhood V of x, contained in U, and an element $t \in \mathcal{F}(V)$ so that for all $y \in V$, the germ t_y is equal to $s(y)$.

Exercise 5.1.57. Check that $\widetilde{\mathcal{F}}$ is actually a sheaf on X.

Exercise 5.1.58. Show that the sheafification of the constant presheaf gives the locally constant sheaf as defined above.

One should think of $\widetilde{\mathcal{F}}$ as the sheaf that best approximates the presheaf $\mathcal{F}$. In particular, one has the following theorem that follows almost immediately from the definition.

Theorem 5.1.59. *Let $\mathcal{F}$ be a presheaf. There is a natural morphism $\theta : \mathcal{F} \to \widetilde{\mathcal{F}}$ of presheaves that satisfies the universal property that given any morphism $\phi : \mathcal{F} \to \mathcal{G}$ of presheaves where $\mathcal{G}$ is a sheaf, there is a unique sheaf homomorphism $\psi : \widetilde{\mathcal{F}} \to \mathcal{G}$ so that $\phi = \psi \circ \theta$. Furthermore, the pair $(\widetilde{\mathcal{F}}, \theta)$ is unique up to unique isomorphism.*

Exercise 5.1.60. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves so that ϕ_U is injective for each U. Show that the induced map $\widetilde{\phi} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ is injective. In particular, use this to show that if $\mathcal{F}$ and $\mathcal{G}$ happen to be sheaves then the sheaf $\operatorname{im}(\phi)$ associated to the presheaf $U \mapsto \operatorname{im}(\phi_U)$ can be naturally identified with a subsheaf of $\mathcal{G}$.

We end this section with a particular type of sheaf that we will need in subsequent sections.

Definition 5.1.61. A sheaf $\mathcal{F}$ on X is *fine* if for each locally finite cover $\mathcal{U} = \{U_i\}$ of X by open sets there exists for each i an endomorphism ϕ_i of $\mathcal{F}$ so that

1. $\operatorname{supp}(\phi_i) \subset U_i$
2. $\sum_i \phi_i = \operatorname{id}$.

We call $\{\phi_i\}$ a *partition of unity for $\mathcal{F}$ with respect to the cover $\mathcal{U}$.*

Example 5.1.62. Let X be a smooth manifold (real or complex) and consider the sheaf $\mathcal{F} = C^\infty_X$. Let $\mathcal{U}$ be a cover and let $\{\phi_i\}$ be a partition of unity with respect to this cover. We define sheaf maps $\widetilde{\phi_i}$ by setting

$$\widetilde{\phi_i}(f) = (\phi_i|_U) \cdot f$$

for $f \in C^\infty(U)$. These form a partition of unity and show that this is a fine sheaf.

Exercise 5.1.63. Show that if $\mathcal{F}$ and $\mathcal{G}$ are fine sheaves over X then $\mathcal{F} \otimes \mathcal{G}$ is itself a fine sheaf.

5.2 Abstract Sheaf Cohomology

Note that the class decided not to see these proofs in class, just the statements so the proofs are omitted for now. They will be added later.

In the next section we will construct Čech cohomology groups, which are very useful for computations. However, in order to show that the Čech cohomology groups agree with the singular and de Rham theories for smooth manifolds, we need an abstract set-up. We provide that in this section.

Let X be a topological space and $\mathcal{U} = \{U_i\}$ an open cover of X. A *refinement* $\mathcal{V} = \{V_j\}$ of $\mathcal{U}$ is an open cover so that for each j there is an i so that $V_j \subset U_i$. We say a collection of subsets $\{U_i\}$ is *locally finite* if for each $x \in X$ there is a neighborhood W_x of x so that $W_x \cap U_i \neq \emptyset$ for only finitely many i. We restrict here to the case that X is *paracompact*, i.e., if every open cover of X has a locally finite refinement. In particular, smooth manifolds are all paracompact so one can restrict to that case.

Definition 5.2.1. Let X be a paracompact space and K a principal ideal domain. A *sheaf cohomology theory $\mathcal{H}$ for X with coefficients in sheaves of K-modules over X* consists of

1. a K-module $\mathrm{H}^n(X, \mathcal{F})$ for each sheaf $\mathcal{F}$ and each integer n,
2. a homomorphism $\phi_* : \mathrm{H}^n(X, \mathcal{F}) \to \mathrm{H}^n(X, \mathcal{G})$ for each sheaf homomorphism $\phi : \mathcal{F} \to \mathcal{G}$ and each integer n,
3. a homomorphism $\delta^n : \mathrm{H}^n(X, \mathcal{F}_3) \to \mathrm{H}^n(X, \mathcal{F}_1)$ for each short exact sequence
$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$
of sheaves,

so that the following properties hold

(a) $\mathrm{H}^n(X, \mathcal{F}) = 0$ for all $n < 0$, and there is an isomorphism $\mathrm{H}^0(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$ so that for each sheaf homomorphism $\phi : \mathcal{F} \to \mathcal{G}$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^0(X, \mathcal{F}) & \xrightarrow{\cong} & \Gamma(X, \mathcal{F}) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(X, \mathcal{G}) & \xrightarrow{\cong} & \Gamma(X, \mathcal{G}). \end{array}$$

(b) If $\mathcal{F}$ is a fine sheaf then $\mathrm{H}^n(X, \mathcal{F}) = 0$ for all $n > 0$.

(c) If
$$0 \to \mathcal{F}_1 \xrightarrow{\phi} \mathcal{F}_2 \xrightarrow{\psi} \mathcal{F}_3 \to 0$$
is an exact sequence of sheaves, then one the following long exact sequence in cohomology is exact:
$$\cdots \to \mathrm{H}^n(X, \mathcal{F}_1) \xrightarrow{\phi_*} \mathrm{H}^n(X, \mathcal{F}_2) \xrightarrow{\psi_*} \mathrm{H}^n(X, \mathcal{F}_3) \xrightarrow{\delta^n} \mathrm{H}^{n+1}(X, \mathcal{F}_1) \to \cdots .$$

(d) The identity sheaf homomorphism $\mathrm{id} : \mathcal{F} \to \mathcal{F}$ induces the identity homomorphism $\mathrm{H}^n(X, \mathcal{F}) \to \mathrm{H}^n(X, \mathcal{F})$ for all n.

(e) If the diagram

commutes, then for each n so does the diagram

(f) For each homomorphism of short exact sequences of sheaves

the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 \\ & \searrow & \downarrow \\ & & \mathcal{F}_3 \end{array}$$

$$\begin{array}{ccc} \mathrm{H}^n(X, \mathcal{F}_1) & \longrightarrow & \mathrm{H}^n(X, \mathcal{F}_2) \\ & \searrow & \downarrow \\ & & \mathrm{H}^n(X, \mathcal{F}_3). \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G}_1 & \longrightarrow & \mathcal{G}_2 & \longrightarrow & \mathcal{G}_3 & \longrightarrow & 0, \end{array}$$

$$\begin{array}{ccc} \mathrm{H}^n(X, \mathcal{F}_3) & \longrightarrow & \mathrm{H}^{n+1}(X, \mathcal{F}_1) \\ \downarrow & & \downarrow \\ \mathrm{H}^n(X, \mathcal{G}_3) & \longrightarrow & \mathrm{H}^{n+1}(X, \mathcal{G}_1). \end{array}$$

A bunch of material needs to be added here, but the result we are after is the following.

Theorem 5.2.2. *Any two cohomology theories on X with coefficients in sheaves of K-modules over X are uniquely isomorphic.*

Add stuff about singular and de Rham cohomology here. Provide proofs for de Rham, probably not singular...

5.3 Čech Cohomology

In this section we define the the Čech cohomology groups. Again we assume that X is paracompact. Let $\mathcal{F}$ be a sheaf of abelian groups on X. Let $\mathcal{U} = \{U_i\}$ be an open cover of X. Fix an integer n. Given a set of indices $(i_0, i_1, \ldots, i_n)$. Set

$$U_{i_0,\ldots,i_n} = U_{i_0} \cap \cdots \cap U_{i_n}.$$

Observe that we have

$$U_{i_0,\ldots,i_n} \subset U_{i_0,\ldots,\widehat{i_k},\ldots,i_n}$$

for any $0 \le k \le n$.

Definition 5.3.1. A *Čech n-cochain* for the sheaf $\mathcal{F}$ over the open cover $\mathcal{U}$ is a collection of sections of $\mathcal{F}$, one over each $U_{i_0,\ldots,i_n}$. The space of Čech n-cochains for $\mathcal{F}$ over $\mathcal{U}$ is denoted by $\check{C}^n(\mathcal{U},\mathcal{F})$, i.e.,

$$\check{C}^n(\mathcal{U},\mathcal{F}) = \prod_{(i_0,\dots,i_n)} \mathcal{F}(U_{i_0,\dots,i_n}).$$

We denote a n-cochain by $(f_{i_0,\dots,i_n})$.

From the definition we see that a 0-chain is a collection of sections $f_i \in \mathcal{F}(U_i)$, i.e., a section for each open set in the cover.

We define the coboundary map $d^n : \check{C}^n(\mathcal{U},\mathcal{F}) \to \check{C}^{n+1}(\mathcal{U},\mathcal{F})$ by setting

$$d((f_{i_0,\dots,i_n})) = (g_{i_0,\dots,i_{n+1}})$$

where

$$g_{i_0,\dots,i_{n+1}} = \sum_{k=0}^{n+1} (-1)^k \rho(f_{i_0,\dots,\widehat{i}_k,\dots,i_{n+1}})$$

where the ρ here is the restriction map from $U_{i_0,\dots,\widehat{i}_k,\dots,i_{n+1}}$ to $U_{i_0,\dots,i_{n+1}}$. In general we will drop the restriction maps from the notation in this case and they will be understood to be there. Note that $d^0((f_i)) = (g_{i,j})$ where $g_{i,j} = f_j - f_i$ and $d^1((f_{i,j})) = (g_{i,j,k})$ where $g_{i,j,k} = f_{j,k} - f_{i,k} + f_{i,j}$.

Definition 5.3.2. Let $c \in \check{C}^n(\mathcal{U},\mathcal{F})$ be a n-cochain satisfying $d^n(c) = 0$. We call such a cochain a *n-cocycle*. The space of n-cocycles is denoted by $\check{Z}^n(\mathcal{U},\mathcal{F})$. If $c \in \check{C}^n(\mathcal{U},\mathcal{F})$ is in the image of d^{n-1} we call c a *n-coboundary*. The space of n-coboundaries is denoted by $\check{B}^n(\mathcal{U},\mathcal{F})$.

Exercise 5.3.3. Check that $d^{n+1} \circ d^n = 0$ for all n.

Using the exercise, we obtain the Čech cochain complex

$$0 \longrightarrow \check{C}^0(\mathcal{U},\mathcal{F}) \xrightarrow{d^0} \check{C}^1(\mathcal{U},\mathcal{F}) \xrightarrow{d^1} \check{C}^2(\mathcal{U},\mathcal{F}) \longrightarrow \cdots .$$

Definition 5.3.4. The nth Čech cohomology group of $\mathcal{F}$ with respect to the cover $\mathcal{U}$ is given by

$$\check{\mathrm{H}}^n(\mathcal{U},\mathcal{F}) = \check{Z}^n(\mathcal{U},\mathcal{F})/\check{B}^n(\mathcal{U},\mathcal{F}).$$

Of course, at this point the cohomology groups depend on the open cover $\mathcal{U}$. We will return to this momentarily.

Lemma 5.3.5. *Let $\mathcal{U}$ be an open cover of X. Then we have*

$$\check{\mathrm{H}}^0(\mathcal{U},\mathcal{F}) = \Gamma(X,\mathcal{F}).$$

Proof. We know that $\check{B}^0(\mathcal{U},\mathcal{F}) = 0$, so it only remains to compute $\check{Z}^0(\mathcal{U},\mathcal{F})$. Define a map

$$\alpha : \Gamma(X,\mathcal{F}) \to \check{C}^0(\mathcal{U},\mathcal{F})$$

by sending a global section f to $f_i = \rho^X_{U_i}(f)$. Observe that $d^0(f_i) = 0$ for all i because $d^0(f_i) = (f_j - f_i)$ and since these are just restrictions of a global section, they are equal on overlaps. Thus, the image of α lies in $\check{Z}^0(\mathcal{U},\mathcal{F})$. The fact that α is injective and surjective is exactly the definition of $\mathcal{F}$ being a sheaf. ☐

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a sheaf homomorphism. Clearly this induces a map on cochains

$$\phi : \check{C}^n(\mathcal{U}, \mathcal{F}) \to \check{C}^n(\mathcal{U}, \mathcal{G})$$

given by

$$(f_{i_0,\ldots,i_n}) \mapsto (\phi(f_{i_0,\ldots,i_n})).$$

Moreover, since the coboundary map commutes with any map induced by a sheaf homomorphism, we have an induced map on the cohomology groups

$$\phi_* : \check{\mathrm{H}}^n(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^n(\mathcal{U}, \mathcal{G}).$$

Example 5.3.6. Consider $X = S^1$ and let $\mathcal{F} = \underline{\mathbb{Z}}$. Consider the open cover $\mathcal{U}$ consisting of two sets U and V that overlap on small intervals as when we computed the cohomology of S^1 before. In this case, we have

$$C^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U) \times \mathcal{F}(V) = \underline{\mathbb{Z}}(U) \times \underline{\mathbb{Z}}(V) \cong \mathbb{Z} \times \mathbb{Z},$$

$$C^1(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U \cap V) = \underline{\mathbb{Z}}(U \cap V) \cong \mathbb{Z} \times \mathbb{Z},$$

and

$$C^n(\mathcal{U}, \mathcal{F}) = 0$$

for all $n \geq 2$. Furthermore, we see that the map $d^0 : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$ takes (a, b) to $(b - a, b - a)$. Thus, we have that

$$\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) = \ker(d^0) \cong \mathbb{Z}$$

and

$$\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{F}) = \ker(d^1)/\operatorname{im}(d^0) \cong \mathbb{Z}.$$

Thus, we see that the cohomology groups agree with those computed using singular cohomology with $\mathbb{Z}$-coefficients in this case.

We would like to have cohomology groups attached to X and $\mathcal{F}$ that do not depend on the choice of cover $\mathcal{U}$. To do this, we must introduce the notion of refinements.

Definition 5.3.7. Let $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ be two open covers of X. Recall that we say $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if every open set $V_j \in \mathcal{V}$ is contained in some open set $U_i \in \mathcal{U}$. We write $\mathcal{V} \prec \mathcal{U}$ to denote that $\mathcal{V}$ is a refinement of $\mathcal{U}$.

Let $\mathcal{V}$ be a refinement of $\mathcal{U}$. We can define a function $r : J \to I$ by $r(j) = i$ where $V_j \subset U_i$. We call such a function a *refining map*. Note that the refining map is not unique!

Exercise 5.3.8. Let X be a Hausdorff space. Let $\mathcal{U}$ be an open cover of X. Show that for any point $x \in X$ there is a refinement $\mathcal{V}$ of $\mathcal{U}$ so that x is in only one open set of $\mathcal{V}$.

Exercise 5.3.9. Show that any two open coverings have a common refinement.

Let $\mathcal{V}$ be a refinement of $\mathcal{U}$ and let r be a refining map. We obtain an induced map on cochains

$$\widetilde{r} : \check{C}^n(\mathcal{U}, \mathcal{F}) \to \check{C}^n(\mathcal{V}, \mathcal{F})$$

as follows. Let $f_{i_0,\ldots,i_n} \in \mathcal{F}(U_{i_0,\ldots,i_n})$. Since $\mathcal{V}$ is a refinement, there are indices $j_0, \ldots, j_n$ so that $r(j_k) = i_k$. Set $g_{j_0,\ldots,j_n}$ to be $f_{i_0,\ldots,i_n}$ restricted to $V_{j_0,\ldots,j_n}$ so that $g_{j_0,\ldots,j_n} \in \mathcal{F}(V_{i_0,\ldots,i_n})$. Doing this for each $f_{i_0,\ldots,i_n}$ for a cochain $(f_{i_0,\ldots,i_n})$ gives the desired map. This map induces a map on cohomology. We leave the proof of the following proposition to the reader.

Proposition 5.3.10. *The map* $\widetilde{r}$ *induces a map*

$$\mathrm{H}(r) : \check{\mathrm{H}}^n(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^n(\mathcal{V}, \mathcal{F})$$

for each n.

This gives a way to compare cohomology groups with respect to different covers, at least if one is a refinement of the other. However, it is not very useful if it depends upon the refining map. It turns out that it only depends on the covers $\mathcal{U}$ and $\mathcal{V}$ and not the refining map r!

Proposition 5.3.11. *The map* $\mathrm{H}(r)$ *depends only upon* $\mathcal{U}$ *and* $\mathcal{V}$ *and not the refining map* r.

Proof. Let $\mathcal{V}$ be a refinement of an open cover $\mathcal{U}$ of X. Let r and r' be refinements of $\mathcal{V}$. First, since we know that $\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}) = \check{\mathrm{H}}^0(\mathcal{V}, \mathcal{F})$, we have that $\mathrm{H}\, r$ is just the identity map on the 0-cocycles so there is nothing to prove in that case.

Let $h \in \check{\mathrm{H}}^n(\mathcal{U}, \mathcal{F})$ and let h be represented by the cocycle $(f_{i_0,\ldots,i_n})$. Then we have that $\mathrm{H}(r)(h)$ is represented by the n-cocycle $(g_{j_0,\ldots,j_n})$ where

$$g_{j_0,\ldots,j_n} = f_{r(j_0),\ldots,r(j_n)}$$

and $\mathrm{H}(r')(h)$ is represented by the n-cocycle $(g'_{j_0,\ldots,j_n})$ where

$$g'_{j_0,\ldots,j_n} = f_{r'(j_0),\ldots,r'(j_n)}$$

for every set of $(n+1)-$indices $(j_0, \ldots, j_n)$. In order to show that $\mathrm{H}(r)$ is independent of the choice of r, it is enough to show that the difference $g'_{j_0,\ldots,j_n} - g_{j_0,\ldots,j_n}$ is a n-coboundary.

Consider the n-cochain $(h_{k_0,\ldots,k_{n-1}})$ defined by

$$h_{k_0,\ldots,k_{n-1}} = \sum_{i=0}^{n-1} (-1)^i f_{r(k_0),\ldots,r(k_i),r'(k_i),\ldots,r'(k_{n-1})}.$$

One now uses that $(f_{i_0,\ldots,i_n})$ is a cocycle to see that

$$d^{n-1}((h_{k_0,\ldots,k_{n-1}})) = (g'_{j_0,\ldots,j_n} - g_{j_0,\ldots,j_n}).$$

Thus, we have that the two cocycles differ by a coboundary and so are equal as cohomology classes, as desired. □

As the map $\mathrm{H}(r)$ depends only on the open covers $\mathcal{U}$ and $\mathcal{V}$, we denote it as $\mathrm{H}^{\mathcal{U}}_{\mathcal{V}}$ from now on. Note that if $\mathcal{W} \prec \mathcal{V} \prec \mathcal{U}$, then

$$\mathrm{H}^{\mathcal{U}}_{\mathcal{W}} = \mathrm{H}^{\mathcal{V}}_{\mathcal{W}} \circ \mathrm{H}^{\mathcal{U}}_{\mathcal{V}}.$$

It is also not hard to check that these maps commute with any ϕ_* induced by a map ϕ of sheaves.

Definition 5.3.12. Let $\mathcal{F}$ be a sheaf on X. For any $n \geq 0$, we define the n^{th} *Čech cohomology group of $\mathcal{F}$ on X* to be

$$\check{\mathrm{H}}^n(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^n(\mathcal{U}, \mathcal{F}).$$

Note that we have $\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$ for all open covers $\mathcal{U}$. Thus, we have that $\check{\mathrm{H}}^n(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ as well. This combined with the fact that we clearly have $\check{\mathrm{H}}^n(X, \mathcal{F}) = 0$ for all $n < 0$ gives that these cohomology groups satisfy condition (a) of Definition 5.2.1.

We have natural maps from $\check{\mathrm{H}}^n(\mathcal{U}, \mathcal{F})$ to $\check{\mathrm{H}}^n(X, \mathcal{F})$ for each open cover $\mathcal{U}$ and each $n \geq 0$ by the definition of a direct limit. Thus, given a cohomology class $h \in \check{\mathrm{H}}^n(\mathcal{U}, \mathcal{F})$, we obtain a cohomology class in $\check{\mathrm{H}}^n(X, \mathcal{F})$. Moreover, the image of h in $\check{\mathrm{H}}^n(X, \mathcal{F})$ is zero precisely if there is a refinement $\mathcal{V}$ of $\mathcal{U}$ so that $\mathrm{H}^{\mathcal{U}}_{\mathcal{V}}(h) = 0$.

As one should expect, there is a universal property for direct limits. One can use this to see that given a sheaf map $\phi : \mathcal{F} \to \mathcal{G}$, the induced maps $\phi_* : \check{\mathrm{H}}^n(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^n(\mathcal{V}, \mathcal{F})$ induce a map

$$\phi_* : \check{\mathrm{H}}^n(X, \mathcal{F}) \longrightarrow \check{\mathrm{H}}^n(X, \mathcal{G})$$

for each $n \geq 0$. This induced map is functorial, i.e., we have that $\mathrm{id}_* = \mathrm{id}$ and $(\phi \circ \psi)_* = \phi_* \circ \psi_*$. This gives conditions (d) and (e) of Definition 5.2.1.

Our next step is to give a long exact sequence in cohomology.

Let

$$0 \longrightarrow \mathcal{K} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\phi} \mathcal{G} \longrightarrow 0$$

be a short exact sequence of sheaves. From this we obtain, for each n and each open cover $\mathcal{U}$, an exact sequence

$$0 \longrightarrow \check{C}^n(\mathcal{U}, \mathcal{K}) \xrightarrow{\psi} \check{C}^n(\mathcal{U}, \mathcal{F}) \xrightarrow{\phi} \check{C}^n(\mathcal{U}, \mathcal{G}).$$

As usual, we do not know that the last map is surjective. However, we can write exact sequences

$$0 \longrightarrow \check{C}^n(\mathcal{U}, \mathcal{K}) \xrightarrow{\psi} \check{C}^n(\mathcal{U}, \mathcal{F}) \xrightarrow{\phi} \phi(\check{C}^n(\mathcal{U}, \mathcal{F})) \longrightarrow 0$$

for each $\mathcal{U}$ and each n. Thus, we obtain a short exact sequence of chain complexes

$$0 \longrightarrow \check{C}^*(\mathcal{U}, \mathcal{K}) \xrightarrow{\psi} \check{C}^*(\mathcal{U}, \mathcal{F}) \xrightarrow{\phi} \phi(\check{C}^*(\mathcal{U}, \mathcal{F})) \longrightarrow 0.$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \check{C}^*(\mathcal{U},\mathcal{K}) & \xrightarrow{\psi} & \check{C}^*(\mathcal{U},\mathcal{F}) & \xrightarrow{\phi} & \phi(\check{C}^*(\mathcal{U},\mathcal{F})) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \check{C}^*(\mathcal{V},\mathcal{K}) & \xrightarrow{\psi} & \check{C}^*(\mathcal{V},\mathcal{F}) & \xrightarrow{\phi} & \phi(\check{C}^*(\mathcal{V},\mathcal{F})) & \longrightarrow & 0.
\end{array}$$

Let $\mathcal{V}$ be a refinement of $\mathcal{U}$. Then we obtain a commutative diagram

Thus, from our general cohomology theory in § 3.2 we see that there exists a map δ^n so that the following diagram commutes:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \mathrm{H}^{n-1}(\phi(\check{C}^*(\mathcal{U},\mathcal{F}))) & \xrightarrow{\delta^{n-1}} & \check{\mathrm{H}}^n(\mathcal{U},\mathcal{K}) & \xrightarrow{\psi_*} & \check{\mathrm{H}}^n(\mathcal{U},\mathcal{F}) & \xrightarrow{\phi_*} & \mathrm{H}^n(\phi(\check{C}^*(\mathcal{U},\mathcal{F}))) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathrm{H}^{n-1}(\phi(\check{C}^*(\mathcal{V},\mathcal{F}))) & \xrightarrow{\delta^{n-1}} & \check{\mathrm{H}}^n(\mathcal{V},\mathcal{K}) & \xrightarrow{\psi_*} & \check{\mathrm{H}}^n(\mathcal{V},\mathcal{F}) & \xrightarrow{\phi_*} & \mathrm{H}^n(\phi(\check{C}^*(\mathcal{V},\mathcal{F}))) & \longrightarrow & \cdots .
\end{array}$$

Thus, taking direct limits we obtain the long exact sequence

$$\cdots \to \varinjlim_{\mathcal{U}} \mathrm{H}^{n-1}(\phi(\check{C}^*(\mathcal{U},\mathcal{F}))) \xrightarrow{\delta^{n-1}} \check{\mathrm{H}}^n(X,\mathcal{K}) \xrightarrow{\psi_*} \check{\mathrm{H}}^n(X,\mathcal{F}) \xrightarrow{\delta^n} \varinjlim_{\mathcal{U}} \mathrm{H}^n(\phi(\check{C}^*(\mathcal{U},\mathcal{F}))) \to \cdots .$$

Thus, to obtain the desired long exact sequence it remains to show that

$$\check{\mathrm{H}}^n(X,\mathcal{G}) \cong \varinjlim_{\mathcal{U}} \mathrm{H}^n(\phi(\check{C}^*(\mathcal{U},\mathcal{F}))).$$

Set $\widetilde{C}^n(\mathcal{U}) = \check{C}^n(\mathcal{U},\mathcal{G})/\phi(\check{C}^n(\mathcal{U},\mathcal{F}))$. From this we obtain an exact sequence of cochain complexes

$$0 \to \phi(\check{C}^*(\mathcal{U},\mathcal{F}) \longrightarrow \check{\mathrm{H}}^*(X,\mathcal{G}) \longrightarrow \widetilde{C}^*(\mathcal{U}) \to 0.$$

By taking the long exact sequence in cohomology for this short exact sequence of chain complexes we see it is enough to show that

$$\varinjlim_{\mathcal{U}} \mathrm{H}^n(\widetilde{C}^*(\mathcal{U})) = 0$$

for all n. We will show this by showing that for each $f \in \check{C}^n(\mathcal{U},\mathcal{G})$, there a refinement $\mathcal{V}$ of $\mathcal{U}$ so that the restriction of f lies in $\phi(\check{C}^n(\mathcal{V},\mathcal{F}))$. Let $\mathcal{V} = \{V_i\}$ be a refinement of $\mathcal{U}$ so that $\mathrm{Cl}(V_i) \subset U_i$ for each $i \in I$. For each $x \in X$, choose an open neighborhood W_x satisfying

1. $W_x \subset V_i$ for some i.
2. If $W_x \cap V_i \neq \emptyset$ then $W_x \subset U_i$.

3. W_x lies in the intersection of all the U_i containing x.

4. Let $U_{i_0}, \ldots, U_{i_n}$ be in $\mathcal{U}$ so that $U_{i_0,\ldots,i_n} \neq \emptyset$. If $x \in U_{i_0,\ldots,i_n}$, then the restriction of f from $U_{i_0,\ldots,i_n}$ to W_x is the image of a section of $\mathcal{F}$ over W_x.

Note that it is possible to satisfy the last condition since there are only finitely many sets $U_{i_0}, \ldots, U_{i_n}$ so that $x \in U_{i_0,\ldots,i_n}$. Let $\mathcal{W} = \{W_x\}$. For every $x \in X$, choose $V_x \in \mathcal{V}$ and $U_x \in \mathcal{U}$ so that $W_x \subset V_x \subset U_x$. Thus, we have that $\mathcal{W}$ is a refinement of $\mathcal{U}$. Let $W_{x_0}, \ldots, W_{x_n}$ be in $\mathcal{W}$ so that $W_{x_0,\ldots,x_n} \neq \emptyset$. Then for $0 \leq i \leq n$ we have $W_{x_0} \cap U_{x_i} \neq \emptyset$ and so by the above conditions we see that $W_{x_0} \subset U_{x_i}$ and so $W_{x_0} \subset U_{x_0,\ldots,x_n}$. Now we have that by (4) above that $\mathcal{W}$ is the refinement we seek so that the restriction of f lies in $\phi(\check{C}^n(\mathcal{W}, \mathcal{F}))$. This gives condition (d) of Definition 5.2.1. We can also use this construction to get (f) as well.

It only remains to prove condition (b) to see that the Čech cohomology groups as defined give a sheaf cohomology theory as in Definition 5.2.1. Let $\mathcal{F}$ be a fine sheaf and let $n > 0$. It is enough to prove that $\check{\mathrm{H}}^n(\mathcal{U}, \mathcal{F}) = 0$ for a locally finite cover $\mathcal{U}$. Let $\{\phi_i\}$ be a partition of unity for $\mathcal{F}$ with respect to the cover $\mathcal{U}$. We will define homomorphisms $\Psi_n : \check{C}^n(\mathcal{U}, \mathcal{F}) \to \check{C}^{n-1}(\mathcal{U}, \mathcal{F})$ for each $n \geq 1$. Let $f \in \check{C}^n(\mathcal{U}, \mathcal{F})$ and let $\{U_0, \ldots, U_{n-1}\}$ be sets in $\mathcal{U}$ so that $U_{0,\ldots,n-1} \neq \emptyset$. We have that $\phi_j \circ f_{j,0,\ldots,n-1}$ has support in $U_{j,0,\ldots,n-1}$. Thus, we can extend $\phi_j \circ f_{j,0,\ldots,n-1}$ to a continuous section of $\mathcal{F}$ over $U_{j,0,\ldots,n-1}$. Consider $\phi_j \circ f_{j,0,\ldots,n-1}$ as this section over $U_{0,\ldots,n-1}$. Define

$$\Psi_n(f_{0,\ldots,n-1}) = \sum_j \phi_j \circ f_{j,0,\ldots,n-1}.$$

Then it follows that

$$d^{n-1} \circ \Psi_n + \Psi_{n+1} \circ d^{n-1} = \mathrm{id}$$

for all $n \geq 1$. Thus, if f is a n-cocycle with $n > 0$, there is a $(n-1)$-cochain $\Psi_n(f)$ so that $d^{n-1}\Psi_n(f) = f$. Hence we have that $\check{\mathrm{H}}^n(\mathcal{U}, \mathcal{F}) = 0$ and so we have the result.

Thus, we have that Čech cohomology gives a sheaf cohomology theory for X as given in Definition 5.2.1. In particular, we have the following results (need to justify why the others give sheaf cohomology theories as well).

Theorem 5.3.13. *Let X be a smooth manifold. Given an abelian group G we have*

$$\check{\mathrm{H}}^n(X, \underline{G}) \cong \mathrm{H}^n_{\mathrm{B}}(X; G)$$

where we denote the singular cohomology group as H_{B} to represent the fact that these groups are often referred to as the Betti cohomology groups. If we define singular cohomology in terms of differentiable simplices instead of continuous ones, we obtain cohomology groups denoted by $\mathrm{H}_{\Delta^\infty}$. In this case we have

$$\check{\mathrm{H}}^n(X, \underline{\mathbb{R}}) \cong \mathrm{H}^n_{\mathrm{dR}}(X) \cong \mathrm{H}^n_{\Delta^\infty}(X; \mathbb{R}).$$

Even though the previous result allows us to compute many Čech cohomology groups from what we have already done, we add a few examples here before moving on to algebraic sheaves in the next section.

Theorem 5.3.14. *Let X be a Riemann surface. For any $n \geq 1$ we have*

1. $\check{\mathrm{H}}^n(X, C^\infty) = 0$,
2. $\check{\mathrm{H}}^n(X, \Omega^1_X) = 0$,
3. $\check{\mathrm{H}}^n(X, \Omega^{1,0}_X) = 0$,
4. $\check{\mathrm{H}}^n(X, \Omega^{0,1}_X) = 0$,
5. $\check{\mathrm{H}}^n(X, \Omega^2_X) = 0$.

Proof. We prove that $\check{\mathrm{H}}^1(\mathcal{U}, C^\infty) = 0$ for every open covering. The same argument works for $\check{\mathrm{H}}^n(\mathcal{U}, C^\infty)$ by just keeping track of more indices. Furthermore, the other results follow with similar arguments and are left as an exercise. Alternatively, one can view that each of these sheaves is fine and so the result follows from what was shown above.

Fix an open cover $\mathcal{U} = \{U_i\}$ of X. Let (f_{ij}) be a 1-cocycle for the sheaf C^∞ and the covering $\mathcal{U}$. Let $\{\phi_j\}$ be a partition of unity with respect to the cover $\mathcal{U}$. Consider the function $\phi_j f_{ij}$ and extend it by 0 outside of $\mathrm{supp}(\phi_i)$ and consider it as a C^∞ function on all of U_i. Set $g_i = -\sum_j \phi_j f_{ij}$, which is also a C^∞ function defined on U_i. Now, we can use that (f_{ij}) is a 1-cocycle to see that

$$\begin{aligned} g_j - g_i &= -\sum_k \phi_k f_{jk} + \sum_k \phi_k f_{ik} \\ &= \sum_k \phi_k (f_{ik} - f_{jk}) \\ &= \sum_k \phi_k f_{ij} \\ &= f_{ij}. \end{aligned}$$

However, we know that $d^0((g_k)) = (g_i - g_j)$ and so (f_{ij}) is a coboundary. Since every cocycle is a coboundary, we obtain the result. □

We can use a variation of the above method to prove the analogous result for skyscraper sheaves. However, we first need an integer-valued version of a partition of unity.

Lemma 5.3.15. *Let X be a topological space and $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. There is a collection of integer-valued functions $\{\phi_i\}$ on X satisfying*

1. *every point $x \in X$ lies in only finitely many of the support sets of the ϕ_i,*
2. *for every $x \in X$, $\sum_i \phi_i(x) = 1$,*

3. $\mathrm{supp}(\phi_i) \subset U_i$ *for every* $i \in I$.

Proof. Put an order on the index set I. Define

$$\phi_i(x) = \begin{cases} 1 & \text{if } x \in U_i - \bigcup_{j<i} U_j \\ 0 & \text{otherwise.} \end{cases}$$

These functions work. □

Theorem 5.3.16. *Let* X *be a topological space and* $\mathcal{F}$ *a skyscraper sheaf on* X. *Then* $\check{\mathrm{H}}^n(X, \mathcal{F}) = 0$ *for all* $n \geq 1$.

Proof. Again we only prove this for $n = 1$ as the general case follows from the same arguments, one just needs to keep track of more indices.

Note that if f is a section of $\mathcal{F}$ over U and ϕ is any $\mathbb{Z}$-valued function defined on U, then ϕf is also a section of $\mathcal{F}$ over U. This allows us to use the integer-valued partition of unity constructed above. This statement would not be true if one used a regular partition of unity as ϕf would not necessarily be a section of $\mathcal{F}$ over U anymore.

Let $\mathcal{U} = \{U_i\}$ be an open cover of X and let $\{\phi_i\}$ be an integer-valued partition of unity. Let (f_{ij}) be a 1-cocycle for $\mathcal{F}$ for this covering. Consider the section $\phi_j f_{ij}$ and extend it by zero outside of $\mathrm{supp}(\phi_i)$, considering it as a section of $\mathcal{F}$ over U_i. As above, set $g_i = -\sum_j \phi_j f_{ij}$. Then g_j is also a section of $\mathcal{F}$ defined over U_i. We have that $(f_{ij}) = d^0(g_i)$ exactly as above, which gives the result. □

In particular, we have the following corollary which lists the cases of most interest.

Corollary 5.3.17. *Let* X *be a Riemann surface. Then:*

1. *for any* $x \in X$, $\check{\mathrm{H}}^n(X, \mathbb{C}_x) = 0$ *for all* $n \geq 1$,
2. $\check{\mathrm{H}}^n(X, Div_X) = 0$ *for all* $n \geq 1$,
3. *for any divisor* D *on* X, $\check{\mathrm{H}}^n(X, \mathcal{T}_X[D]) = 0$ *for all* $n \geq 1$,
4. *for any pair of divisors* D_1 *and* D_2 *with* $D_1 \leq D_2$, $\check{\mathrm{H}}^n(X, \mathcal{T}_X[D_1/D_2]) = 0$ *for all* $n \geq 1$.

We can use Theorem 5.3.13 and our previous results calculating singular cohomology groups to give the Čech cohomology groups for locally constant sheaves.

Corollary 5.3.18. *Let* X *be a contractible Riemann surface and let* G *be an abelian group. Then*

1. $\check{\mathrm{H}}^0(X, \underline{G}) \cong G$
2. $\check{\mathrm{H}}^n(X, \underline{G}) = 0$ *for all* $n \geq 1$.

Corollary 5.3.19. *Let X be a compact Riemann surface of genus g, i.e., the torus with g holes. Let G be an abelian group. Then*

1. $\check{\mathrm{H}}^0(X, \underline{G}) \cong G$,

2. $\check{\mathrm{H}}^1(X, \underline{G}) \cong G^{2g}$,

3. $\check{\mathrm{H}}^2(X, \underline{G}) \cong G$,

4. $\check{\mathrm{H}}^n(X, \underline{G}) = 0$ *for all* $n > 2$.

We can use these results to show that $\check{\mathrm{H}}^n(X, \mathcal{O}_X[D]) = 0$ for $n \geq 2$.

Theorem 5.3.20. *Let X be a Riemann surface and let D be a divisor on X. Then for $n \geq 2$ we have $\check{\mathrm{H}}^n(X, \mathcal{O}_X[D]) = 0$.*

Proof. We begin with the case that $D = 0$ so we are just dealing with the sheaf of holomorphic functions $\mathcal{O}_X$. Given a form $\omega = \sum_{I,J} f_{I,J} dz_I \wedge d\overline{z}_J \in \Omega_X^{p,q}(U)$, we define

$$\overline{d} : \Omega_X^{p,q}(U) \to \Omega_X^{p,q+1}(U)$$

by setting

$$\overline{d}(\omega) = \sum_{I,J,j} \frac{\partial}{\partial \overline{z}_j} f_{I,J}(z) d\overline{z}_j \wedge dz_I \wedge d\overline{z}_J.$$

This gives the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow C_X^\infty \xrightarrow{\overline{d}} \Omega_X^{0,1} \longrightarrow 0.$$

We now use the long exact sequence in cohomology to obtain sequences

$$\check{\mathrm{H}}^n(X, \Omega_X^{0,1}) \xrightarrow{\delta^n} \check{\mathrm{H}}^{n+1}(X, \mathcal{O}_X) \longrightarrow \check{\mathrm{H}}^{n+1}(X, C_X^\infty)$$

for every $n \geq 0$. However, we know that the first and last terms of the sequence vanish for $n \geq 1$, so the result follows in this case.

For the general case we consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X[D_1] \longrightarrow \mathcal{O}_X[D_2] \xrightarrow{\alpha_{D_1/D_2}} \mathcal{T}_X[D_1/D_2] \longrightarrow 0$$

for $D_1 \leq D_2$. Using the associated long exact sequence we obtain

$$\check{\mathrm{H}}^{n-1}(X, \mathcal{T}_X[D_1/D_2]) \to \check{\mathrm{H}}^n(X, \mathcal{O}_X[D_1]) \to \check{\mathrm{H}}^n(X, \mathcal{O}_X[D_2]) \to \check{\mathrm{H}}^n(X, \mathcal{T}_X[D_1/D_2]).$$

Now for $n \geq 2$ we know that the two sequences on the end vanish. Thus, we have

$$\check{\mathrm{H}}^n(X, \mathcal{O}_X[D_1]) \cong \check{\mathrm{H}}^n(X, \mathcal{O}_X[D_2])$$

for all $n \geq 2$ as long as $D_1 \leq D_2$.

Let D be a divisor and write $D = D_1 - D_2$ with $D_1, D_2 \geq 0$. Then we have

$$\begin{aligned}\check{\mathrm{H}}^n(X, \mathcal{O}_X[D]) &\cong \check{\mathrm{H}}^n(X, \mathcal{O}_X[D_1]) \qquad (\text{since } D \leq D_1)\\ &\cong \check{\mathrm{H}}(X, \mathcal{O}_X) \qquad (\text{since } 0 \leq D_1)\\ &= 0\end{aligned}$$

where the last equality uses the base case $D = 0$. Thus, we have the result. ☐

Corollary 5.3.21. *Let X be a Riemann surface and D a divisor on X. Then for $n \geq 2$ we have $\check{\mathrm{H}}^n(X, \Omega^1_{X,\mathrm{hol}}[D]) = 0$.*

Proof. Let ω be a meromorphic 1-form on X and consider the *canonical divisor* associated to ω given by

$$K = \sum_x \mathrm{ord}_x(\omega) \cdot x.$$

Then one can check that there is an isomorphism of sheaves

$$\mathcal{O}_X[K + D] \xrightarrow{\simeq} \Omega^1_{X,\mathrm{hol}}[D]$$

given by multiplication by ω. Combining this with the previous proposition gives the result. ☐

5.4 Algebraic Sheaves

In this section we will study the analogous algebraic theory for Riemann surfaces. As such, we will consider our spaces with the Zariski topology. We say $U \subset X$ is a *cofinite* if $X - U$ is a finite set.

Definition 5.4.1. Let X be a compact Riemann surface. The *Zariski topology* on X is the topology where open sets are given by cofinite sets along with the empty set. When we consider X with the Zariski topology we write X_{Zar}.

We recall the following basic facts about the Zariski topology that will be useful:

1. X_{Zar} is not Hausdorff,
2. X_{Zar} is compact,
3. Any two nonempty open sets of X_{Zar} intersect nontrivially.

Note that if U is open in the Zariski topology, it also open in the classical topology. Thus, for a compact manifold we have that the Zariski topology is a subtopology of the classical topology. In particular, given a sheaf $\mathcal{F}$ on X, we obtain an *algebraic sheaf* $\mathcal{F}_{\mathrm{alg}}$ on X_{Zar} by restricting the sheaf to the Zariski open sets. We can determine what these algebraic sheaves look like fairly easily.

Example 5.4.2. The sheaf $\mathcal{O}_{X,\mathrm{alg}}$ on X is given by

$$\mathcal{O}_{X,\mathrm{alg}}(U) = \{f \in \mathcal{M}(X) : f \in \mathcal{O}_X(U)\}.$$

Note here that we are requiring the functions to be globally meromorphic and holomorphic on U, where the sheaf $\mathcal{O}_X$ only required the functions to be holomorphic on U and made no global constraints. This follows because of the property above that any two nonempty open sets in X_{Zar} have nontrivial intersection. We call $\mathcal{O}_{X,\mathrm{alg}}$ the *sheaf of regular functions* on X. The terminology arises from algebraic geometry, but applies here as well. Observe that we have an inclusion of sheaves given by

$$\mathcal{O}_{X,\mathrm{alg}} \subset \mathcal{O}_X.$$

Example 5.4.3. Consider a divisor D on X. We define the *sheaf of rational functions with poles bounded by* D on X by setting

$$\mathcal{O}_{X,\mathrm{alg}}[D] = \{f \in \mathcal{M}(X) : \mathrm{div}(f) \geq -D \text{ for all points of } U\}.$$

We have a natural inclusion here as well given by

$$\mathcal{O}_{X,\mathrm{alg}}[D] \subset \mathcal{O}_X[D].$$

Note that in each of the definitions above of the associated algebraic sheaves we had that the functions were globally meromorphic. The algebraic version of $\mathcal{M}_X$ is $\mathcal{M}_{X,\mathrm{alg}}$, and is a constant sheaf since every two open sets intersect in X_{Zar}. Thus, the sections of $\mathcal{M}_{X,\mathrm{alg}}$ are given by $\mathcal{M}_X(X)$ for any open set U.

Example 5.4.4. We can also consider the algebraic forms as well. For instance, consider the group of meromorphic 1-forms $\mathcal{M}_X^{(1)}(X)$. This is a 1-dimensional vector space over the field $\mathcal{M}_X(X)$ generated by any non-zero 1-form. One can associate an algebraic sheaf to $\mathcal{M}_X^{(1)}$, which is again a constant sheaf. The global sections of $\mathcal{M}_{X,\mathrm{alg}}^{(1)}$ is given by $\mathcal{M}_X^{(1)}$ for any open set. Similarly, we have the *sheaf of regular 1-forms*

$$\Omega^1_{X,\mathrm{alg}}(U) = \{\omega \in \mathcal{M}_X^{(1)}(X) : \omega \in \Omega^1_{X,\mathrm{hol}}(U)\}$$

and given a divisor D on X the *sheaf of rational 1-forms with poles bounded by* D is given by

$$\Omega^1_{X,\mathrm{alg}}[D](U) = \{\omega \in \mathcal{M}_X^{(1)}(X) : \mathrm{div}(\omega) \geq -D \text{ for all points in } U\}.$$

We again have the natural inclusions

$$\Omega^1_{X,\mathrm{alg}} \hookrightarrow \Omega^1_{X,\mathrm{hol}}$$
$$\Omega^1_{X,\mathrm{alg}}[D] \hookrightarrow \Omega^1_{X,\mathrm{hol}}[D]$$
$$\mathcal{M}^{(1)}_{X,\mathrm{alg}} \hookrightarrow \mathcal{M}^{(1)}_X.$$

Exercise 5.4.5. Check that the algebraic "sheaves" defined above are actually sheaves.

Exercise 5.4.6. Show that the stalk $\mathcal{O}_{X,\mathrm{alg},x}$ of the sheaf $\mathcal{O}_{X,\mathrm{alg}}$ at the point x is the subring of the rational function field $\mathcal{M}_X(X)$ consisting of those rational functions which are holomorphic at the point x.

We can now given the motivation for the definition of the Zariski topology. The following proposition shows that when considering sections of algebraic sheaves, ones only needs the cofinite sets.

Proposition 5.4.7. *Let D be a divisor on X and consider the algebraic sheaf $\mathcal{O}_{X,\mathrm{alg}}[D]$. For any open set U and any $f \in \mathcal{O}_{X,\mathrm{alg}}[D](U)$ there is a cofinite open set V with $U \subset V \subset X$ so that the restriction map*

$$\rho_U^V : \mathcal{O}_{X,\mathrm{alg}}[D](V) \longrightarrow \mathcal{O}_{X,\mathrm{alg}}[D](U)$$

so that f lies in the image of the restriction map. The same statement holds for the algebraic sheaves $\Omega^1_{X,\mathrm{alg}}[D]$.

Proof. Let $f \in \mathcal{O}_{X,\mathrm{alg}}[D](U)$. Since $f \in \mathcal{M}_X(X)$, we know that f has a finite number of poles overall, and so in particular a finite number of poles not lying in U. Let $x_1, \dots, x_m$ be the poles of f outside U. The divisor D has finite support by definition, so there are finitely many points $y_1, \dots, y_n$ outside of U with $D(y_i) < 0$.

Set V to be the complement of $\{x_i\} \cup \{y_j\}$. By construction we have $\mathrm{div}(f) \geq -D$ on all of V since it is on all of U and at any point $x \in V - U$ we have $\mathrm{div}(f)(x) \geq 0$ and $D(x) \geq 0$. Thus, we have $f \in \mathcal{O}_{X,\mathrm{alg}}[D](V)$.

The same proof works for $\Omega^1_{X,\mathrm{alg}}[D]$. □

Let f be a meromorphic function on X. We say that f has *multiplicity 1 at* $x \in X$ if either f is holomorphic at x and $\mathrm{ord}_x(f - f(x)) = 1$ or f has a simple pole at x.

Definition 5.4.8. Let $\mathcal{S}$ be a set of meromorphic functions on X. We say that $\mathcal{S}$ *separates points of* X if for every pair of points $x, y \in X$ with $x \neq y$ there is a meromorphic function $f \in \mathcal{S}$ so that $f(x) \neq f(y)$. We say that $\mathcal{S}$ *separates tangents of* X if for every $x \in X$ there is a meromorphic function $f \in \mathcal{S}$ which has multiplicity 1 at x.

We call X an *algebraic curve* if the field $\mathcal{M}(X)$ of global meromorphic functions separates the points and tangents of X. The following is a deep theorem, but one that we will assume.

Theorem 5.4.9. *Every compact Riemann surface is an algebraic curve.*

We can construct Čech cohomology on X_{Zar} in the exact same manner as was used in § 5.3, the only difference being the open sets under consideration here. In this way we obtain the Čech cohomology groups of the sheaf $\mathcal{F}$ on X_{Zar}

$$\check{\mathrm{H}}^n(X_{\mathrm{Zar}}, \mathcal{F}).$$

Proposition 5.4.10. *Let G be an abelian group and $\underline{G}$ the associated sheaf on X_{Zar}. For every $n \geq 1$ we have*

$$\check{\mathrm{H}}^n(X_{\mathrm{Zar}}, \underline{G}) = 0.$$

Proof. Note that since all open sets in X_{Zar} have nontrivial intersection, the locally constant sheaf $\underline{G}$ is actually a constant sheaf and $\underline{G}(U) = G$ for all open U in X_{Zar}. We prove the result in the case that $n = 1$. As in the computations in § 5.3, the arguments for general n are the same up to keeping track of more indices.

Let f be a cohomology class in $\check{\mathrm{H}}^1(X_{\mathrm{Zar}}, \underline{G})$. We can represent f as (f_{ij}) for some open cover $\mathcal{U} = \{U_i\}$. We can assume that $\mathcal{U}$ is a finite open cover since X_{Zar} is compact. Write $\mathcal{U} = \{U_0, \dots, U_n\}$. Since f is a cocycle we have $f_{ii} = 0$ for $0 \leq i \leq n$ and $f_{ij} = -f_{ji}$ for all $i \neq j$. Thus, the cocycle f is completely determined by the f_{ij} with $i < j$. In fact, one can do better. The cocycle condition gives that for $i < j < k$,

$$f_{ik} = f_{ij} + f_{jk}$$

and each of these elements make sense since all open sets intersect. Moreover, if one has $f_{i,i+1}$ chosen arbitrarily in G one recovers the cocycle condition by setting

$$f_{ij} = \sum_{k=i}^{j-1} f_{k,k+1}$$

for all $i < j$. Thus, the cocycle f is completely determined by the $f_{i,i+1}$ for each $0 \leq i \leq n-1$.

Set $g_0 = 0$ and for $i \geq 1$, set

$$g_i = \sum_{k=0}^{i-1} f_{k,k+1}.$$

Then we have that (g_i) is a 0-cocycle for the sheaf $\underline{G}$ and we clearly have $f_{ij} = g_i - g_j$ for $i < j$. Thus, f is coboundary and so is zero in $\check{\mathrm{H}}^1(X_{\mathrm{Zar}}, \underline{G})$. $\square$

The particular case of this proposition we are interested in is the following corollary.

Corollary 5.4.11. *For $n \geq 1$ we have*

$$\check{\mathrm{H}}^n(X_{\mathrm{Zar}}, \mathcal{M}_{X,\mathrm{alg}}) = 0$$

and

$$\check{\mathrm{H}}^n(X_{\mathrm{Zar}}, \mathcal{M}^{(1)}_{X,\mathrm{alg}}) = 0.$$

It turns out that in this case one does not necessarily obtain a long exact sequence of cohomology groups for X_{Zar} from a short exact sequence of sheaves on X_{Zar} because X_{Zar} is not paracompact. One does have a long exact sequence up through $\check{\text{H}}^1$, i.e., given a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

on X_{Zar}, one obtains a long exact sequence

$$0 \to \Gamma(X_{\text{Zar}}, \mathcal{K}) \to \Gamma(X_{\text{Zar}}, \mathcal{F}) \to \Gamma(X_{\text{Zar}}, \mathcal{G}) \to \check{\text{H}}^1(X_{\text{Zar}}, \mathcal{K}) \to \check{\text{H}}^1(X_{\text{Zar}}, \mathcal{F}) \to \check{\text{H}}^1(X_{\text{Zar}}, \mathcal{G}).$$

We now have two ways to view the space X, as a Riemann surface with standard topology or with the Zariski topology. As is standard when working in both settings, we write X_{an} when we are working in the analytic setting considering X with standard topology. Given a divisor D on X, it would be nice if there was a way to compare the groups $\check{\text{H}}^n(X_{\text{an}}, \mathcal{O}_X[D])$ and $\check{\text{H}}^n(X_{\text{Zar}}, \mathcal{O}_{X,\text{alg}}[D])$ as well as the groups $\check{\text{H}}^n(X_{\text{an}}, \Omega^1_{X,\text{hol}}[D])$ and $\check{\text{H}}^n(X_{\text{Zar}}, \Omega^1_{X,\text{alg}}[D])$.

Recall that $\mathcal{O}_{X,\text{alg}}[D]$ is a subsheaf of $\mathcal{O}_X[D]$. The inclusion map induces a map on cohomology

$$j_1 : \check{\text{H}}^n(X_{\text{an}}, \mathcal{O}_{X,\text{alg}}[D]) \longrightarrow \check{\text{H}}^n(X_{\text{an}}, \mathcal{O}_X[D]).$$

We also have that the Zariski topology is a subtopology of the standard topology in this case, so any Zariski open cover is a classical open cover and so any cochain for the Zariski topology is a cochain for the standard topology as well. Thus, we obtain a map

$$j_2 : \check{\text{H}}^n(X_{\text{Zar}}, \mathcal{O}_{X,\text{alg}}[D]) \longrightarrow \check{\text{H}}^n(X_{\text{an}}, \mathcal{O}_{X,\text{alg}}[D]).$$

We can compose j_2 with j_1 to obtain a map

$$j : \check{\text{H}}^n(X_{\text{Zar}}, \mathcal{O}_{X,\text{alg}}[D]) \longrightarrow \check{\text{H}}^n(X_{\text{an}}, \mathcal{O}_X[D]).$$

The same construction yields a map

$$j^1 : \check{\text{H}}^n(X_{\text{Zar}}, \Omega^1_{X,\text{alg}}[D]) \longrightarrow \check{\text{H}}^n(X_{\text{an}}, \Omega^1_{X,\text{hol}}[D]).$$

It turns out that a deep result of Serre, known as a GAGA theorem (Geometrie Analytique et Geometrie Algebrique), gives that these maps are actually isomorphisms. Serre's theorem is much more general, but for our set-up it is given as follows.

Theorem 5.4.12. *([10]) Let X be an algebraic curve. Then for any divisor D, the comparison maps*

$$j : \check{\text{H}}^n(X_{\text{Zar}}, \mathcal{O}_{X,\text{alg}}[D]) \longrightarrow \check{\text{H}}^n(X_{\text{an}}, \mathcal{O}_X[D]).$$

and

$$j^1 : \check{\text{H}}^n(X_{\text{Zar}}, \Omega^1_{X,\text{alg}}[D]) \longrightarrow \check{\text{H}}^n(X_{\text{an}}, \Omega^1_{X,\text{hol}}[D]).$$

are group isomorphisms for all n.

Note that the theorem does not say we get the same thing for any sheaf! For example, if we look at the sheaf $\underline{G}$, we have seen that

$$\check{\mathrm{H}}^1(X_{\mathrm{Zar}}, \underline{G}) = 0$$

but

$$\check{\mathrm{H}}^1(X, \underline{G}) = G^{2g}$$

if g is the genus of X.

5.5 Further applications and computations

In this section we summarize some applications of the previous sections. Many of the results will be statements without proofs as they require a more thorough study of Riemann surfaces than the previous chapters provide at this point. As in the previous section, we assume X is a compact Riemann surface throughout this section.

Let D be a divisor on X. As in the analytic setting, we obtain a map of sheaves

$$\alpha_{D,\mathrm{alg}} : \mathcal{M}_{X,\mathrm{alg}} \longrightarrow \mathcal{T}_{X,\mathrm{alg}}[D]$$

given by truncation of Laurent series. In terms of the classical theory, we consider only the global sections for a moment. In this case we have a map

$$\alpha_D : \mathcal{M}_X(X) \longrightarrow \mathcal{T}_X[D](X).$$

Given a Laurent tail $f \in \mathcal{T}_X[D](X)$, a natural question to ask is if it is in the image of α_D. Note that f is a collection of Laurent tails, one for each point. So we are asking if there is a global meromorphic function that when expanded in a Laurent series at each point gives the Laurent tail at that point. The problem of constructing a meromorphic function $g \in \mathcal{M}_X(X)$ so that $\alpha_D(g) = f$ is known as the *Mittag-Leffler problem.* We set

$$\mathrm{H}^1(D) = \mathrm{coker}(\alpha_D).$$

This can be studied classically. For instance, it is known that $\mathrm{H}^1(D)$ is a finite dimensional $\mathbb{C}$-vector space ([7], Chapter VI, Proposition 2.7.)

Recall that a canonical divisor K on X is the divisor associated to a nonzero $\omega \in \mathcal{M}_X^{(1)}(X)$. One has the following version of Serre duality. For a more general version, see ([4], Chapter III, Theorem 7.6).

Theorem 5.5.1. *([7], Theorem 3.3) For D a divisor on X, and K a canonical divisor on X, one has that there is an isomorphism*

$$L^{(1)}(-D) \cong \mathrm{H}^1(D)^\vee.$$

In particular,

$$\dim_{\mathbb{C}} \mathrm{H}^1(D) = \dim L^{(1)}(-D) = \dim_{\mathbb{C}} L(K - D).$$

This theorem provides a crucial step in the proof of the Riemann-Roch theorem.

Theorem 5.5.2. *([7], Theorem 3.11) Let X have genus g. Then for any divisor D and any canonical divisor K, we have*

$$\dim_{\mathbb{C}} L(D) - \dim_{\mathbb{C}} L(K - D) = \deg(D) + 1 - g$$

where the degree of a divisor $D = \sum_x n_x \cdot x$ is given by $\sum_x n_x$.

Corollary 5.5.3. *With X and K as in the previous theorem, let D be a divisor of degree at least $\deg(K) + 1$. Then $\mathrm{H}^1(D) = 0$ and*

$$\dim_{\mathbb{C}} L(D) = \deg(D) + 1 - g.$$

Proof. The fact $\deg(D) > \deg(K)$ implies that $L(K - D) = 0$. To see this, we prove the following more general result: if D_1 is a divisor on X with $\deg(D_1) < 0$, then $L(D_1) = 0$. Suppose that $f \in L(D_1)$ and f is not identically 0. Consider the divisor $E = \mathrm{div}(f) + D_1$. Since $f \in L(D_1)$, $E \geq 0$ and so $\deg(E) \geq 0$. However, since $\deg(\mathrm{div}(f)) = 0$ we have $\deg(E) = \deg(D) \leq 0$. This contradiction gives the result modulo the result that $\deg(\mathrm{div}(f)) = 0$, which we omit the proof of as it would take us too far afield.

Thus, we have by Serre-duality that $\mathrm{H}^1(D) = 0$. The other result is immediate from the Riemann-Roch theorem. □

One should note that $\deg(K) = 2g - 2$ for any canonical divisor K, so we can make the previous corollary more precise if we grant this result.

Example 5.5.4. Let X be an algebraic curve of genus $g = 1$ and let P be a point on X. Observe that $\deg(K) = 0$ in this case. Thus, if D is a divisor with $\deg(D) > 0$, then $L(K - D) = 0$. Thus, Riemann-Roch in this case reads:

$$\dim L(D) = \deg(D).$$

Consider the divisor $D = P$. Then $\deg(P) = 1$ and so $L(P)$ is the field $\mathbb{C}$. We can take 1 as a basis for $L(P)$. Now consider the divisor $D = 2P$, which has degree 2 and so $\dim(2P) = 2$. Thus, there is a nonconstant function $x \in L(2P)$, i.e., x is a meromorphic function with a degree two pole at P. We can take $\{1, x\}$ as a basis for $L(2P)$. We have that $L(3P)$ has dimension 3. Since $\{1, x\}$ are both in $L(3P)$, we have a function $y \in L(3P)$ with a degree 3 pole at P. Continuing in this pattern, we have $L(4P)$ is spanned by $\{1, x, x^2, y\}$ and $L(5P)$ is spanned by $\{1, x, x^2, y, xy\}$. Things change when we reach $L(6P)$. We have that the set $\{1, x, x^2, x^3, y, y^2, xy\}$ is contained in $L(6P)$. However, since $\dim L(6P) = 6$, this set cannot be linearly independent. Thus, there are constants in $\mathbb{C}$ so that

$$y^2 + a_1 xy + a_2 y + a_3 x^3 + a_4 x^2 + a_5 x + a_6 = 0.$$

Note that this is precisely the equation giving an elliptic curve.

We can give a cohomological interpretation of $\mathrm{H}^1(D)$ via the following proposition. Note that this proposition is really saying that if one allows an arbitrarily bad pole outside of U, then one can arrange for any finite set of Laurent tails inside U.

Proposition 5.5.5. *Let D be a divisor on X. The map $\alpha_{D,\mathrm{alg}}$ is an onto map of sheaves on X_{Zar} with kernel $\mathcal{O}_{X,\mathrm{alg}}[D]$. Thus, we have a short exact sequence of sheaves*

$$0 \longrightarrow \mathcal{O}_{X,\mathrm{alg}}[D] \longrightarrow \mathcal{M}_{X,\mathrm{alg}} \xrightarrow{\alpha_{D,\mathrm{alg}}} \mathcal{T}_{X,\mathrm{alg}}[D] \longrightarrow 0.$$

Proof. The fact that $\mathcal{O}_{X,\mathrm{alg}}[D]$ is the kernel of $\alpha_{D,\mathrm{alg}}$ is clear. The real issue is showing that $\alpha_{D,\mathrm{alg}}$ is surjective. We will show that $\alpha_{D,\mathrm{alg}}$ is surjective on any open set U that is properly contained in X. This is clearly enough to show surjectivity.

Let $x \in X - U$ and let $f \in \mathcal{T}_{X,\mathrm{alg}}[D](U)$, i.e., f is a finite Laurent tail divisor supported on U. Let D_n be the divisor given by $D_n = D + n \cdot x$. Now, for large n we know that $\mathrm{H}^1(D_n) = 0$ by Corollary 5.5.3. Thus, for large n we have that the map α_{D_n} on global sections is surjective. Observe that $f \in \mathcal{T}_X[D_n](X)$ since f does not have x in its support. Thus, there is a global meromorphic function g with $\alpha_{D_n}(g) = f$. Thus, if we restrict g to U, then g is a preimage of f as well. ∎

Recall that we do not have a long exact sequence of cohomology for algebraic sheaves, but that we do have a long exact sequence through the first cohomology groups. Thus, the short exact sequence gives rise to

$$0 \longrightarrow L(D) \longrightarrow \mathcal{M}_X(X) \xrightarrow{\alpha_D} \mathcal{T}_X[D](X) \longrightarrow \check{\mathrm{H}}^1(X_{\mathrm{Zar}}, \mathcal{O}_{X,\mathrm{alg}}[D]) \longrightarrow 0$$

where we have used that $\check{\mathrm{H}}^1(X_{\mathrm{Zar}}, \mathcal{M}_{X,\mathrm{alg}}) = 0$. This gives that $\check{\mathrm{H}}^1(X_{\mathrm{Zar}}, \mathcal{O}_{X,\mathrm{alg}}[D])$ is isomorphic to the cokernel of α_D, i.e., that we have

$$\mathrm{H}^1(D) \cong \check{\mathrm{H}}^1(X_{\mathrm{Zar}}, \mathcal{O}_{X,\mathrm{alg}}[D]).$$

Proposition 5.5.6. *Let X be an algebraic curve of genus g. Let D be a divisor on X. The spaces $\check{\mathrm{H}}^1(X_{\mathrm{an}}, \mathcal{O}_X[D])$ and $\check{\mathrm{H}}^1(X_{\mathrm{an}}, \Omega^1_{X,\mathrm{hol}}[D])$ are finite dimensional. Moreover,*

$$\dim \check{\mathrm{H}}^1(X_{\mathrm{an}}, \mathcal{O}_X) = g$$

and

$$\dim \check{\mathrm{H}}^1(X_{\mathrm{an}}, \Omega^1_{X,\mathrm{hol}}) = 1.$$

If $\deg(D) \geq 2g - 1$, *then* $\check{\mathrm{H}}^1(X_{\mathrm{an}}, \mathcal{O}_X[D]) = 0$.

Proof. Observe that by Serre's GAGA theorem we have that

$$\check{\mathrm{H}}^1(X_{\mathrm{an}}, \mathcal{O}_X[D]) \cong \check{\mathrm{H}}^1(X_{\mathrm{Zar}}, \mathcal{O}_{X,\mathrm{alg}}[D]).$$

Combining this with the fact that

$$\check{\mathrm{H}}^1(X_{\mathrm{Zar}}, \mathcal{O}_{X,\mathrm{alg}}[D]) \cong \mathrm{H}^1(D)$$

and $\mathrm{H}^1(D)$ is finite dimensional, gives the first result. For the second statement, recall that for any divisor E and K a canonical divisor associated to a global nonzero meromorphic 1-form ω, we have an isomorphism of sheaves $\mathcal{O}_X[E] \to \Omega^1_{X,\mathrm{hol}}[E-K]$. In particular we have

$$\check{\mathrm{H}}^1(X_{\mathrm{an}}, \mathcal{O}_X[D+K]) \cong \check{\mathrm{H}}^1(X_{\mathrm{an}}, \Omega^1_{X,\mathrm{hol}}[D]).$$

Using that $\check{\mathrm{H}}^1(X_{\mathrm{an}}, \mathcal{O}_X[E])$ is finite dimensional for any divisor E gives the result.

In the case that $D = 0$, we apply Theorem 5.5.1 to see that $\dim L(K) = \dim \mathrm{H}^1(0) = \dim \check{\mathrm{H}}^1(X_{\mathrm{an}}, \mathcal{O}_X)$. We have that $\dim L(0) = 1$ because $f \in L(0)$ means that f has no poles, which on a compact Riemann surface means it must be a constant function. Thus, applying Riemann-Roch we obtain

$$\dim L(K) = g.$$

Thus, we have $\dim \check{\mathrm{H}}^1(X_{\mathrm{an}}, \mathcal{O}_X) = g$ as claimed. Furthermore, we have that

$$\check{\mathrm{H}}^1(X_{\mathrm{an}}, \Omega^1_{X,\mathrm{hol}}) \cong \check{\mathrm{H}}^1(X_{\mathrm{an}}, \mathcal{O}_X[K]) \cong \mathrm{H}^1(K).$$

Now apply Theorem 5.5.1 again to see that

$$\dim \mathrm{H}^1(K) = \dim L(0),$$

and so has dimension 1 as claimed.

Finally, we must deal with the case when $\deg(D) \geq \deg(K)+1$. We again use that $\dim \mathrm{H}^1(D) = \dim L(K-D)$. The latter space vanishes since $\deg(K-D) < 0$ by the assumption on D, and so using the above isomorphisms we have the result. □

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X[D] \longrightarrow \mathcal{M}_X \xrightarrow{\alpha_D} \mathcal{T}_X[D] \longrightarrow 0.$$

Taking the associated long exact sequence we obtain

$$0 \to L(D) \to \mathcal{M}_X(X) \to \mathcal{T}_X[D](X) \to \check{\mathrm{H}}^1(X, \mathcal{O}_X[D]) \to \check{\mathrm{H}}^1(X, \mathcal{M}_X) \to 0$$

where we have used that $\check{\mathrm{H}}^1(X, \mathcal{T}_X[D]) = 0$. For D a divisor with large enough degree, the previous proposition gives that $\check{\mathrm{H}}^1(X, \mathcal{O}_X[D]) = 0$. Thus, we obtain that

$$\check{\mathrm{H}}^1(X, \mathcal{M}_X) = 0,$$

a result we already had for the pair $(X_{\mathrm{Zar}}, \mathcal{M}_{X,\mathrm{alg}})$, but did not know for the analytic space and sheaf.

Our next application is to discuss Abel's theorem. Recall that $\mathrm{Div}(X) = \mathrm{Div}_X(X)$ is the group of divisors on the Riemann surface X. The divisors D that satisfy $\deg(D) = 0$ form a subgroup of $\mathrm{Div}(X)$ denoted by $\mathrm{Div}^0(X)$. A divisor D is said to be a *principal divisor* if there exists a meromorphic function $f \in \mathcal{M}_X(X)$ so that $D = \mathrm{div}(f)$. The set of principal divisors forms a subgroup $\mathrm{PDiv}(X)$ of $\mathrm{Div}(X)$. In the case that X is compact we have that $\mathrm{PDiv}(X) \subset \mathrm{Div}^0(X)$. A natural question to ask is when is a degree 0 divisor a principal divisor? This is answered via Abel's theorem. We need to introduce some more concepts before we can state the theorem.

Our first step is to define a map from $\mathrm{H}_1(X;\mathbb{Z})$ to $\Omega^1_{X,\mathrm{hol}}(X)^\vee$. Let ω be a smooth closed 1-form on X. Let U be a triangulated subset of X, i.e., U can be covered by simplices. Applying Stoke's theorem to this setting we have

$$\int_{\partial U} \omega = \int\int_U d^1\omega = \int\int_U 0 = 0.$$

Thus, we see that the integral of ω around a boundary chain is 0 and so the integral of ω around any closed chain depends only on the homology class of the chain. We see that for any homology class $[c] \in \mathrm{H}_1(X,\mathbb{Z})$, the integral

$$\int_{[c]} \omega = \int_c \omega$$

is well-defined. Note that since X is a Riemann surface, i.e., a complex manifold of dimension 1, we have that every holomorphic 1-form is closed. Thus, given any $\omega \in \Omega^1_{X,\mathrm{hol}}(X)$ and any homology class $[c] \in \mathrm{H}_1(X,\mathbb{Z})$, the integral

$$\int_{[c]} \omega$$

is well-defined. Thus, we obtain a map from $\mathrm{H}_1(X,\mathbb{Z})$ to $\Omega^1_{X,\mathrm{hol}}(X)^\vee$ defined by

$$[c] \mapsto \left(\int_{[c]} : \Omega^1_{X,\mathrm{hol}}(X) \to \mathbb{C} \right).$$

We denote the image of this map by Λ and refer to it as the subgroup of *periods of 1-forms.* This construction allows us to attach a group called the Jacobian to any algebraic curve.

Definition 5.5.7. Let X be a compact Riemann surface. The *Jacobian of* X, denoted by $\mathrm{Jac}(X)$, is the quotient group

$$\mathrm{Jac}(X) = \frac{\Omega^1_{X,\mathrm{hol}}(X)^\vee}{\Lambda}.$$

Note that by choosing a basis of $\Omega^1_{X,\mathrm{hol}}(X)$ and $\mathrm{H}_1(X,\mathbb{Z})$, we can identify the Jacobian of X with

$$\mathrm{Jac}(X) \cong \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}}.$$

Exercise 5.5.8. Let X be the complex torus $\mathbb{C}/L$ for some lattice L. Show that $\mathrm{Jac}(X) \cong X$. In particular, this gives that the Jacobian of an elliptic curve is just the elliptic curve itself.

We can now define the Abel-Jacobi map. Fix a basepoint $x_0 \in X$. For each $x \in X$ we choose a path γ_x from x_0 to x. The Abel-Jacobi map

$$\mathrm{AJ} : X \to \Omega^1_{X,\mathrm{hol}}(X)^\vee$$

is defined by sending

$$x \mapsto \left(\omega \mapsto \int_{\gamma_x} \omega\right).$$

This map is not well-defined because it depends on the choice of γ_x; if one chooses a different path one will obtain a different integral. However, we can remedy this by considering the map into $\mathrm{Jac}(X)$ instead of $\Omega^1_{X,\mathrm{hol}}(X)^\vee$. In this case the map is well-defined. It is easy to extend this from a map on X to a map on $\mathrm{Div}(X)$ by setting

$$\mathrm{AJ}\left(\sum_x n_x \cdot x\right) = \sum_x n_x\,\mathrm{AJ}(x).$$

Thus, we obtain a group homomorphism from $\mathrm{Div}(X)$ to $\mathrm{Jac}(X)$. We can restrict this map to $\mathrm{Div}^0(X)$, which we denote as AJ_0.

Lemma 5.5.9. *The Abel-Jacobi map* AJ_0 *is independent of the basepoint* x_0.

Proof. Set AJ_{0,x_0} to be the Abel-Jacobi map restricted to $\mathrm{Div}^0(X)$ defined relative to the basepoint x_0. Let x_0' be a different basepoint. For $x \in X$, let γ_x be a path from x_0 to x and γ_x' a path from x_0 to x_0'. Let γ be a path from x_0' to x_0. Then we have $\gamma_x - \gamma_x' - \gamma = 0$ in $\mathrm{H}_1(X,\mathbb{Z})$. Thus, we have

$$\begin{aligned}\mathrm{AJ}_{0,x_0}(x)(\omega) - \mathrm{AJ}_{0,x_0'}(x)(\omega) &= \int_{\gamma_x} \omega - \int_{\gamma_x'} \omega \\ &= \mathbb{Z}_{\gamma_x - \gamma_{x'}}\omega \\ &= \int_\gamma \omega.\end{aligned}$$

Observe that the element $\int_\gamma \omega$ is independent of the point x. Now if $D =$

$\sum_x n_x \cdot x \in \mathrm{Div}^0(X)$, then

$$\begin{aligned}
\mathrm{AJ}_{0,x_0}(D)(\omega) - \mathrm{AJ}_{0,x_0'}(D)(\omega) &= \sum_x n_x \int_\gamma \omega \\
&= \left(\int_\gamma \omega\right) \sum_x n_x \\
&= \left(\int_\gamma \omega\right) \cdot 0 \\
&= 0.
\end{aligned}$$

Thus, we have lemma. ◻

We can now state Abel's theorem.

Theorem 5.5.10. *Let X be a compact Riemann surface of genus g. Let $D \in$ Div$^0(X)$. Then $D \in \mathrm{PDiv}(X)$ if and only if $\mathrm{AJ}_0(D) = 0$ in $\mathrm{Jac}(X)$.*

Consider the exact sequence arising from the exponential map:

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0.$$

Using the fact that the exponential map from $\mathbb{C}$ to $\mathbb{C}^*$ is surjective, the long exact sequence in cohomology gives

$$0 \to \check{\mathrm{H}}^1(X, \underline{\mathbb{Z}}) \to \check{\mathrm{H}}^1(X, \mathcal{O}_X) \to \check{\mathrm{H}}^1(X, \mathcal{O}_X^*) \to \check{\mathrm{H}}^2(X, \underline{\mathbb{Z}}).$$

In particular, we can write this exact sequence as an exact sequence

$$0 \to \check{\mathrm{H}}^1(X, \mathcal{O}_X)/\check{\mathrm{H}}^1(X, \underline{\mathbb{Z}}) \to \check{\mathrm{H}}^1(X, \mathcal{O}_X^*) \to \check{\mathrm{H}}^2(X, \underline{\mathbb{Z}}).$$

We also have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}_X^* \longrightarrow \mathrm{Div}_X \longrightarrow 0$$

where the map $\mathcal{M}_X^* \longrightarrow \mathrm{Div}_X$ is given by sending a meromorphic function to its divisor. The associated long exact sequence here begins as

$$0 \to \mathbb{C}^* \to \mathcal{M}_X(X)^* \to \mathrm{Div}(X) \to \check{\mathrm{H}}^1(X, \mathcal{O}_X^*).$$

Note that the image of the map $\mathcal{M}_X(X)^* \longrightarrow \mathrm{Div}(X)$ is precisely the set of principal divisors. Moreover, using that $\check{\mathrm{H}}^2(X, \underline{\mathbb{Z}}) \cong \mathbb{Z}$, we have a map

$$\mathrm{Div}(X) \to \check{\mathrm{H}}^1(X, \mathcal{O}_X^*) \to \check{\mathrm{H}}^2(X, \underline{\mathbb{Z}}) \cong \mathbb{Z}$$

In particular, one can check that this map is given by sending a divisor to its degree. Combining all of this gives the following commutative diagram:

$$\begin{CD}
@. 0 @. 0 @. @. \\
@. @VVV @VVV @. @. \\
@. \mathrm{PDiv}(X) @>{=}>> \mathrm{PDiv}(X) @. @. \\
@. @VVV @VVV @. @. \\
0 @>>> \mathrm{Div}^0(X) @>>> \mathrm{Div}(X) @>{\deg}>> \mathbb{Z} @>>> 0 \\
@. @. @VVV @VV{=}V @. \\
0 @>>> \check{\mathrm{H}}^1(X,\mathcal{O}_X)/\check{\mathrm{H}}^1(X,\underline{\mathbb{Z}}) @>>> \check{\mathrm{H}}^1(X,\mathcal{O}_X^*) @>{\deg}>> \check{\mathrm{H}}^2(X,\underline{\mathbb{Z}}).
\end{CD}$$

We can apply Serre duality (Theorem 5.5.1) to see that $\check{\mathrm{H}}^1(X,\mathcal{O}_X) \cong \Omega^1_{X,\mathrm{hol}}(X)^\vee$, and so $\check{\mathrm{H}}^1(X,\mathcal{O}_X)/\check{\mathrm{H}}^1(X,\underline{\mathbb{Z}}) \cong \mathrm{Jac}(X)$. Composition of this map with the Abel-Jacobi map gives a map

$$\mathrm{Div}^0(X) \to \check{\mathrm{H}}^1(X,\mathcal{O}_X)/\check{\mathrm{H}}^1(X,\underline{\mathbb{Z}}).$$

We denote this map as AJ_0 as well. One then can work out that this map fits into the diagram so that the diagram commutes:

$$\begin{CD}
@. @. 0 @. @. \\
@. @. @VVV @. @. \\
@. \mathrm{PDiv}(X) @>{=}>> \mathrm{PDiv}(X) @. @. \\
@. @VVV @VVV @. @. \\
0 @>>> \mathrm{Div}^0(X) @>>> \mathrm{Div}(X) @>>> \mathbb{Z} @>>> 0 \\
@. @VVV @VVV @VV{=}V @. \\
0 @>>> \check{\mathrm{H}}^1(X,\mathcal{O}_X)/\check{\mathrm{H}}^1(X,\underline{\mathbb{Z}}) @>>> \check{\mathrm{H}}^1(X,\mathcal{O}_X^*) @>>> \check{\mathrm{H}}^2(X,\underline{\mathbb{Z}}) @>>> 0.
\end{CD}$$

It is now clear that if $D \in \mathrm{Div}^0(X)$ then $D \in \mathrm{PDiv}(X)$ since $\mathrm{Div}^0(X)$ injects into $\mathrm{Div}(X)$ and the above diagram is commutative. Similarly, it is clear that if $D \in \mathrm{PDiv}(X)$ then D is in the kernel of the map AJ_0. Thus, we have Abel's theorem up to the nontrivial checking that the maps are the appropriate ones and that the above diagram commutes after inserting the map AJ_0.

5.6 The Hodge Conjecture

The Hodge conjecture is one of the Clay Mathematics Institute's Millenium problems. It is probably more difficult to state than any of the other millenium problems as it states a deep relationship between analysis, algebraic geometry, and topology. Very little is known about this conjecture. In this section we outline the statement of the conjecture.

Let X be a complex manifold of dimension n. Viewing X as a real manifold of dimension $2n$, for any $x \in X$ we have an associated tangent space $T_x(X)$ as defined in § 3.5. By choosing coordinates $z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n$, we can realize $T_x(X)$ as the space of $\mathbb{R}$-linear derivations on the ring of $C^\infty(U, \mathbb{R})$ for U an open neighborhood of x, i.e., $T_x(X)$ is generated over $\mathbb{R}$ by the operators $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$ for $1 \leq i \leq n$. As $T_x(X)$ is a real vector space, it is natural to consider the complexified vector space

$$T_{\mathbb{C},x}(X) = T_x(X) \otimes_{\mathbb{R}} \mathbb{C}.$$

Choosing coordinates, this vector space can be realized as the space of $\mathbb{C}$-linear derivations on the ring of smooth complex valued functions in a neighborhood of x. In particular, we have that $T_{\mathbb{C},x}(X)$ is generated over $\mathbb{C}$ by $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$ for $1 \leq i \leq n$. If we write

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right)$$

and

$$\frac{\partial}{\partial \overline{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right),$$

then $T_{\mathbb{C},x}(X)$ is generated over $\mathbb{C}$ by $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial \overline{z}_i}$ for $1 \leq i \leq n$. Finally, we define the *holomorphic tangent space to* X *at* x by setting $T_{x,\mathrm{hol}}(X)$ to be the vector space over $\mathbb{C}$ generated by the $\frac{\partial}{\partial z_i}$, i.e., the space of derivations that vanish on antiholomorphic functions (functions f where $\overline{f}$ is holomorphic.) Similarly, one can define the antiholomorphic tangent space, which is isomorphic to $\overline{T_{x,\mathrm{hol}}(X)}$. Thus, we can write

$$T_{\mathbb{C},x}(X) = T_{x,\mathrm{hol}}(X) \oplus \overline{T_{x,\mathrm{hol}}(X)}.$$

Recall, given a complex vector space V, a hermitian inner product is a bilinear form

$$\langle\, , \,\rangle : V \otimes V \to \mathbb{C}$$

that for any $\alpha \in \mathbb{C}$ and $u, v, w \in V$ satisfies

- $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$
- $\langle u, v + w\rangle = \langle u, w\rangle + \langle u, v\rangle$
- $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$
- $\langle u, \alpha v\rangle = \overline{\alpha}\langle u, v\rangle$
- $\langle u, v\rangle = \overline{\langle v, u\rangle}$.

A *hermitian metric* on X is a positive definite hermitian inner product

$$\langle \, , \, \rangle_z : T_{z,\mathrm{hol}}(X) \otimes \overline{T_{z,\mathrm{hol}}(X)} \to \mathbb{C}$$

that depends smoothly on z, i.e., if we choose local coordinates around z as above and set $h_{ij}(z) = \left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right\rangle_z$, then the $h_{ij}(z)$ are smooth functions of z. If we write $\langle \, , \, \rangle_z$ in terms of the basis $\{dz_i \otimes d\overline{z}_j\}$ of $(T_{z,\mathrm{hol}}(X) \otimes \overline{T_{z,\mathrm{hol}}(X)})^\vee = T_{z,\mathrm{hol}}(X)^\vee \otimes \overline{T_{z,\mathrm{hol}}}(X)^\vee$, then the hermitian metric is given by

$$ds^2 = \sum_{i,j} h_{ij}(z) dz_i \otimes d\overline{z}_j.$$

A *coframe* for the hermitian metric ds^2 is a n-tuple of forms $(\omega_1, \ldots, \omega_n)$ with $\omega_i \in \Omega_X^{1,0}(X)$ so that

$$ds^2 = \sum_i \omega_i \otimes \overline{\omega}_i.$$

We say that the metric ds^2 is *Kähler* if the $(1,1)$-form

$$\omega = \frac{\sqrt{-1}}{2} \sum_i \omega_i \wedge \overline{\omega}_i$$

is d-closed, i.e., $d^2(\omega) = 0$. A complex manifold X is said to be a *Kähler manifold* if it admits a Kähler metric. There are many other equivalent conditions one can give on ds^2 to ensure it is Kähler, including many that are more useful for geometric insight. However, we choose the easiest version to state to keep from going to far afield.

We have already seen that for any m, we have

$$\Omega^m_{X,\mathrm{hol}}(X) \cong \bigoplus_{p+q=m} \Omega_X^{p,q}(X).$$

In general one does not have the same decomposition for the de Rham cohomology groups. However, on a Kähler manifold one does!

Theorem 5.6.1. *(Hodge Decomposition Theorem) Let X be a compact Kähler manifold. Then we have for the complex de Rham cohomology groups*

$$\mathrm{H}^m_{\mathrm{dR}}(X, \mathbb{C}) \cong \bigoplus_{p+q=m} \mathrm{H}^{p,q}_{\mathrm{dR}}(X)$$
$$\mathrm{H}^{p,q}_{\mathrm{dR}}(X) \cong \overline{\mathrm{H}^{q,p}_{\mathrm{dR}}(X)}.$$

One does not get the same decomposition in terms of the real cohomology, in this case one gets

$$\mathrm{H}^m_{\mathrm{dR}}(X, \mathbb{R}) \cong \left(\bigoplus_{\substack{p+q=m \\ p \leq q}} (\mathrm{H}^{p,q}_{\mathrm{dR}}(X) \oplus \mathrm{H}^{q,p}_{\mathrm{dR}}(X)) \right) \cap \mathrm{H}^m_{\mathrm{dR}}(X, \mathbb{R}).$$

We define the *rational Hodge classes* to be the set

$$\mathrm{Hdg}(X) = \bigcup_p \left(\check{\mathrm{H}}^{2p}(X, \underline{\mathbb{Q}}) \cap \mathrm{H}^{p,p}_{\mathrm{dR}}(X) \right).$$

These are the classes in the de Rham cohomology groups that have any hope of being "algebraic".

For the algebraic side of things, we now let X be a smooth projective algebraic variety of dimension n. The space of $\mathbb{C}$-valued points $X(\mathbb{C})$ is a complex n-manifold, which we denote as X_{an}. Given a subvariety $Y \subset X$ of codimension p, we obtain a submanifold Y_{an} of X_{an} of codimension p by considering the $\mathbb{C}$-valued points. Since Y_{an} is a submanifold of X_{an}, so we have a natural injection

$$Y_{\mathrm{an}} \hookrightarrow X_{\mathrm{an}},$$

which gives a natural map of sheaves

$$\Omega^m_{X,\mathrm{hol}} \longrightarrow \Omega^m_{Y,\mathrm{hol}}$$

for each $m \geq 0$. Consequently, we have a map

$$\mathrm{H}^m_{\mathrm{dR}}(X_{\mathrm{an}}, \mathbb{C}) \longrightarrow \mathrm{H}^m_{\mathrm{dR}}(Y_{\mathrm{an}}, \mathbb{C}).$$

We specialize to the case that $m = n - p$. We apply the fact that Y_{an} has dimension $n - p$ to see that $\mathrm{H}^{n-p}_{\mathrm{dR}}(Y_{\mathrm{an}}, \mathbb{C}) \xrightarrow{\simeq} \mathbb{C}$. Combining this with the map above, we have a map

$$\mathrm{H}^{n-p}_{\mathrm{dR}}(X_{\mathrm{an}}, \mathbb{C}) \longrightarrow \mathbb{C},$$

i.e., for each subvariety Y of X of codimension p we obtain an element of $\mathrm{H}^{n-p}_{\mathrm{dR}}(X_{\mathrm{an}}, \mathbb{C})^\vee$. We now apply Poincare duality to obtain an element $[Y]$ in $\mathrm{H}^p_{\mathrm{dR}}(X_{\mathrm{an}}, \mathbb{C})$. We call these elements *algebraic cycles*.

Conjecture 5.6.2. (Hodge Conjecture) Let X be a smooth complex projective algebraic variety. Every Hodge class can be written as a rational sum of alegebraic cycles.

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