Anticyclotomic μ-invariants of elliptic curves:

(Tom Weston) 1-22-7

(joint w/ Rob Pollack)

$E/\mathbb{Q}$ elliptic curve

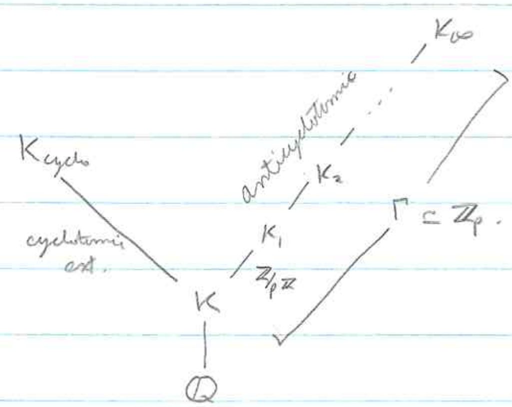
N = conductor (square free)

Fix a prime $p \geq 5$, $p \nmid N$, p is ordinary for E. (s.s is ok, but this is simpler).

K = imag. quadratic field.

$(D_K, Np) = 1$ $\quad$ D_K = discriminant of K.

$\Rightarrow$ $N = N^+N^-$ $\quad$ N^+ primes split in K, N^- primes inert in K.



complex conjugation acts by -1 on the anticyclotomic ext.

Algebraic object of interest: $E(K_\infty) \circlearrowleft \Gamma$

To get the Iwasawa algebra to act nicely, we instead study $Sel_p(K_\infty, E)$.

The Selmer group is a p-adic object, so we get an action of $\mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T]]$.

Analytic object of study: $L_p(K_\infty, E) \in \Lambda$

This p-adically interpolates $L(E, \chi, 1)$.

There are two cases depending on N^-:

① N^- has even # of prime divisors $\Rightarrow$ Heegner points exist in $E(K_\infty)$.

Cornut - Vatsal using ergodic theory show $rk_\Lambda \, Sel_p(K_\infty, E)^\vee \geq 1$.

$\Rightarrow L_p(K_\infty, E) = 0$ (forced by BSD, but it is known w/o BSD).

(2) N^- has odd # of prime divisors (no Heegner points)

Vatsal: $L_p(K_\infty, E) \neq 0$. In fact, $p \nmid L_p(K_\infty, E)$. Say

$\mu^{an}(K_\infty, E) = 0$.

Bertolini - Darmon: $Sel_p(K_\infty, E)$ is Λ-torsion.

$$Sel_p(K_\infty, E) \simeq \Lambda/p^{n_1} \oplus \cdots \oplus \Lambda/p^{n_r} \oplus \Lambda/f_1 \oplus \cdots \oplus \Lambda/f_s.$$

$L_p^{alg}(K_\infty, E) = p^{n_1 + \cdots + n_r} f_1 \cdots f_s$. , $\mu^{alg}(K_\infty, E) = n_1 + \cdots + n_r$.

Further, we know $L_p^{alg} \mid L_p^{an}$ in Λ. Cor: $\mu^{alg} = 0$.

<u>Main conjecture:</u> $L_p^{alg} = L_p^{an}$ (up to $\Lambda^\times$)

<u>Remark:</u> Assume $\bar\rho_{E,p} : G_{\mathbb{Q}} \twoheadrightarrow GL_2(\mathbb{F}_p)$.

$\leftarrow$ no other E' congruent to E mod p w/ same cond.

B-D also assume E is "p-isolated".

P-W weaken this condition to $q \mid N^-$ and $q \equiv \pm 1 \pmod p \Rightarrow E[p]$ is ramified at q.

<u>The p-adic $\mathcal{L}$-function:</u>

$L_p(K_\infty, E) \in \Lambda = \mathbb{Z}_p[[\Gamma]]$

$\chi : \Gamma \twoheadrightarrow Gal(K_n/K) \hookrightarrow \mathbb{C}_p^\times$.

$\rightsquigarrow \chi : \Lambda \to \mathbb{C}_p^\times$.

Interpolation:

$$\chi(L_p(K_\infty, E)) = \alpha^{-2n} \frac{L(E, \chi, 1)}{\Omega} \sqrt{D_K} \cdot p^n$$

α = unit root of $x^2 - a_p(E) x + p$.

$$L(E,\chi,s) = \sum_{n\geq 1} \frac{a_n(E)\chi(n)}{n^s}$$

Ω = period

Periods:

f = newform of wt 2 and level N assoc to E.

$\mathbb{T}(N)$ = Hecke algebra acting on $S_2(N)$. (p-adic completion of this)

$$f = \sum_{n\geq 1} a_n(E) q^n$$

$$\rightsquigarrow \quad \pi_f : \mathbb{T}(N) \longrightarrow \mathbb{Z}_p$$

$$T_n \longmapsto a_n(E).$$

Def: $\eta_f(N) = \pi_f(\operatorname{ann}_{\mathbb{T}(N)} \operatorname{Ker} \pi_f)$

"measures congruences to f"

Example: $\mathbb{T}(N) = \mathbb{Z}_p \times \mathbb{Z}_p$

π_f = proj_1

$\eta_f(N) = 1$ (no congruences mod p)

Example: $\mathbb{T}(N) = \{(a,b) \in \mathbb{Z}_p \times \mathbb{Z}_p \mid a \equiv b\ (p)\}$

π_f = proj_1

$\eta_f(N) = p$. (so there is a mod p congruence)

$$\Omega^{can} = \frac{(f,f)}{\eta_f(N)}$$ ← Petersson product.

Hida's canonical form.

Not the one used by Vatsal or B-D.

$\mathbb{T}(N^+, N^-)$ = qt. of $\mathbb{T}(N)$ acting on $S_2(N)^{N^-\text{-new}}$.

Jacquet-Langlands : $\mathbb{T}(N^+, N^-)$ acts on X_{N^+, N^-} = Shimura curve $\leftarrow$ level N^+ structure

par. abelian surfaces with quaternionic-mult
$\uparrow$ ram at $N^-\infty$

$M = \mathrm{Pic}(X_{N^+,N^-}) \otimes \mathbb{Z}_p$.

$\langle\cdot,\cdot\rangle : M \times M \to \mathbb{Z}_p$ arith. intersection pairing.

$M_f \subseteq M$ subspace where $\mathbb{T}(N^+, N^-)$ acts via π_f.

This M_f is free of rank one over $\mathbb{Z}_p$.

Fix generator g_f,

$$\Omega = \frac{(f,f)}{\langle g_f, g_f\rangle} \quad \leftarrow \text{used by V. and B-D}$$

Not the same as other period.

Remark: Often $\langle g_f, g_f\rangle = \eta_f(N^+, N^-)$ measures congruences of f w/ forms new at N^-.

$L_p(K_\infty, E) \longleftrightarrow \Omega$

$\mathcal{L}_p(K_\infty, E) \longleftrightarrow \Omega^{can}$

Advantages of $\mathcal{L}_p$:

① Behaves well under congruences.

② Has more info. at primes div. N^-.

Does $L_p(K_\infty, E)$ have a Selmer group?

Yes: Greenberg's def..

$$\mathrm{Sel}_p(K_\infty, E) = \mathrm{Ker}\left(H^1(K_\infty, E[p^\infty]) \to \prod_{w \nmid p} H^1(I_w, E[p^\infty]) \times (\text{condition at } p)\right)$$

Remark: $H^1(I_w) = H^1(K_{\infty,w})$ if $w \nmid N^-$

Prop: $0 \to Sel_p \to \mathcal{S}el_p \longrightarrow \prod_{q | N^-} \mathbb{Z}/p^{t_E(q)} \otimes \Lambda^\vee \to 0$

$t_E(q)$ = largest t s.t. $E[p^t]$ is unram. at q.

Cor: $\mu(\mathcal{S}el_p) = \sum_{q | N^-} t_E(q)$.

However, $\mu(\mathcal{L}_p) = \mathrm{ord}_p \dfrac{\eta_E(N)}{\langle g_f, g_f \rangle}$

Do these agree? Yes!

Intuition: Pretend $\langle g_f, g_f \rangle = \eta_f(N^+, N^-)$.

Suppose $t_E(q) > 0$, $q | N^-$. $\Rightarrow E[p]$ unram. at $q | N^-$

$\Rightarrow \exists\ g \equiv f \pmod p$, g of level N/q.

$\Rightarrow g$ congruent to $\eta_f(N)$ but not $\eta_f(N^+, N^-)$

$\Rightarrow \mathrm{ord}_p \left(\dfrac{\eta_f(N)}{\eta_f(N^+, N^-)} \right) > 0.$