Euler systems for Rankin-Selberg convolutions and generalizations

I. Euler Systems:

Let V = p-adic representation of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \supseteq T$ $G_{\mathbb{Q}}$-stable lattice

Ex: $\mathbb{Q}_p(1)$, $V_pE = T_pE \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, E elliptic curve$/\mathbb{Q}$

p-adic cycl. char.

V_pf, f mod form of wt ≥ 2.

One can attach to V a p-Selmer group $\mathrm{Sel}_{p^\infty}(V) \subset H^1(G_{\mathbb{Q}}, V/T)$.

Fact: $\mathrm{Sel}_{p^\infty}(V)$ contains interesting arithmetic information about V.

Ex: $V = V_pE$, $\quad 0 \to E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{p^\infty}(V_pE) \to Ш_{p^\infty}(E/\mathbb{Q}) \to 0$.

Euler system is a tool for controlling the size of the Selmer group.

Def: Assume V is unramified outside a finite set of primes $\Sigma \supseteq \{p\}$.

An Euler system for V is a collection $(z_m)_{m\geq 1}$,

$z_m \in H^1(\mathbb{Q}(\mu_m), V^*(1))$ $\quad$ ($V^*(1)$ = twisted dual of V)

s.t. • z_m takes values in $G_{\mathbb{Q}}$-stable lattice of $V^*(1)$ independent of m

• satisfy Euler system norm relations

$$\mathrm{cores}^{\mathbb{Q}(\mu_{m\ell})}_{\mathbb{Q}(\mu_m)} z_{m\ell} = \begin{cases} z_m & \text{if } \ell \mid m \text{ or } \ell \in \Sigma \\ P_\ell(\sigma_\ell^{-1}) z_m \end{cases}$$

where $P_\ell(X) = \det(1 - X\sigma_\ell^{-1} \mid V)$ where σ_ℓ = Frobenius.

Thm (Rubin): If $z_1 \neq 0$ and technical hypotheses ($G_{\mathbb{Q}}$ is large in $GL(V)$)

then $\mathrm{Sel}_{p^\infty}(V)$ is finite.

Remarks: • Similar definition of Euler system over number fields (needs Zerbes

<u>Thm (Ash-Stevens, Bellaiche):</u> For $\mathcal{U}$ sufficiently small, $\exists\, M_{\mathcal{U}}(\mathcal{F})$ free of rank $2/\Lambda_{\mathcal{U}}$ of specializations at $k \in \mathbb{Z}_{\geq 0} \cap \mathcal{U}$ recovers $V_p(\mathcal{F}_k)$.

$\Gamma = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$

$D(\Gamma) = \mathbb{Q}_p$ - valued distributions on Γ

<u>Theorem (KLZ in Hida families, LZ for Coleman families):</u> ~~There exist~~ $\mathcal{F}, \mathcal{G}$ Coleman families over $\mathcal{U}, \mathcal{U}'$ tame level N. Then there exist $(BF_m^{(\mathcal{F},\mathcal{G},\mathrm{cycl})})_{m\geq 1}$, $BF_m^{(\mathcal{F},\mathcal{G},\mathrm{cycl})} \in H^1(\mathbb{Q}(\mu_m), M_{\mathcal{U}}(\mathcal{F})^* \hat{\otimes} M_{\mathcal{U}'}(\mathcal{G})^* \hat{\otimes} D(\Gamma))$.

interpolating $(BF_{mp^r}^{(\mathcal{F}_k, \mathcal{G}_{k'}, j)})$, $k \in \mathbb{Z}_{\geq 0} \cap \mathcal{U}$, $k' \in \mathbb{Z}_{\geq 0} \cap \mathcal{U}'$, $0 \leq j \leq \min(k, k')$.

Get Euler systems for critical twists.

Relation to L-values (explicit reciprocity law): Let $k \in \mathbb{Z}_{\geq 0} \cap \mathcal{U}$, $k' \in \mathbb{Z}_{\geq 0} \cap \mathcal{U}'$, say $k' \leq k$. let $k'+1 \leq j \leq k$. Then $\exp^*(BF_1^{(\mathcal{F}_k, \mathcal{G}_{k'}, j)}) = (*)\, L(\mathcal{F}_k, \mathcal{G}_{k'}, 1+j)$

$(*)$ explicit nonzero constant.

<u>Cor:</u> Let $V = V_p\mathcal{F}_k \otimes V_p \mathcal{G}_{k'}(1+j)$. If technical hypotheses are satisfied (in particular $\mathcal{F}_k$ is not a twist of $\mathcal{G}_{k'}$ and $\mathcal{F}_k, \mathcal{G}_{k'}$ are not CM and $L(\mathcal{F}_k, \mathcal{G}_{k'}, 1+j) \neq 0$), then $\mathrm{Sel}_{p^\infty}(V)$ is finite.

<u>III Euler System for $\mathrm{Sym}^2 f(\psi)$</u>

Suppose f weight ≥ 2, not CM. ψ = Dirichlet character, non-trivial, not quadratic, $\psi(p) \neq 1$.

Let $j \in \{k, \dots, 2k-2\}$ s.t. $(-1)^j = \psi(-1)$.

$V = \mathrm{Sym}^2 V_p(f)(j+\psi)$.

<u>Thm (LZ)</u>: If $L(\mathrm{Sym}^2 f, \psi, j) \neq 0$, then $\mathrm{Sel}_{p^\infty}(V)$ is finite.

<u>IV Construction of the Euler System:</u>

$k = k' = j = 0$, $N \geq 1$, $m \geq 1$.

Geometric input Siegel unit

$g_{1/mN} \in \mathcal{O}(Y_1(m^2N))^\times \otimes \mathbb{Q} = H^1_{\mathcal{M}}(Y_1(m^2N), \mathbb{Q}(1))$

define embedding $Y_1(m^2N) \xrightarrow{\iota_{m,N}} Y_1(N)^2$ defined over $\mathbb{Q}(\mu_m)$

$z \longmapsto (z, z + \frac{1}{m})$

$$
\begin{tikzcd}
H^1_{\mathcal{M}}(Y_1(m^2N), \mathbb{Q}(1)) \arrow[r, "(\iota_{m,N})_*"] \arrow[d, phantom, "\ni"] & H^2_{\mathcal{M}}(Y_1(N)^2, \mathbb{Q}(2)) & \\
g_{1/m^2N} \arrow[r, mapsto, "r_{et}"] \arrow[dr, bend right] & H^3_{et}(Y_1(N)^2, \mathbb{Q}_p(2)) & \\
\arrow[r, "HS"] & H^1(\mathbb{Q}(\mu_m), H^2_{et}(\overline{Y_1(N)}^2, \mathbb{Q}_p(2))) \arrow[r, "pr_{(f,g)}"] \arrow[d, phantom, "\in"] & H^1(\mathbb{Q}(\mu_m), V_f^* \otimes V_g^*) \arrow[dl, phantom, "\in"] \\
 & BF_m^{(f,g,0)} &
\end{tikzcd}
$$

f, g modular forms of weight 2, level dividing N then

$V_f^* \otimes V_g^*$ arises as a quotient of H^2_{et}

<u>Remark:</u> for modular forms of higher weight we replace g_{1/m^2N} by motivic Eisenstein class (Kings) $\mathrm{Eis}^k_{m^2N} \in H^1_{\mu}(Y_1(m^2N), \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}}(1))$

$\mathcal{H}$ = relative cohom sheaf of univ. elliptic curve $/ Y_1(m^2N)$.

If $k', k \geq 0$, $0 \leq j \leq \min(k, k')$, then exists map (def over $\mathbb{Q}(\mu_m)$) taking

$\mathrm{Eis}^{k+k'-2j}_{m^2N}$ into $H^3_{\mu}(Y_1(N)^2, \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}} \boxtimes \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}}(2-j))$

Zerbes 3-4-16 p.5

If f, g have weight $k+2, k'+2$, levels dividing N, $V_f^* \otimes V_g^*(-j)$ arises as quotient of $H^2_{ét}(\overline{Y_1(N)^2}, \mathrm{Sym}^k \boxtimes \mathrm{Sym}^{k'}(2-j))$.

Crucial idea: embedding $Y_1(m^2N) \hookrightarrow Y_1(N)^2$ to end up in right cohom. degree and pick up $\mathbb{Q}(\mu_m)$.

($GL_2 \hookrightarrow GL_2 \times GL_2$ perturb the diagonal embedding)

IV. Similar constructions

1) Hilbert modular forms

$F/\mathbb{Q}$ real quadratic field, $\sigma_1, \sigma_2 : F \hookrightarrow \mathbb{R}$.

$G = \mathrm{Res}^F_{\mathbb{Q}} GL_2$

$\Rightarrow$ natural embedding $GL_{2/\mathbb{Q}} \hookrightarrow G$

$\Rightarrow$ modular curve $\overset{\iota}{\hookrightarrow}$ Hilbert modular surface

More precisely, $\mathfrak{N}$ = ideal of $\mathcal{O}_F$. $N = \mathfrak{N} \cap \mathbb{Z}$.

$$\Rightarrow \iota : Y_1(N) \to Y_1(\mathfrak{N})$$
$$z \mapsto (z, z).$$

Perturbation: $m \geq 1$, $a \in \mathcal{O}_F/\mathbb{Z}$

$$\iota_{m,a} : Y_1(m^2N) \to Y_1(\mathfrak{N})$$
$$z \longmapsto \left(z + \frac{\sigma_1(a)}{m}, z + \frac{\sigma_2(a)}{m}\right)$$ defined over $\mathbb{Q}(\mu_m)$

$$H^1_\mu(Y_1(m^2N), \mathbb{Q}(1)) \xrightarrow{(\iota_{m,a})_*} H^3_\mu(Y_1(\mathfrak{N}), \mathbb{Q}(2))$$

$$\begin{aligned}
g_{1/m^2N} &\xrightarrow{r_{ét}} H^3_{ét}(Y_1(\mathfrak{N}), \mathbb{Q}_p(2)) \\
&\xrightarrow{HS} H^1(\mathbb{Q}(\mu_m), H^2_{ét}(\overline{Y_1(\mathfrak{N})}, \mathbb{Q}_p(2))) \\
&\xrightarrow{\mathrm{pr}_{\mathcal{F}}} H^1(\mathbb{Q}(\mu_m), V_{\mathcal{F}}^{\mathrm{asai},*}) \\
&\longrightarrow z_m^{\mathcal{F}}
\end{aligned}$$

Zerbes
3-4-16
pg 6

$\mathcal{F}$ Hilbert mod form of parallel weight $(2,2)$ level $\mathfrak{n}$

$V_{\mathcal{F}}$ = p-adic rep. of G_F attached to $\mathcal{F}$

$V_{\mathcal{F}}^{asai} = \otimes\, ind_{\mathbb{Q}}^{F}\, V_{\mathcal{F}}$ 4-dim. p-adic rep of $G_{\mathbb{Q}}$.

Thm (LZ): Assume F has narrow class number 1, $\mathcal{F}$ Hilbert mod. form of weight $(k+2, k'+2) \geq (2,2)$, $0 \leq j \leq \min(k,k')$. Let $V = V_{\mathcal{F}}^{asai}(1+j)$.
Then there exists Euler system $(Z_m^{(\mathcal{F})})_{m \geq 1}$, $Z_m^{(\mathcal{F})} \in H^1(\mathbb{Q}(\mu_m), V^*(1))$.

Conjecture (work in progress w/ Loeffler & Skinner): $Z_1^{(\mathcal{F})}$ is related to a value of p-adic Asai L-function.

2) $GU(2,1)$ (work in progress w/ LS (Loeffler-Skinner):

K imaginary quadratic field.

$GL_{2/\mathbb{Q}} \times Res_{\mathbb{Q}}^{K} G_m \hookrightarrow GU(2,1)$

↑
Siegel units

we can perturb the embedding and construct classes in the Galois cohom. of p-adic rep. arising in $H^2_{et}(\bar{Y}^{\ast}, \mathbb{Q}_p(2))$.

Thm (LSZ): These classes satisfy the ES norm relations.