

Aspects of non-abelian Lubin-Tate Theory:

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Aspects of Non-abelian Lubin-Tate theory

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PS1

(computability of local/global Langlands corresp. for GL_n)

§1 irreducibility of cyclotomic polynomials

Thm: $N \geq 1$ $\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^\times$
 $(\zeta \mapsto \zeta^a) \longmapsto a \pmod{N}$

$p \nmid N$ arithmetic Frobenius

$$\text{Frob}_p \mapsto p \pmod{N}$$

This describes the decomposition law of good primes.

Vary N :

$$\mathbb{Q}(\zeta_\infty) = \bigcup_N \mathbb{Q}(\zeta_N)$$

$$\text{Gal}(\mathbb{Q}(\zeta_\infty)/\mathbb{Q}) \xrightarrow{\sim} \hat{\mathbb{Z}}^\times = \varprojlim_N \mathbb{Z}/N\mathbb{Z}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \text{Gal}(\mathbb{Q}(\zeta_\infty^p)/\mathbb{Q}) & \longrightarrow & \prod_{k \neq p} \mathbb{Z}_k^\times \\ \cup \text{Frob}_p & \longmapsto & (p, p, \dots) \end{array}$$

$$\mathbb{Q}(\zeta_\infty^p) = \bigcup_{p \nmid N} \mathbb{Q}(\zeta_N)$$

$$p \in \mathbb{Q}^\times \quad \hat{\mathbb{Z}}^\times \cap \hat{\mathbb{Z}} = \mathbb{Z}^\times = \{\pm 1\}$$

$$p \in \mathbb{Z} \subset \hat{\mathbb{Z}} \quad \text{but } p \notin \hat{\mathbb{Z}}^\times.$$

To remedy this issue we work with the adèles.

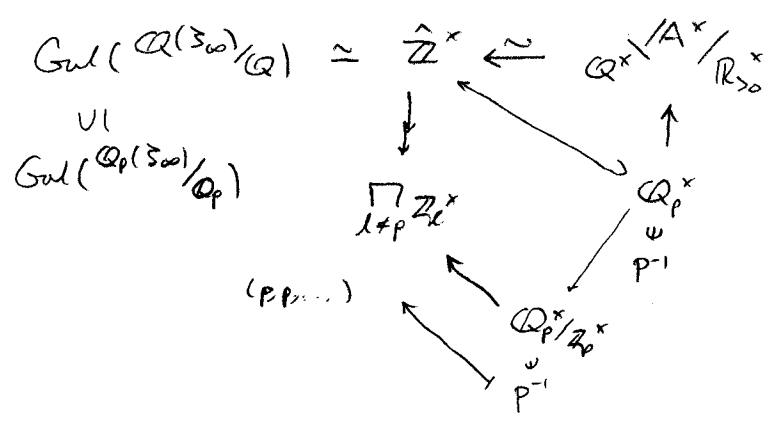
$$\mathbb{A}^\infty = \hat{\mathbb{Z}} \otimes \mathbb{Q}$$

$$(\mathbb{A}^\infty)^\times = \hat{\mathbb{Z}}^\times \otimes \mathbb{Q}^\times.$$

$$A = A^\infty \times \mathbb{R} \quad (A^\infty \text{ } \mathbb{Q}\text{-algebra})$$

$$\hat{\mathbb{Z}}^\times \cap \mathbb{Q}^\times = \mathbb{Z}^\times$$

$$A^\times = \hat{\mathbb{Z}}^\times \times \mathbb{Q}^\times \times \mathbb{R}_{>0}^\times$$



Local CFT:

$$\begin{array}{ccccccc}
 1 \rightarrow & \text{Gal}(\mathbb{Q}_p(\zeta_{\infty})/\mathbb{Q}_p^{ur}) & \rightarrow & \text{Gal}(\mathbb{Q}_p(\zeta_{\infty})/\mathbb{Q}_p) & \rightarrow & \text{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) & \rightarrow 1 \\
 & \downarrow \cong & & \downarrow \text{Art}_{\mathbb{Q}_p} & & \downarrow \cong & \\
 1 \rightarrow & \mathbb{Z}_p^\times & \rightarrow & \mathbb{Q}_p^\times & \rightarrow & \mathbb{Z} & \rightarrow 1
 \end{array}$$

⇒ Continuous local characterization of $\text{Art}_{\mathbb{Q}_p}$?

Via Lubin-Tate theory.

LCFT: K/\mathbb{Q}_p fin. $\exists!$ $\text{Art}_K : K^\times \rightarrow \text{Gal}(K^{ab}/K)$ s.t.

① $\forall \omega \in K$ unif $\text{Art}_K(\omega)|_{K^{ur}} = \text{Frob}_K$ geometric Frob
 char by $\text{Frob}_K^{-1}(x) = x^q \pmod{\mathfrak{p}}$
 $q = |\mathcal{O}_K/\mathfrak{p}|$.

② $\forall K'/K$ fin. abelian $\forall x \in (K')^\times$

$$\text{Art}_K(N_{K'/K}(x))|_{K'} = \text{id.}$$

Behind ② there is

Base change

$$\begin{array}{ccc} (K')^\times & \xrightarrow{\text{Art}_{K'}} & \text{Gal}(K^{ab}/K') \\ \downarrow \text{Nm}_{K'/K} & \circlearrowleft & \downarrow \\ K^\times & \longrightarrow & \text{Gal}(K^{ab}/K) \end{array}$$

Lubin-Tate theory allows one to construct the local Artin map explicitly and one can prove these properties locally.

Remark: Now one recovers

$$\text{Art}_{\mathbb{Q}} : \mathbb{Q}^\times \backslash \mathbb{A}^\times \longrightarrow \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$$

$$\text{as } \prod_p \text{Art}_{\mathbb{Q}_p}$$

• change fields.

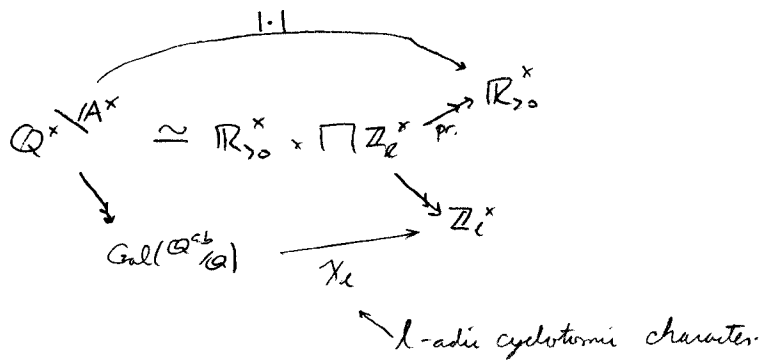
§ 2. GL₁ case:

$\forall l$ prime

$i: \overline{\mathbb{Q}}_l \simeq \mathbb{C}$
as fields

$$\left\{ \begin{array}{l} \text{Alg Hecke char.} \\ \mathbb{Q}^\times \backslash \mathbb{A}^\times \xrightarrow{\pi} \mathbb{C}^\times \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \overline{\mathbb{Q}}_l^\times \\ \text{de Rham or } l \end{array} \right\}$$

$$\pi|_{\mathbb{R}_{>0}^\times} (x) = x^k \quad (k \in \mathbb{Z})$$

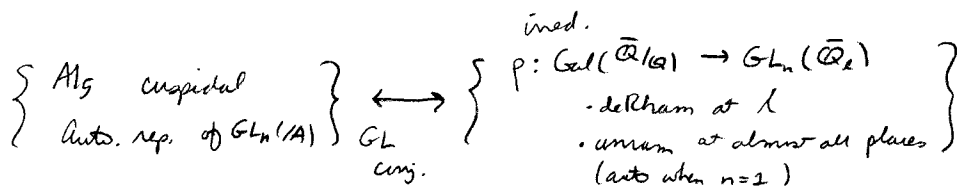


The natural generalization from this GL_1 set-up is as follows:

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Decomp. law of good primes

$$\Pi = \bigotimes_p \Pi_p$$

for a.e. p Π_p unram. spherical

(has $GL_n(\mathbb{Z}_p)$ -fixed vector)

Π_p unram. smooth rep. of $GL_n(\mathbb{Q}_p)$

classified by Satake parameters

$$(\mathbb{C}^\times)^n / G_n$$

eigenvalues of Frob_p on

$$\rho^1 \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \cong \rho^1 \text{Gal}(\bar{\mathbb{Q}}_p^\times/\mathbb{Q}_p^\times)$$

Shimura varieties (say modular curves) / \mathbb{C} :

$$X := \text{moduli of } (E/\mathbb{C}, \eta_{\text{ct}} : (\mathbb{A}^\times)^\alpha \xrightarrow{\sim} H_1^{\text{ct}}(E, \mathbb{A}^\times)) / \text{isog} \text{ all. curv.}$$

$$X(\mathbb{C}) \cong GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}^\times) \times \mathfrak{H}$$

$$\mathfrak{H} = \mathfrak{H}^+ \cup \mathfrak{H}^- = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$$

$$(E, \eta_{\text{ct}}) \longleftrightarrow \left(\begin{array}{ccc} V_B & V_{dR} & V_{\text{ct}} \\ \text{rk } 2/\mathbb{Q} & \text{rk } 2/\mathbb{C} & \text{rk } 2/\mathbb{A}^\times \end{array} \right) \begin{array}{l} \text{CDR}_B : V_B \otimes \mathbb{R} \cong V_{dR} \\ \text{cot}_B : V_B \otimes \mathbb{A}^\times \cong V_{\text{ct}} \cdot \eta_{\text{ct}} \end{array}$$

$$V_B = H_1(E, \mathbb{Q}) \quad \eta_B : \mathbb{Q}^2 \xrightarrow{\sim} V_B$$

$$V_{dR} = \text{Lie } E \quad \eta_{dR} : \mathbb{C} \rightarrow V_{dR}$$

$$GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}^\times) \times \text{Isom}(\mathbb{C}, \mathbb{R}^2) / \mathbb{C}^\times$$

Γ cusp, autom. rep of $GL_2(\mathbb{A})$
 Γ_∞ discrete series of wt $k \geq 2$.

Γ^∞ appear in the space of sections of $\omega^{\otimes k}$ on $X(\mathbb{C})$

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$$\Gamma = \Gamma^\infty \otimes \Gamma_\infty$$

\uparrow \uparrow
 $GL_2(\mathbb{A}^\infty)$ $(\mathfrak{o}_f, \mathfrak{K})$ -mod

$$\Gamma(X(\mathbb{C}), \omega^{\otimes 2}) \rightarrow H^0(X(\mathbb{C}), \Omega)$$

\uparrow \downarrow
 $GL_2(\mathbb{A}^\infty)$ $H^1_B(X(\mathbb{C}), \mathbb{C})$

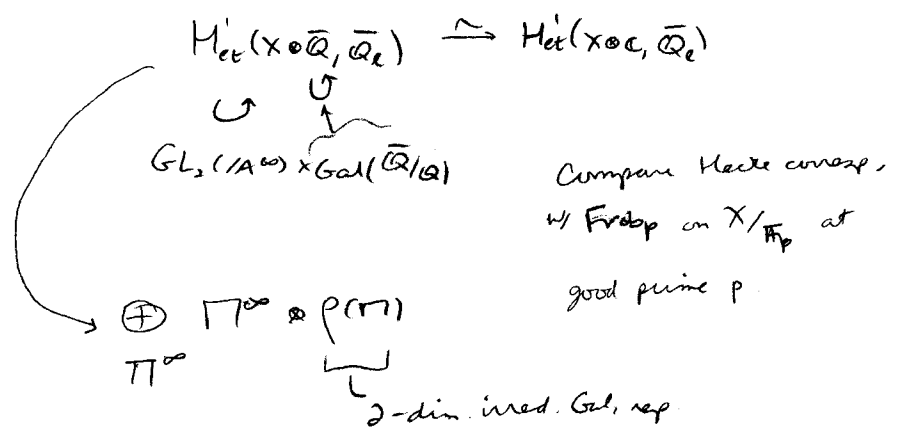
\uparrow

Sh. var / \mathbb{Q}

moduli problem descends to \mathbb{Q}

$(X, \omega) / \mathbb{Q}$

[ignoring the cusps]



What happens at bad primes p ?

$$H^*_{et}(X \otimes \bar{\mathbb{Q}}, \bar{\mathbb{Q}}_x) \simeq H^*_{et}(X \otimes \bar{\mathbb{Q}}_p, \bar{\mathbb{Q}}_x)$$

\uparrow \uparrow
 $GL_2(\mathbb{A}^\infty) \times Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$

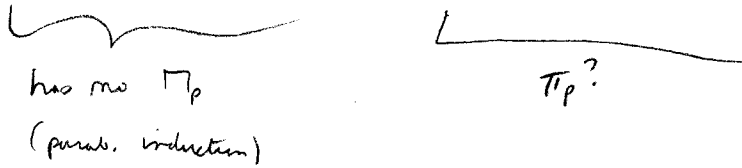
$$X \otimes \bar{\mathbb{Q}}_p = X^{ord} \amalg X^{ss} \quad (\text{as rigid spaces})$$

$$= H^*_{et,c}(X^{ord}_{\bar{\mathbb{Q}}_p}, \bar{\mathbb{Q}}_x) + H^*_{et,c}(X^{ss}_{\bar{\mathbb{Q}}_p}, \bar{\mathbb{Q}}_x)$$

\uparrow
 in $Gr_{\text{cont}}(GL_2(\mathbb{A}^\infty) \times Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p))$

$$= H_{et,c}^*(X_{\overline{\mathbb{F}}_p}^{ord}, R\psi(\overline{\mathbb{Q}}_p)) + H_{et,c}^*(X_{\overline{\mathbb{F}}_p}^{ss}, R\psi(\overline{\mathbb{Q}}_p))$$

Say Π_p
supercomp.



$$X_{\overline{\mathbb{Q}}_p}^{ss} \simeq D_{p,\infty}^x \backslash GL_2(\mathbb{A}^{\infty,p}) \times LT$$

↑

(E, η_{et})

⌊ / $\overline{\mathbb{Q}}_p$ ssing $\overline{E} := E \bmod p, \text{End}(E) = D_{p,\infty}$ quad. alg. rem. at p, ∞ .

$$V_B = \text{End}(E)$$

$$V_{et}^p = H_1^{et}(E, \mathbb{A}^{\infty,p})$$

$$V_{crys} = H_1^{crys} E \text{ rk } 1/D.$$

⌊

$$V_B \otimes \mathbb{Q}_p$$

$$D = D_{p,\infty} \otimes \mathbb{Q}_p \text{ quad. alg. } / \mathbb{Q}_p$$

! up to isom / $\overline{\mathbb{F}}_p$.

Serre-Tate: Deforming $\overline{E} \xleftrightarrow{\text{bij}}$ Deforming $\Sigma_E = \text{formal } \mathbb{Z}_p\text{-mod.}$
ht $\geq \dim 2$

$$D_{\text{oni}}^x \hookrightarrow \widetilde{LT} = \text{moduli of } (\Sigma, i, \alpha)$$

$\supset GL_2(\mathbb{Q}_p)$
mod

$$\eta_{crys} : D \xrightarrow{\sim} V_{crys}$$

↑ ID

$$\overline{\Sigma} \xrightarrow{\sim} \Sigma_E$$

$$i: \overline{\Sigma}_E \xrightarrow{\sim} \overline{\Sigma}_i$$

$$\Sigma: \text{formal mod. } / \mathbb{Q}_p$$

$$\alpha: \mathbb{Q}_p^2 \xrightarrow{\sim} H_1^{et}(\Sigma, \mathbb{Q}_p)$$

$$= \lim_{\leftarrow} \Sigma[\rho^m] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

$$X_{\overline{\mathbb{Q}}_p}^{ss} \simeq D_{p,\infty}^x \backslash GL_2(\mathbb{A}^{\infty,p}) \times \widetilde{LT}$$

⌊ $GL_2(\mathbb{A}^{\infty,p})$ ⌋ $GL_2(\mathbb{Q}_p)$

knows LLC $\Pi_p \leftrightarrow \rho_p$

$$\Psi := \text{Hom}(\tilde{L}_T, \bar{\mathbb{Q}}_p) \hookrightarrow \mathbb{D}^* \times \underbrace{GL_2(\mathbb{Q}_p)}_{\pi_p} \times \underbrace{Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}_{\rho|_{Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}}$$

$$\Psi [JL^{-1}(\pi_p) \otimes \pi_p] = \rho_p$$

gives LLC.

§3 NALT ... to characterize LLC / define LLC in terms of \tilde{L}_T

Open questions:

Define LLC by

$$\pi_p \mapsto \Psi [JL^{-1}(\pi_p) \otimes \pi_p] \text{ and extend suitably to } \text{Rep}(GL_2(\mathbb{Q}_p)).$$

Show this is compatible w/ functoriality.

Change $\left\{ \begin{array}{l} n \text{ (of } GL_n) \\ K/\mathbb{Q}_p \end{array} \right.$	$AI \leftrightarrow \text{Ind}$	}	all unknown in terms of a local proof.
	$\boxplus \leftrightarrow \oplus$		
	$st \leftrightarrow S_p$		
	$St(\pi) \xleftrightarrow{\pi \leftrightarrow \rho} Sp \otimes \rho$		

- using alg. geo. can see log smooths (tame) purely locally.
- mirabolic level \rightarrow compute conductor.

p-adic applications:

$$\left\{ \text{Alg.} \right\} \longleftrightarrow \left\{ \text{deRham } \sigma \right\}_p$$

\cap \cap

$\left. \begin{array}{l} \text{p-adic Auto.} \\ \text{rep. of} \\ \text{GL}_n(\mathbb{A}) \end{array} \right\}$

$\xleftrightarrow[\text{GL}_n]{\text{p-adic}}$

$\left. \begin{array}{l} \text{irred } \rho: \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_n(\bar{\mathbb{Q}}_p) \\ \text{unram at all } p \end{array} \right\}$

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?