Algebraic flows on abelian varieties and Shimura varieties

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Let $\pi: \mathbb{C}^n \to A = \Gamma \backslash \mathbb{C}^n$ be a uniformizing map.

<u>Def:</u> A weakly special subvariety V of A is a variety V of the form $P+B$, $P \in A(\mathbb{C})$, B abelian subvariety.

<u>Thm:</u> (Ax-Lindemann) Let Θ be an irreducible subvariety of $\mathbb{C}^n$. Then $\overline{\pi(\Theta)}^{Zar} = P+B$.

<u>Question:</u> What can be said about the topological closure $\overline{\pi(\Theta)}$?

<u>Def:</u> Let $W \subseteq \mathbb{C}^g$ be an $\mathbb{R}$-vector space s.t. $W \cap \Gamma$ is a lattice in W. Then $W/_{W\cap\Gamma}$ is a real torus. A real analytic subvariety V of A is said to be real weakly special if $V = P + W/_{\Gamma \cap W}$ with $P \in A(\mathbb{C})$, $W/_{\Gamma\cap W}$ a real torus. Let μ_V be the canonical Haar measure on such a V.

<u>Conj:</u> $\pi: \mathbb{C}^g \to A = \Gamma \backslash \mathbb{C}^g$, Θ irreducible algebraic in $\mathbb{C}^g$. There exists a finite number of real weakly special subvarieties of A s.t. $\overline{\pi(\Theta)} = \pi(\Theta) \cup \bigcup_{k=1}^{n} Z_k$.

Let $\omega = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$. Let $R>0$, be the probability measure on $\mathbb{C}^g$

$$\mu_{\Theta,R}(f) = \frac{\int_{\Theta \cap B(0,R)} f\, \omega^d}{\int_{\Theta \cap B(0,R)} \omega} \qquad d = \dim(\Theta).$$

Conj 2: There exists $Z_1, \dots, Z_r$ real weakly special subvarieties of A and $(c_1, \dots, c_r)$, $c_i > 0$, $\sum c_i = 1$ s.t. $\mu_{\Theta, R}$ converges weakly to $\sum c_\alpha \mu_{Z_\alpha}$.

Thm (Yafaev, -): Let C be a curve in $\mathbb{C}^g$. Let $C_1, \dots, C_r$ be the set of branches of C through all points at infinity. For all $\alpha \in \{1, \dots, r\}$ we will define a real weakly special subvariety Z_α. We will call it the asymptotic Mumford-Tate torus of C_α.

(i) $\overline{\pi(C)} = \pi(C) \cup \bigcup_{i=1}^{r} \cancel{} T_\alpha$

(ii) $\mu_{C,R} \xrightarrow{\text{w.c.}} \sum_{\alpha=1}^{r} c_\alpha \mu_{Z_\alpha}$.

Def 1: $\Theta \subseteq \mathbb{C}^g$ irreducible. Assume $0 \in \Theta$. The Mumford-Tate group $MT(\Theta)$ is the smallest $\mathbb{Q}$-vector subspace of $\Gamma \otimes \mathbb{Q}$ s.t. $\Theta \subseteq W \otimes \mathbb{R}$. If $P \in \Theta$, $MT(\Theta)$ is defined $MT(\Theta - P)$.

Let $W_\Theta = MT(\Theta) \otimes \mathbb{R}$. $\qquad T_\Theta = \pi(P) + W_\Theta / \Gamma \cap W_\Theta$.

Remark: $\overline{\pi(\Theta)} \subseteq T_\Theta$

Def 2: C^* the Zariski closure of C in $\mathbb{P}_1^g$.

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$C^\times \setminus C = \{P_1, \dots, P_s\}$. Let C_α be a branch of C near some P_i.

There exists a smallest real affine subspace $Q_\alpha + W_\alpha$ such that $W_\alpha \cap \Gamma$ is a lattice in W_α and such that C_α is asymptotic to $Q_\alpha + W_\alpha$.

We say that $T_\alpha = \pi(Q_\alpha) + W_\alpha/_{\Gamma \cap W_\alpha}$ is the asymptotic M.T. torus of C_α.

Lemma: (a) $\overline{\pi(C_\alpha)} \setminus \pi(C_\alpha) \subset T_\alpha$

(b) $T_\alpha \subset T_{(\omega)}$.

(c) $\overline{\pi(C)} \subset \pi(C) \cup \bigcup_{\alpha=1}^{r} T_\alpha$.

Main result: Let μ_α be the canonical measure on T_α. For $R \gg 0$, let $\mu_{\alpha,R}$ be the probability measure on A s.t. $\forall f \in C^0(A)$,

$$\mu_{\alpha,R}(f) = \frac{\int_{C_\alpha \cap B(0,R)} f\omega}{\int_{C_\alpha \cap B(0,R)} \omega} \xrightarrow{c.w} \mu_\alpha.$$

Ex. 1: $C \subseteq \mathbb{C}^2$

$z_2^2 = z_1^n + a_{n-1} z^{n-1} + \dots + a_0$

$\forall A = \mathbb{C}^2/\Gamma$

$\overline{\pi(C)} = A$

$\mu_{C,R} \to \mu_A$ as $R \to \infty$

$T_C = A = T_\alpha$ for all branches.

$C = \{ (z_1, z_2) \in \mathbb{C}^2, z_1 z_2 = 1 \}$

$\Gamma = \mathbb{Z}[i] \oplus \mathbb{Z}[i]$

$A = E \times E$

$\overline{\pi(C)} = \pi(C) \cup E \times \{0\} \cup \{0\} \times E$

$\mu_{C,R} \longrightarrow \frac{1}{2}(\mu_{\{0\}\times E} + \mu_{E \times \{0\}})$.

(*) For a generic Γ, $\overline{\pi(C)} = A$.

<u>Proof:</u> $\langle , \rangle$ Hermitian product on $\mathbb{C}^g$

$(,) = \mathrm{Re}(\langle , \rangle)$

$\Gamma \subseteq \mathbb{C}^g$

$\hat{\Gamma} := \{ \theta \in \mathbb{C}^g, (\theta, \gamma) \in \mathbb{Z} \ \forall \ \gamma \in \Gamma \}$

$X^*(A) \simeq \hat{\Gamma} \simeq \{ \chi_\theta : \theta \in \hat{\Gamma} \}$

$\chi_\theta : \mathbb{C}^g \to \mathbb{C}^\times$

$z \mapsto \exp(2\pi i (\theta, z))$.

<u>Weyl's criterion</u> Let μ be a measure on A.

$\mu_{\theta,R} \xrightarrow{w.c} \mu \iff \forall \ \theta \in \hat{\Gamma}$

$\mu_{\theta,R}(\chi_\theta) \to \mu(\chi_\theta)$

as $R \to \infty$.

<u>Computation of the asymptotic MT torus</u>

Let C_α be a branch of C.

Let $\underline{Z} = (Z_1, \dots, Z_g)$, let $Z_i = Z$ be a coordinate s.t.

Z is unbounded as $\underline{Z}$ varies in C_α. $\forall j \in \{1, \dots, g\}$

we have a Puiseux expansion

$$Z_j = \sum_{n \geq 0} a_{n,j} Z^{\alpha_j - n/e_j} \qquad \begin{array}{l} \alpha_j \in \mathbb{Q} \\ e_j \in \mathbb{N}^\times . \end{array}$$

Let $a_0(Z_j)$ be the constant term of Z_j in this expansion.

$P_\alpha = (a_0(Z_1), \dots, a_0(Z_g))$.

Let $\Theta_\alpha = \{ \chi_\theta ; \theta \in \hat{\Gamma}$ s.t. $(\theta, \underline{Z})$ is bounded as $\underline{Z}$ varies in $C_\alpha \}$.

There is an exact sequence

$$0 \longrightarrow \Theta_\alpha \longrightarrow \hat{\Gamma} = X^*(A) \longrightarrow X^*(T_\alpha') \longrightarrow 0$$

for a real subtorus T_α' of A.

<u>Prop.:</u> $T_\alpha = \pi(P_\alpha) + T_\alpha'$.

$\mu_{T_\alpha} = \mu_\alpha$ is characterized by

$$\mu_\alpha(\chi_\theta) = \begin{cases} 0 & \text{if } \chi_\theta\big|_{T_\alpha} \neq 1 \\ e^{2\pi i (\theta, P_\alpha)} & \text{if } \chi_\theta\big|_{T_\alpha} = 1 . \end{cases}$$

$$\mu_{\alpha,R}(f) = \frac{\int_{C \cap B(0,R)} f\omega}{\int_{C \cap B(0,R)} \omega}$$

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$\mu_{\alpha, R} \xrightarrow{w.c} \mu_\alpha.$

$\iff \forall \theta \in \hat{\Gamma}, \ \mu_{\alpha, R}(\chi_\theta) \to \mu_\alpha(\chi_\theta).$