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PS1

Kolyvagin's Conjecture on Heegner Points:

I) Definitions

$E: y^2 = f(x)$, $\deg(f) = 3$ $/\mathbb{Q}$

$Ш(E) = \{C : \mathrm{Jac}(C) \simeq E,\ C(\mathbb{Q}_v) \neq \emptyset,\ v \leq \infty\}$

$n \in \mathbb{Z}_{\geq 1}$:

$$0 \to E(\mathbb{Q}) \otimes \mathbb{Z}/n\mathbb{Z} \to \mathrm{Sel}_n(E) \to Ш(E)_n \to 0$$

$n = p^m$, take a limit: cohomological description.

$$0 \to E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{p^\infty}(E) \to Ш(E)_{p^\infty} \to 0.$$

"Hodge Conj."

$X/_{\mathbb{C}}$ smooth proj.

$$0 \to \left\{\begin{matrix}\text{cycle} \\ \text{classes}\end{matrix}\right\} \to \underbrace{H_B^{2i}(X(\mathbb{C}), \mathbb{Z}) \cap H^{i,i}}_{\text{Hodge classes}} \to \text{"analogue" to } Ш \to 0$$

↓ quotient being finite Hodge Conj.

Shows proving $Ш(E)$ is finite is hard!

$$\mathrm{Sel}_{p^\infty}(E) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{r_p} \oplus \text{finite}.$$

$$E(\mathbb{Q}) = \mathbb{Z}^{r_{alg}} \oplus \text{finite}$$

$0 \leq r_{alg} \leq r_p < \infty$

↖ "=" iff $Ш_{p^\infty}$ is finite.

<u>Remark:</u>

$$Sel_p(E)/E(\mathbb{Q})_p \simeq Sel_{p^\infty}(E)_p \leftarrow p\text{-torsion}$$

<u>Lemma:</u> $\dim_{\mathbb{F}_p}(Sel_p(E)/E_p(\mathbb{Q})) = 1$

$\Rightarrow r_p = 1.$

(r_p also called $\mathbb{Z}_p$ - corank)

<u>Conj:</u> $r_p = 1 \Rightarrow r_{alg} = 1$

(and $\# Ш_{p^\infty} < \infty$)

<u>Consequences:</u>

1) Assume the conj for E/K = number field $\overset{\text{Mazur-Rubin}}{\Rightarrow}$ Hilbert 10 has negative answer for ring of integers.

(only need for $p = 2$)

2) Assume the conjecture for $p = 3$ E/K = quadratic.

$\Rightarrow$ by Swinnerton-Dyer diagonal cubic 3-folds satisfy Hasse principle

3) Congruent numbers problem. ($p = \cancel{3}\ 2$)

II) <u>Theorem:</u> $E/\mathbb{Q}$, N = conductor * Assume

1) $\bar{\rho}_{E,p} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_p) = Aut(E_p)$

satisfies: • $\bar{\rho}_{E,p}$ is surjective

- $\overline{\rho}_{E,p}$ is ramified at all $\ell \| N$ and $\ell \equiv \pm 1 \pmod p$ and at least 2 prime $\ell \| N$.

2) $p \geq 5$ good ordinary

Then $\overset{a)}{} r_p = 1 \overset{b)}{\iff} r_{an} = \operatorname{ord}_{s=1} L(E,s) = 1$

$\overset{c)}{\iff} r_{alg} = 1$ and $\# Ш < \infty$.

Remark:

1) b) $\Rightarrow$ c) Gross-Zagier, Kolyvagin
2) Theorem also holds if one replaces 1 by 0 in a), b), c) this is known by Kato, Skinner-Urban.
3) w/ Skinner $p \| N$, + extra assumptions
4) Skinner c) $\Rightarrow$ b)
5) $\left(\overline{\rho}_{E,p} \text{ is ramified at } \ell \right.$ $\left. \ell \| N \right) \iff p \nmid v_\ell(\Delta_E)$

$\mathrm{Sel}_p(E)$ average size $= p+1$ $\quad p \leq 5$ (Bhargava-Shankar)

$\Rightarrow$ high percentage of E

(root # equidist) $\quad \mathrm{Sel}_p(E) = \begin{cases} 0 \\ \mathbb{Z}/p\mathbb{Z}. \end{cases}$

$\Rightarrow \quad r_p = \begin{cases} 0 \\ 1. \end{cases}$

Theorem (Bhargava-Skinner-Z): at least 66.48% of elliptic curves satisfy BSD (rank part.) (indexed by height)

<u>Remark:</u> 1) If one shows $av(Sel_p(E)) = p+1$ for all p, then BSD would be true for "100%".

2) $\geq 20\%$ rank $= 1$

about the same for rank 0.

Rest have 0 or 1, but can't pin it down.

$av(\text{rank}) \geq 0.20$.

III.) Kolyvagin conjecture:

K_n ring class field

$$K = \mathbb{Q}(\sqrt{-D})$$
$$|$$
$$\mathbb{Q}$$

$\mathfrak{H}$ upper half plane

X Shimura curve

$$\begin{array}{c} \mathfrak{H} \\ \downarrow \\ X \\ \downarrow \varphi \\ E \end{array}$$

Gross-Zagier, Kolyvagin

$$K_+ \subseteq \mathfrak{H}$$

$$\varphi(K_+) \subseteq E(K^{ab})$$

$$Gal(K_n/K) \simeq Pic(\mathcal{O}_{n,K})$$

$$\mathcal{O}_{n,K} = \mathbb{Z} + n\mathcal{O}_K$$

($n=1$ get Hilbert class field)

$$y_n \in E(K_n)$$

$$y_K = tr^{K_1}_{K}\, y(1) \in E(K).$$ (don't get anything new for larger n)

Gross-Zagier: y_K non-torsion $\Leftrightarrow$ $\operatorname{ord} L(E/K, s) = 1$
(root # $= -1$).

Kolyvagin: y_K non-torsion $\Rightarrow$ $r_p = 1$
($\Rightarrow$ Ш$_{p^\infty}$ is finite)

$$\{y_n\} \rightsquigarrow \{c(n) \in H^1(K, E_p)\} = \kappa$$

$n = \prod_\lambda \lambda$ Kolyvagin system.
sq-free, λ inert

$c(1) \leftrightarrow y_K$
$\neq 0 \leftrightarrow p \nmid y_K \in E(K)$

<u>Conj:</u> $\kappa \neq \{0\}$

<u>Def:</u> $\operatorname{ord} \kappa = \min_{c(n) \neq 0} \nu(n)$, $\nu(n)$ number of prime factors of n.

($\operatorname{ord} \kappa = 0 \Leftrightarrow c(1) \neq 0 \Leftarrow y_K$ non-torsion)

<u>Thm (Kolyvagin 1990):</u> Assume conjecture.

$$\operatorname{ord} \kappa = \max\{r_p(E/K)^+, r_p(E/K)^-\} - 1.$$

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$\left(\mathrm{ord}\, \kappa = 0 \Leftrightarrow r_p(E/K) = 1 \right)$

<u>Thm (Z.):</u> Conjecture holds for (p, E, K) satisfying same hypotheses in earlier theorem.

$$\Rightarrow \text{(Thm earlier)} \quad \begin{array}{ccc} r_p(E/K) = 1 & \Leftrightarrow & \mathrm{ord}\, L(E/K, s) = 1 \\ \Downarrow & & \Updownarrow \text{ G-Z} \\ \mathrm{ord}\, \kappa = 0 & & y_K \text{ non-torsion} \\ & & \cancel{\Updownarrow} \end{array}$$

$$\begin{tikzcd} & \text{Heegner points } (y_n)_n \arrow[dl, "\text{Kolyvagin}"'] \arrow[dr, "\text{G-Z. Walds.}"] & \\ (\text{Selmer grp}) \arrow[rr, leftrightarrow, "\text{BSD}"] & & (L\text{-fctn}) \end{tikzcd}$$

$r_{an} \le 1$ can this be removed now?