Aspects of non-abelian Lubin-Tate Theory:

Teruyoshi Yoshida

SCNTD 2008

3 - 8 - 2008

Yoshida

Aspects of Non-abelian Lubin-Tate theory

(compatibility of local/global Langlands corresp. for GL_n)

§1 Irreducibility of cyclotomic polynomials

Thm: $N \geq 1$ $\quad \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^\times$

$$(\zeta \mapsto \zeta^a) \longmapsto a \pmod N$$

$p \nmid N$ arithmetic Frobenius

$$\mathrm{Frob}_p \mapsto p \pmod N$$

This describes the decomposition law of good primes.

Vary N:

$$\mathbb{Q}(\zeta_\infty) = \bigcup_N \mathbb{Q}(\zeta_N)$$

$$\begin{array}{ccccc} \mathrm{Gal}(\mathbb{Q}(\zeta_\infty)/\mathbb{Q}) & \xrightarrow{\sim} & \hat{\mathbb{Z}}^\times & = & \varprojlim_N \mathbb{Z}/N\mathbb{Z}. \\ \downarrow & & \downarrow & & \\ \mathrm{Gal}(\mathbb{Q}(\zeta_\infty^p)/\mathbb{Q}) & \longrightarrow & \prod_{\ell \neq p} \mathbb{Z}_\ell^\times & & \\ \rotatebox{90}{$\in$} & & & & \\ \mathrm{Frob}_p & \longmapsto & (p, p, \dots) & & \end{array}$$

$$\mathbb{Q}(\zeta_\infty^p) = \bigcup_{p \nmid N} \mathbb{Q}(\zeta_N)$$

$p \in \mathbb{Q}^\times \qquad \hat{\mathbb{Z}}^\times \cap \hat{\mathbb{Z}} = \mathbb{Z}^\times = \{\pm 1\}$

$p \in \mathbb{Z} \subset \hat{\mathbb{Z}} \quad$ but $p \notin \hat{\mathbb{Z}}^\times$.

To remedy this issue we work with the adeles.

$$\mathbb{A}^\infty = \hat{\mathbb{Z}} \otimes \mathbb{Q}$$
$$(\mathbb{A}^\infty)^\times = \hat{\mathbb{Z}}^\times \mathbb{Q}^\times .$$

Yoshida
3-8-08
p52

$\mathbb{A} = \mathbb{A}^\infty \times \mathbb{R}$ (∞ $\mathbb{Q}$-algebras) $\qquad \hat{\mathbb{Z}}^\times \cap \mathbb{Q}^\times = \mathbb{Z}^\times$

$\mathbb{A}^\times = \hat{\mathbb{Z}}^\times \times \mathbb{Q}^\times \times \mathbb{R}_{>0}^\times$

$$\begin{tikzcd}
\mathrm{Gal}(\mathbb{Q}(\zeta_\infty)/\mathbb{Q}) \arrow[r, "\simeq"] & \hat{\mathbb{Z}}^\times \arrow[d] & \mathbb{Q}^\times\backslash\mathbb{A}^\times/\mathbb{R}_{>0}^\times \arrow[l, "\sim"'] \\
\mathrm{Gal}(\mathbb{Q}_p(\zeta_\infty)/\mathbb{Q}_p) \arrow[u, phantom, "\cup"] & \prod_{\ell\neq p}\mathbb{Z}_\ell^\times & \mathbb{Q}_p^\times \arrow[u] \arrow[ul] \arrow[dl] \\
(p,p,\dots) & \mathbb{Q}_p^\times/\mathbb{Z}_p^\times \arrow[u] &
\end{tikzcd}$$

$$p^{-1} \in \mathbb{Q}_p^\times \mapsto p^{-1} \in \mathbb{Q}_p^\times/\mathbb{Z}_p^\times \mapsto (p,p,\dots)$$

__Local CFT:__

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Gal}(\mathbb{Q}_p(\zeta_\infty)/\mathbb{Q}_p^{ur}) \arrow[r] & \mathrm{Gal}(\mathbb{Q}_p(\zeta_\infty)/\mathbb{Q}_p) \arrow[r] & \mathrm{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{Z}_p^\times \arrow[r] \arrow[u, phantom, "\wr|"] & \mathbb{Q}_p^\times \arrow[r] \arrow[u, "\mathrm{Art}_{\mathbb{Q}_p}"'] & \mathbb{Z} \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

$\mathrm{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \cong \hat{\mathbb{Z}}$

$\Rightarrow$ Intrinsic local characterization of $\mathrm{Art}_{\mathbb{Q}_p}$?

Via Lubin-Tate theory.

__LCFT:__ $K/\mathbb{Q}_p$ fin. $\exists!\ \mathrm{Art}_K : K^\times \to \mathrm{Gal}(K^{ab}/K)$ s.t.

① $\forall\, \varpi \in K$ unif $\quad \mathrm{Art}_K(\varpi)|_{K^{ur}} = \mathrm{Frob}_K$ $\quad$ geometric Frob
char by $\mathrm{Frob}_K^{-1}(x) = x^q \pmod{\mathfrak{p}}$
$q = |\mathcal{O}_K/\mathfrak{p}|$.

② $\forall\, K'/K$ fin. abelian $\quad \forall\, x \in (K')^\times$

$$\mathrm{Art}_K(\mathrm{Nm}_{K'/K}(x))|_{K'} = id.$$

Behind ③ there is

Base change

$$\begin{tikzcd} (K')^\times \arrow[r, "\mathrm{Art}_{K'}"] \arrow[d, "\mathrm{Nm}_{K'/K}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Gal}(K^{ab}/K') \arrow[d] \\ K^\times \arrow[r] & \mathrm{Gal}(K^{ab}/K) \end{tikzcd}$$

Lubin-Tate theory allows one to construct the local Artin map explicitly and one can prove these properties locally.

<u>Remark</u>: Now one recovers

$$\mathrm{Art}_{\mathbb{Q}} : \mathbb{Q}^\times \backslash \mathbb{A}^\times \longrightarrow \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$$

as $\prod_p \mathrm{Art}_{\mathbb{Q}_p}$

· change fields.

<u>§2. GL_1 case:</u>

$\forall \ell$ prime

$i : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ as fields

$$\left\{ \begin{array}{c} \text{(Alg) Hecke char.} \\ \mathbb{Q}^\times \backslash \mathbb{A}^\times \xrightarrow{\pi} \mathbb{C}^\times \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \overline{\mathbb{Q}}_\ell^\times \\ \text{de Rham at } \ell \end{array} \right\}$$

$\pi|_{\mathbb{R}_{>0}^\times}(x) = x^k \ (k \in \mathbb{Z})$

$$\begin{tikzcd} \mathbb{Q}^\times \backslash \mathbb{A}^\times \arrow[rr, bend left, "|\cdot|"] \arrow[d] & \simeq \mathbb{R}_{>0}^\times \times \prod \mathbb{Z}_\ell^\times \arrow[r, "pr."] \arrow[dr] & \mathbb{R}_{>0}^\times \\ \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \arrow[rr, "\chi_\ell"'] & & \mathbb{Z}_\ell^\times \end{tikzcd}$$

ℓ-adic cyclotomic character.

The natural generalization from this GL_1 set-up is as follows:

Yoshida
3-8-08
p54

$$\left\{ \begin{array}{l} \text{AFs cuspidal} \\ \text{Auto. rep. of } GL_n(\mathbb{A}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{irred.} \\ \rho : Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\bar{\mathbb{Q}}_\ell) \\ \quad \cdot \text{ deRham at } \ell \\ \quad \cdot \text{ unram at almost all places} \\ \quad (\text{auto when } n=1) \end{array} \right\} \text{ GL conj.}$$

Decomp. law of good primes $\qquad \Pi = \bigotimes_p \Pi_p$.

for a.e. p $\quad \Pi_p$ unram. spherical

(has $GL_n(\mathbb{Z}_p)$ - fixed vector)

Π_p irred, smth. rep. of $GL_n(\mathbb{Q}_p)$

classified by Satake parameters $(\mathbb{C}^\times)^n / \mathfrak{S}_n \longleftrightarrow$ eigenvalues of Frob_p on $\rho|_{Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$

$(\bar{\mathbb{Q}}_\ell^\times) / \mathfrak{S}_n$.

Shimura varieties (say modular curves)$/\mathbb{C}$:

$$X := \text{moduli of } \left(\underset{\text{ell. curve}}{E/\mathbb{C}}, \eta_{et} : (\mathbb{A}^\infty)^2 \xrightarrow{\sim} H_1^{et}(E, \mathbb{A}^\infty) \right) / \text{isog}$$

$$H_1^{et}(E,\mathbb{A}^\infty) = \varprojlim E[N] \otimes_{\mathbb{Z}} \mathbb{A}^\infty$$

$$X(\mathbb{C}) \simeq GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}^\infty) \times \mathfrak{h} \qquad \mathfrak{h} = \mathfrak{h}^+ \sqcup \mathfrak{h}^- = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$$

$$(E, \eta_{et}) \longleftrightarrow \left(\underset{\text{rk } 2/\mathbb{Q}}{V_B}, \underset{\text{rk } 1/\mathbb{C}}{V_{dR}}, \underset{\text{rk } 2/\mathbb{A}^\infty}{V_{et}}, \begin{array}{l} c_{dRB} : V_B \otimes \mathbb{R} \simeq V_{dR} \\ c_{etB} : V_B \otimes \mathbb{A}^\infty \simeq V_{et} \end{array}, \eta_{et} \right)$$

$V_B = H_1(E, \mathbb{Q}) \qquad \eta_B : \mathbb{Q}^2 \xrightarrow{\sim} V_B$

$V_{dR} = \mathrm{Lie}\, E \qquad \eta_{dR} : \mathbb{C} \to V_{dR}$.

$$GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}^\infty) \times \underbrace{\mathrm{Isom}(\mathbb{C}, \mathbb{R}^2) / \mathbb{C}^\times}_{\mathfrak{h}}$$

Yoshida
3-8-08
p. 5

$$\left.\begin{array}{l} \Pi \quad \text{cusp. autom. rep of } GL_2(\mathbb{A}) \\ \Pi_\infty \quad \text{discrete series of wt } k \geq 2, \end{array}\right\} \longrightarrow \Pi^\infty \text{ appear in the space of sections of } \omega^{\otimes k} \text{ on } X(\mathbb{C})$$

$$\Pi = \Pi^\infty \otimes \Pi_\infty$$

$\Pi^\infty \circlearrowleft GL_2(\mathbb{A}^\infty)$, $\Pi_\infty \circlearrowleft (\mathfrak{g}, K)$-mod

$$\Gamma(X(\mathbb{C}), \omega^{\otimes 2}) \to H^0(X(\mathbb{C}), \Omega) \hookrightarrow H^1_B(X(\mathbb{C}), \mathbb{C}).$$

$\Gamma(X(\mathbb{C}), \omega^{\otimes 2}) \circlearrowleft GL_2(\mathbb{A}^\infty)$

Sh. var$/\mathbb{Q}$

moduli problem descends to $\mathbb{Q}$

$(X, \omega)/\mathbb{Q}$

[ignoring the cusps]

$$H^1_B(X(\mathbb{C}), \mathbb{C}) \xleftarrow{\ \iota\ } H^1_{et}(X \otimes \mathbb{C}, \overline{\mathbb{Q}}_\ell) \xleftarrow{\ \sim\ } H^1_{et}(X \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$$

$H^1_{et}(X \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell) \circlearrowleft GL_2(\mathbb{A}^\infty) \times Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

$$H^1_{et}(X \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell) = \bigoplus_{\Pi^\infty} \Pi^\infty \otimes \underbrace{\rho(\Pi)}_{\text{2-dim. irred. Gal. rep}}$$

Compare Hecke corresp. w/ Frob_p on $X/_{\overline{\mathbb{F}}_p}$ at good prime p

What happens at bad primes p?

$$H^*_{et}(X \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell) \simeq H^*_{et}(X \otimes \overline{\mathbb{Q}}_p, \overline{\mathbb{Q}}_\ell)$$

$H^*_{et}(X \otimes \overline{\mathbb{Q}}_p, \overline{\mathbb{Q}}_\ell) \circlearrowleft GL_2(\mathbb{A}^\infty) \times Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

$$X \otimes \overline{\mathbb{Q}}_p = X^{ord} \amalg X^{ss} \quad \text{(as rigid spaces)}$$

$$= H^*_{et,c}(X^{ord}_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell) + H^*_{et,c}(X^{ss}_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell)$$

in Groth $(GL_2(\mathbb{A}^\infty) \times Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p))$

Yoshida
3-8-08
pg 6

Say Π_p supercusp.

$$= \underbrace{H^*_{et,c}(X^{ord}_{\overline{\mathbb{F}}_p}, R\Psi\overline{\mathbb{Q}}_\ell)}_{\substack{\text{has no } \Pi_p \\ \text{(parab. induction)}}} + \underbrace{H^*_{et,c}(X^{ss}_{\overline{\mathbb{F}}_p}, R\Psi\overline{\mathbb{Q}}_\ell)}_{\pi_p ?}$$

$$X^{ss}(\overline{\mathbb{Q}}_p) \simeq D_{p,\infty}^\times \backslash GL_2(\mathbb{A}^{\infty,p}) \times LT$$

$\uparrow$

(E, η_{et})

$\uparrow$ $/\overline{\mathbb{Q}}_p$ ssing

$\overline{E} := E \bmod p$, $\mathrm{End}(\overline{E}) = D_{p,\infty}$ quat. alg. ram. at p, ∞.

$V_B = \mathrm{End}(E)$

$V^p_{et} = H_1^{et}(\overline{E}, \mathbb{A}^{\infty,p})$

$V_{crys} = H_1^{crys}\overline{E}$ rk $1/D$.

$\|$

$V_B \otimes \mathbb{Q}_p$

$D = D_{p,\infty} \otimes \mathbb{Q}_p$ quat. alg $/\mathbb{Q}_p$

<u>Serre-Tate:</u> Deforming $\overline{E} \underset{\text{bij}}{\longleftrightarrow}$ Deforming $\underbrace{\Sigma_{\overline{E}}}$ = formal $\mathbb{Z}_p$-mod. ht 2 dim 1

! up to isom $/\overline{\mathbb{F}}_p$.

$D^\times$ on i $\circlearrowright$ $\widetilde{LT}$ $\circlearrowleft$ $GL_2(\mathbb{Q}_p)$ on α

$\widetilde{LT}$ = moduli of (Σ, i, α)

$i: \overline{\Sigma}_E \xrightarrow{\sim} \overline{\Sigma}$

Σ : formal mod. $/\overline{\mathbb{Q}}_p$

$\alpha: \mathbb{Q}_p^2 \xrightarrow{\sim} H_1^{et}(\Sigma, \mathbb{Q}_p)$

$= \varprojlim \Sigma[p^n] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$

$\eta_{crys} : D \xrightarrow{\sim} V_{crys}$.

$\uparrow \mathbb{D}$

$\overline{\Sigma} \xrightarrow{\sim} \Sigma_E$

$$X^{ss}(\overline{\mathbb{Q}}_p) \simeq D_{p,\infty}^\times \backslash GL_2(\mathbb{A}^{\infty,p}) \times \widetilde{LT}$$

$GL_2(\mathbb{A}^\infty)$ $\circlearrowright$; $\circlearrowleft$ $GL_2(\mathbb{Q}_p)$

knows LLC

$\pi_p \leftrightarrow \rho_p$

Non-abelian LT-theory

Yoshida
3-8-08
p57

$$\Psi := H^1_{et}(\widetilde{LT}, \overline{\mathbb{Q}}_\ell) \circlearrowleft D^\times \times \underbrace{GL_2(\mathbb{Q}_p)}_{\pi_p} \times \underbrace{Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}_{\rho|_{Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}}$$

$$\Psi[JL^{-1}(\pi_p) \otimes \pi_p] = \rho_p$$

gives LLC.

§3 NALT ... to characterize LLC / define LLC in terms of $\widetilde{LT}$

Open questions:

Define LLC by

$\pi_p \longmapsto \Psi[JL^{-1}(\pi_p) \otimes \pi_p]$ and extend suitably to

$Rep(GL_2(\mathbb{Q}_p))$.

Show this is compatible w/ functoriality.

Change $\begin{cases} n & (\text{of } GL_n) \\ K/\mathbb{Q}_p \end{cases}$

$$\left.\begin{array}{l} AI \longleftrightarrow Ind \\ \boxplus \longleftrightarrow \oplus \\ St \longleftrightarrow Sp \\ St(\pi) \xleftrightarrow{\pi \leftrightarrow \rho} Sp \otimes \rho. \end{array}\right\} \text{all unknown in terms of a local proof.}$$

- using alg. geo. can see log smooth (tame) purely locally.
- mirahoric level $\longrightarrow$ compute conductor.

p-adic applications:

$$\begin{array}{ccc} \{Alg.\} & \longleftrightarrow & \{\text{deRham at } p\} \\ \cap & & \cap \end{array}$$

$$\left\{\begin{array}{l}\text{p-adic Auto.}\\ \text{rep. of}\\ \quad GL_n(\mathbb{A})\end{array}\right\} \xleftrightarrow{\text{p-adic GLC}} \left\{\begin{array}{c}\text{irred } \rho : Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\bar{\mathbb{Q}}_p)\\ \text{unram at all } p\end{array}\right\}$$

?