The mean number of 3-torsion elements in ray class groups of quadratic fields.

K = # field, $\mathcal{O}_K$ = ring of integers

I_K = group of invertible fractional ideals of $\mathcal{O}_K$

P_K = group of principal ideals of I_K

$Cl(K) = I_K/P_K$ class group (finite, abelian)

$h_K = \# Cl(K)$ "class number"

<u>Motivating Question:</u> How are class groups of quadratic fields distributed over all quadratic fields ordered by discriminant?

<u>Cohen-Lenstra:</u> (conjecture) Outside 2, the groups are "random", i.e., a group G occurs in such a family with frequency proportional to $\frac{1}{\# Aut(G)}$.

<u>Cohen-Lenstra heuristics:</u> $p > 2$

① The average number of p-torsion elements in class groups of real quadratic fields is $1 + \frac{1}{p}$.

② $$\lim_{X \to \infty} \frac{\displaystyle\sum_{\substack{D \text{ neg} \\ \text{fund. disc}}} \# Cl(\mathbb{Q}(\sqrt{D}))[p]}{\displaystyle\sum_{\substack{D \text{ neg} \\ \text{fund. disc}}} 1} = 2.$$

<u>Thm (Davenport-Heilbronn):</u> True for $p = 3$.

Pf: (sketch): K_2 = quadratic field.

① #(3-torsion elements in $Cl(K_2)$)

$\overset{CFT}{\longleftrightarrow}$ number of cubic fields w/ same discriminant as K_2

② parameterize cubic rings in terms of binary cubic forms $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$, $a,b,c,d \in \mathbb{Z}$.

Then $f(x,y) \longleftrightarrow$ ring w/ basis $R_f = \langle 1, \omega, \theta \rangle$ where one can write down the mult. table R_f in terms of a,b,c,d.

In the other direction, if R is a cubic ring w/

$R = \langle 1, \omega, \theta \rangle$

$$R/\langle 1 \rangle \cong \mathbb{Z}^2 \longrightarrow \Lambda^2 R/\langle 1 \rangle \cong \mathbb{Z}.$$
$$(x,y) \longmapsto f(x,y)$$

This map is cubic and can be represented as a binary cubic form.

③ Count binary cubic forms which are assoc. to cubic fields on RHS of ①.

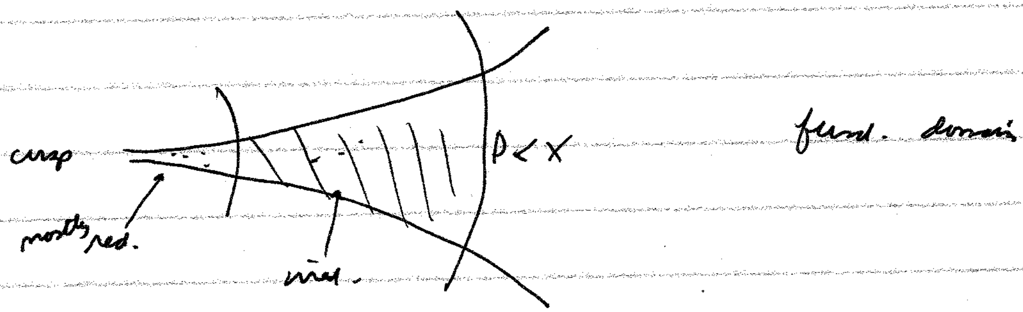
② is due to Delone-~~Faddeev~~ Faddeev (Gan-Gross-Savin)

$$\left\{ \begin{matrix} \text{cubic rings} \\ \text{up to isom} \end{matrix} \right\} \overset{1\text{-}1}{\longleftrightarrow} \left\{ \begin{matrix} GL_2(\mathbb{Z}) \text{ equiv. classes of} \\ \text{integer binary cubic forms} \end{matrix} \right\}$$

③ $\exists$ a non-compact fundamental domain for the action of $GL_2(\mathbb{Z})$ on real binary cubic forms

$GL_2(\mathbb{Z}) \circlearrowleft (a,b,c,d) \in \mathbb{R}^4$

Fact: $\left\{\begin{matrix}\text{orders in} \\ \text{cubic fields}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}GL_2(\mathbb{Z}) \text{ equiv. classes} \\ \text{of irred. integral binary} \\ \text{cubic fields}\end{matrix}\right\}$



volume of shaded $\sim$ # of $GL_2(\mathbb{Z})$ irred. classes of binary cubic forms $f(x,y)$ where disc $f(x,y) < X$.

$\longrightarrow$ volume of shaded region = # of cubic orders w/ disc $<|X|$

$$\left\{\begin{matrix} = \pi^2/72\, X & (X \text{ pos, real quad.}) \\ = \pi^2/24\, X & (X \text{ neg., imag. quad.}) \end{matrix}\right.$$

Fix K_2 a quadratic field

$[I]$ is a 3-torsion ideal class of K_2.

$$\#\{\text{3-torsion elts}\} \longleftrightarrow 2\cdot\#\left\{\begin{matrix}\text{order 3} \\ \text{subgroups}\end{matrix}\right\} + 1$$

b/c 3-torsion elt and their square constitute same thing.

from identity

$$\left\{\begin{matrix}\text{order 3-subgroups} \\ \text{of } Cl(K_2)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{index 3-subgroups} \\ \text{of } Cl(K_2)\end{matrix}\right\}$$

$$\begin{tikzcd} & H_{K_2} \arrow[dl, dash] \arrow[dr, dash] & & \\ & & K_6 \arrow[dr, dotted, dash, "2"] & \\ K_2 \arrow[uur, dash, bend left, "Cl(K_2)"] \arrow[urr, dash, "3"'] \arrow[dr, dash, "2"'] & & & K_3 \\ & \mathbb{Q} \arrow[urr, dotted, dash, "3"'] & & \end{tikzcd}$$

Hilbert class field

By CFT $\{$index 3 subgroups of $Cl(K_2)\}$

$\longleftrightarrow \{$cubic extensions of K_2 unramified $/K_2\}$

<u>Fact:</u> All such extensions K_6 are Galois over $\mathbb{Q}$.

K_6 is unramified iff K_3 is nowhere totally ramified (as ~~there does not exist a prime~~ $_{K_3}$ no prime $p \in \mathbb{Z}$ splits as $\mathfrak{p}^3$ in $\mathcal{O}_{K_3}$.)

① $\#\{$3-torsion ideal class of $K_2\}$

$\updownarrow$

$2 \cdot \#\{$nowhere totally ramified cubic fields K_3 w/ $\mathrm{disc}(K_2) = \mathrm{disc}(K_3)\} + 1$

<u>Sieve:</u> shaded region

$\#\{$nowhere totally ramified maximal cubic orders w/ $\mathrm{Disc} < |X|\}$

$$\sim \begin{cases} \cancel{3}\, \dfrac{1}{2\pi^2} X & \text{real} \\ \dfrac{3}{2\pi^2} X & \text{imag} \end{cases}$$

$$\lim_{X\to\infty} \frac{\sum_{0<\mathrm{Disc}(K_2)<X} \#\mathrm{Cl}(K_2)[3]}{\sum_{0<\mathrm{Disc}(K_2)<X} 1} = 2 \lim_{X\to\infty} \frac{\frac{1}{2\pi^2} X}{\sum_{\substack{K_2 \text{ w/} \\ 0<\mathrm{Disc}(K_2)<X}} 1} + 1$$

$$= \begin{cases} 2 & \text{complex} \\ 4/3 & \text{real} \end{cases}$$ (written down only for real here though)

<u>Ray class groups:</u>

$c \in \mathbb{Z}$ conductor

K = # field

$$I_c(K) = \left\{ \begin{array}{l} \text{invertible fractional ideals of } \mathcal{O}_K \text{ generated by prime} \\ \text{ideals coprime to } c \end{array} \right\}$$

$$P_{1,c\infty}(K) = \left\{ \begin{array}{l} (\alpha) \text{ principal of } \mathcal{O}_K \\ \alpha \equiv 1 \pmod{c} \\ \alpha \text{ is totally positive} \end{array} \right\}$$

$$\mathrm{Cl}(K,c) = I_c(K) / P_{1,c\infty}(K)$$

<u>Prop:</u> If c is a square-free integer coprime to 3

$$\#\mathrm{Cl}_3(K,c) = 2 \cdot \# \left\{ \begin{array}{l} \text{cubic exts of } K \\ \text{unram away from } c \end{array} \right\} + 1$$

$$\begin{tikzcd}
 & \text{Ray class field } (K,c) \arrow[dl, no head] \arrow[dr, no head] & \\
K_2 \arrow[rr, no head, "3"] \arrow[d, no head, "2"'] & & K_6 \\
\mathbb{Q} & &
\end{tikzcd}$$

Fixing $K_2 = K$:

K_6/K_2 unramified away from c, $\mathrm{Gal}(\overset{\text{normal closure}}{\tilde{K}_6}/\mathbb{Q}) \cong \begin{cases} S_3 \longleftrightarrow \text{unram}/K_2 \\ C_6 \\ S_3 \times C_3 \end{cases}$

$\longleftrightarrow$ K_3 nowhere tot. ramified

fix $c \in \mathbb{Z}$ squarefree indivisible by 3

The average number of 3-torsion elements in the ray class group of conductor c is

$j = \#$ of $1 \pmod 3$ primes $\mid c$.

$$\begin{cases} 3^j\left(1 + \frac{1}{3}\prod_{p \mid c}\left(1 + \frac{p}{p+1}\right)\right) & \text{real case} \\ 3^j\left(1 + \prod_{p \mid c}\left(1 + \frac{p}{p+1}\right)\right) & \text{imag case.} \end{cases}$$