

Congruence ideals and Hecke groups for symmetric powers:

Thouine

2-5-16

PS1

Joint with H. Hida, in progress.

①  $N > 1, p \times N \quad \Lambda_1 = \mathbb{Z}_p[[x]] \quad p > 2$

$$h_1 = \varprojlim_m h_2(\Gamma_0(m) \cap \Gamma_1(p^m), \mathbb{Z}_p).$$

generated by  $T_x, U_p, \chi \otimes N_p$ .

finite flat over  $\Lambda_1$ , reduced.

$A_1 =$  domain, finite torsion free over  $\Lambda_1$ .

$$\mu: h_1 \rightarrow A_1 \quad \Lambda_1\text{-alg. homom.}$$

$\mu =$  fixed Hecke family

Assume  $\bar{\rho}_\mu$  irred.  $\rho_\mu: G_{\mathbb{Q}} \rightarrow GL_2(A_1)$  rep. assoc. to  $\mu$ .

$$j \geq 1 \quad a^j = \text{Sym}^{2j} St_2 \otimes \det^{-j} St_2 \hookrightarrow^{GL_2/\mathbb{Z}_p} \quad p \geq 2j$$

$a_\mu^j = a^j \circ \rho_\mu$  Galois rep. of dim  $2j+1$ .

$$p\text{-ord: } G_{\mathbb{Q}, p} \curvearrowright \{F^k a_\mu^j\}_k$$

$$\text{gr}^k a_\mu^j \curvearrowright \mathbb{Z}_p = X^k \quad (A_1\text{-free})$$

$$X^k: G_{\mathbb{Q}, p} \rightarrow \Lambda_1^X \downarrow \mathbb{Z}_p^X$$

$$\chi(\sigma) = \omega(\sigma) \cdot u^{\ell(\sigma)}$$

$u = 1+p$  top. gen. of  $1+p\mathbb{Z}_p$

$$X(\sigma) = \omega(\sigma)(1+x)^{\ell(\sigma)}$$

$$\text{Ad}(a_\mu^j) = H_{\text{minord}}^1(\mathbb{Q}, a_\mu^j \otimes_{A_1} \tilde{A}_1^*)$$

$$(M^* = \text{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p))$$

$$= \ker(H' \rightarrow \bigoplus_{l \neq p} H'(\mathbb{I}_l, -) \oplus H'(\mathbb{I}_p, \frac{F \cdot a_\mu^j}{F' \cdot a_\mu^j} \otimes \tilde{A}_1^{\wedge}))$$

Tilouine

2-5-16

192

$\tilde{A}_1 =$  normal closure of  $A_1$ , flat over  $\Lambda_1$ .

$A_1$ -mod.  $\text{Sel}(a_\mu^j)$  is f.g.  $\tilde{A}_1$ .

Conj. (duzawa-Greuter me):  $\text{Sel}(a_\mu^j)^*$  is torsion over  $\tilde{A}_1$

and

$$\chi_{\text{Sel}(a_\mu^j)^*} \stackrel{!}{=} L_p(a_\mu^j)$$

interpolates  $L^{int}(a_{f_{k^j}}^j, 1)$  for  $k \in \mu$ .

## ② Congruence ideals

$$h_i \xrightarrow{\mu} A_i$$

$$\downarrow T_i \uparrow$$

loc. of  $h_i$  at the max. ideal assoc. to  $\bar{p}_\mu$

$$\tilde{T}_i = T_i \otimes_{\Lambda_1} \tilde{A}_1 \xrightarrow{\tilde{\mu}} \tilde{A}_i$$



reduced

$$\mathcal{L}_1 = \text{Frac}(\Lambda_1)$$

$$\tilde{T}_i \otimes_{\Lambda_1} \mathcal{L}_1 = (\tilde{A}_i \otimes_{\Lambda_1} \mathcal{L}_1) \times \tilde{T}'_i \otimes_{\Lambda_1} \mathcal{L}_1$$

$$\tilde{T}_i \hookrightarrow \tilde{A}_i \times \tilde{T}'_i \quad \text{torsion cokernel}$$

$$p_{\tilde{T}_i} = \bar{p}_\mu$$

$$L_{\tilde{\mu}} = \tilde{T}_1 \cap (\tilde{A}_1 \times 0)$$

$L_{\tilde{\mu}} \subset \tilde{A}_1$  ideal: congruence ideal of  $\tilde{\mu}$ .

Thm (Wiles + Kida): Assume  $N$  is sq. free,  $\text{Im } \bar{\rho}_{\mu} \cong \text{SL}_2(\mathbb{F}_p)$ ,  
 $\alpha = \mu(U_p)$ ,  $\alpha^2 \not\equiv 1 \pmod{m_A}$ ,  $\forall \lambda | N$ ,  $\bar{\rho}_{\mu}|_{\mathbb{Z}_{\lambda}} \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$  } (\*),  $\leftarrow \neq 0$

1)  $R_1 = T_1$  is real complete int. over  $\Lambda$ .

"  
 Univ. mod  
 def. by  $\bar{\rho}_{\mu}$

2)  $\text{Sel}(\alpha_{\mu}^{\pm})^*$  is torsion,  $L_{\tilde{\mu}}$  is principal and

(a)  $L_{\tilde{\mu}} = (\chi_{\text{Sel}(\alpha_{\mu}^{\pm})^*})$

(b)  $L_{\tilde{\mu}} = (L_p(\alpha_{\mu}^{\pm}))$

$\alpha_{\mu}^{\pm} = \text{Ad}_{\text{Sel}_2}(\rho_{\mu})$

Thm 1:  $p \geq 13$

$(*)_q = (x)_1 + \alpha^{12} \not\equiv 1 \pmod{m_A}$

Assume  $(*)_q$  then (a) is true for  $\alpha_{\mu}^j$   $j = 3, 2, 4$ .

for some congruence ideals.

③ Precise statements and sketch of proof.

$j = 3, 2$

Consider  $\text{Sym}^3 \rho_{\mu} : G_{\mathbb{Q}} \rightarrow \text{GSp}_4(A_1)$

$\text{GL}_2/\mathbb{Q} \xrightarrow{\text{Sym}^3 B_C} \text{GSp}_4/\mathbb{Q}$

established by H. Kim and

$\mathfrak{h}_2 = \mathfrak{h}_2^S$  : Hecke algebra for  $GSym(\mathbb{Q})$

finite tensor free over  $\Lambda_2 = \mathbb{Z}_p \llbracket x_1, x_2 \rrbracket$

$$\begin{array}{ccc} T_{1,1}, T_{1,2} & \mathfrak{h}_2 & \xrightarrow{\Theta} \mathfrak{h}_1 \xrightarrow{\mu} A_1 \\ \text{LXNP} & | & | \nearrow \\ U_{p,1}, U_{p,2} & \Lambda_2 & \rightarrow \Lambda_1 \end{array}$$

$$1+x_1 \mapsto (1+x)^2$$

$$1+x_2 \mapsto (1+x)$$

$$\begin{array}{ccc} & \curvearrowright & \\ T_2 & \xrightarrow{\Theta} T_1 & \xrightarrow{\mu} A_1 \\ | & & | \nearrow \\ \Lambda_2 & \rightarrow & \Lambda_1 \end{array}$$

$$\begin{array}{ccc} \tilde{T}_2 & \xrightarrow{\tilde{\Theta}} \tilde{T}_1 & \xrightarrow{\tilde{\mu}} \tilde{A}_1 \\ & \searrow & | \nearrow \\ & & \tilde{A}_1 \end{array}$$

$$\tilde{T}_2 = T_2 \otimes_{\Lambda_2} \tilde{A}_1$$

$$\tilde{T}_1 = T_1 \otimes_{\Lambda_1} \tilde{A}_1$$

$$\tilde{T}_2 \hookrightarrow \tilde{A}_1 \times \tilde{T}'_1$$

$$\tilde{T}_2 \hookrightarrow \tilde{T}_1 \times \tilde{T}'_0$$

$$\tilde{T}_1 \hookrightarrow \tilde{A}_1 \times \tilde{T}'_1$$

$$[\tilde{\lambda}] = \tilde{T}_2 \cap (\tilde{A}_1 \times 0)$$

$$[\tilde{\theta}] = \tilde{T}_2 \cap (\tilde{T}_1 \times 0)$$

$\Gamma_{\tilde{\mu}}$ 

Thm (Pollin, Cralle): Assume  $(*)_4$  then  $R_2 = T_2$  rel c. i./ $\Lambda_2$

↑  
univ. def. mod and  
def. rel to  $\text{Sym}^3 \tilde{\rho}_{\mu}$

Cor 1:  $\text{Sel}(\text{Ad}_{\text{Sp}_4}^{\text{Sym}^3} \rho_{\mu})^*$  torsion and  $(\chi_{\text{Sel}(\text{Ad}_{\text{Sp}_4}^{\text{Sym}^3} \rho_{\mu})^*}) = \Gamma_{\tilde{\lambda}}$ .

Cor 2:  $\Gamma_{\tilde{\lambda}} = \tilde{\lambda}(\Gamma_{\tilde{\theta}}) \Gamma_{\tilde{\mu}}$  and these ideals are all principal.

p&gt;2n

$$\text{Ad}_{\text{Sp}_{2n}}(\text{Sym}^{2n-1} St_2) = \bigoplus_{j=0}^{n-1} a^{2j+1} \hookrightarrow GL_2/\mathbb{Z}_p$$

n=2

$$(*)_4 \quad \text{Ad}_{\text{Sp}_4}(\text{Sym}^3 \rho_{\mu}) = a_{\mu}^1 \oplus a_{\mu}^3$$

$$\text{Sel} \text{Ad}(\quad) = \text{Sel}(a_{\mu}^1) \oplus \text{Sel}(a_{\mu}^3)$$

$$\downarrow$$

$$\Gamma_{\tilde{\lambda}}$$

$$\Gamma_{\tilde{\mu}}$$

$$\tilde{\lambda}(\Gamma_{\tilde{\theta}})$$

$$\tilde{\lambda}(\Gamma_{\tilde{\theta}}) = (\chi_{\text{Sel}(a_{\mu}^3)^*})$$

$$\begin{array}{ccc} & \nearrow & \\ \Gamma_2 & \xrightarrow{\theta} \Gamma_1 & \xrightarrow{\mu} A_1 \end{array}$$

$$j=2 \quad \text{Ad}_{\text{SL}_{2n}}(\text{Sym}^{2n-1}) = \bigoplus_{i=1}^{2n-1} a^i$$

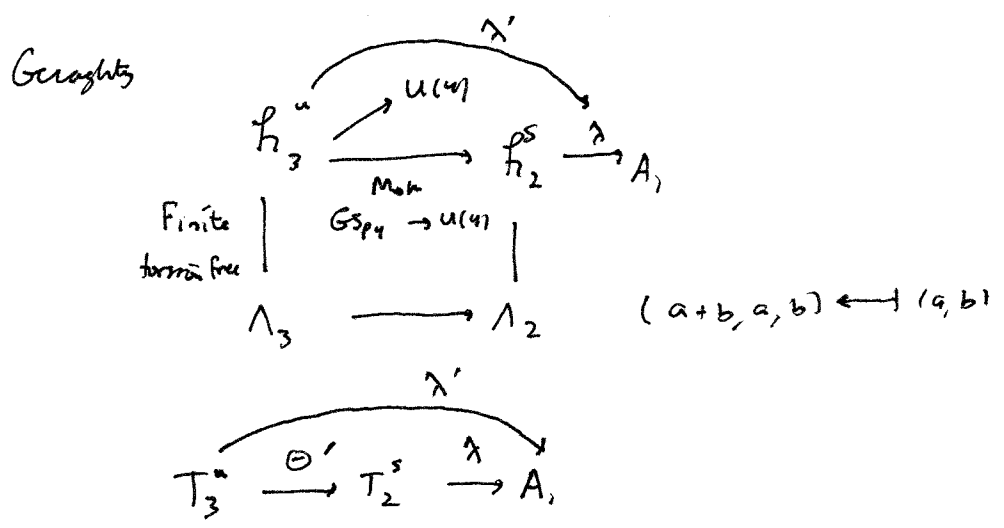
$n=2$

Tildene  
2-5-76  
pg 6

$$a^1 \oplus a^3 \oplus a^2 \quad (*)$$

$K =$  unreg. quad field in which  $p$  split  
 $q_1, q_2 \in \mathbb{N}$  are inert in  $K$

$U(4)_{/\mathbb{Z}}$  Compact at  $\infty$   
 $q$  split at all inert primes except  $q_1, q_2$ .



Thm (Hida-T.): Under  $(*)$ , we have  $R_3 = T_3$  complete intersections /  $\Lambda_3$ .

Corollary:  $[\tilde{\lambda}] = [\tilde{\lambda} \circ \Theta']$

$(*)$  says  $\text{Ad}_{\text{SL}_4} = \text{Ad}_{\text{Sp}_4} \oplus a^2$

Thus,

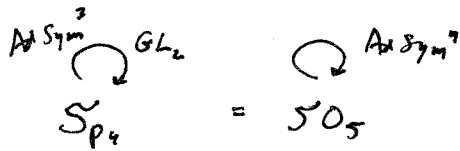
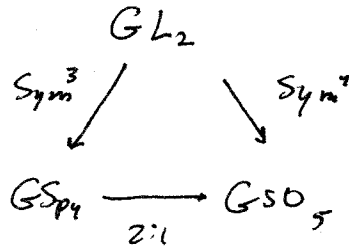
$$[\tilde{\chi}] = [\tilde{\chi}] \tilde{\chi}([\tilde{\theta}'])$$

" "  $\Rightarrow$  "  $\leftarrow$  Second term

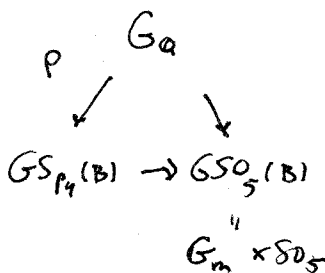
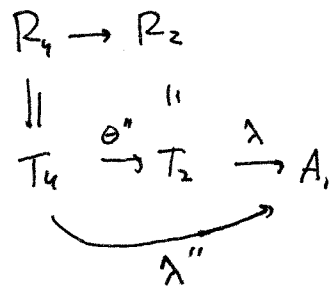
$$\chi_{\text{Ad Ad}_{SU_4}} \quad \chi_{\text{Ad Ad}_{Sp}} \quad \chi_{\text{Ad}(a_p^2)^*}$$

$\tilde{\chi}([\tilde{\theta}']) \leftarrow$  congruence between  $Sym^3$  and families in  $U(4)$   
 that are not symplectic.

$j=4:$

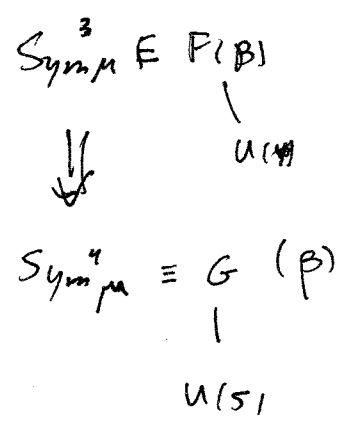


$$\mathcal{X}_5 = SO_5 \oplus a^2 \oplus a^4$$

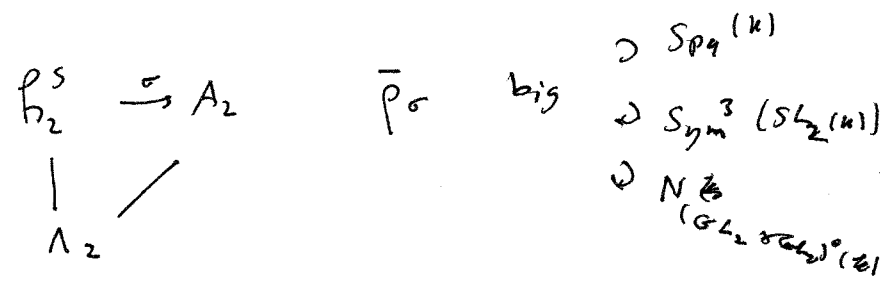


$$\begin{aligned}
 [ \tilde{\lambda}^n ] &= [ \tilde{\lambda} \tilde{\lambda} (L_{\tilde{G}^n}) ] \\
 \parallel & \\
 \lambda_{\text{Ad}(Ad_{Sym^3} S^2_5)} & \parallel \lambda_{\text{Ad} Ad_{Sym^3} P^2} \\
 \parallel & \\
 \lambda_{\text{Ad}(a^2)^*} & \parallel \lambda_{\text{Ad}(a^2)^*} \\
 \parallel & \\
 \lambda(L_{\tilde{G}^n}) &
 \end{aligned}$$

$$\lambda_{\text{Ad}(a^2)^*} = \frac{\tilde{\lambda}(L_{\tilde{G}^n})}{\tilde{\lambda}(L_{\tilde{G}^1})}$$

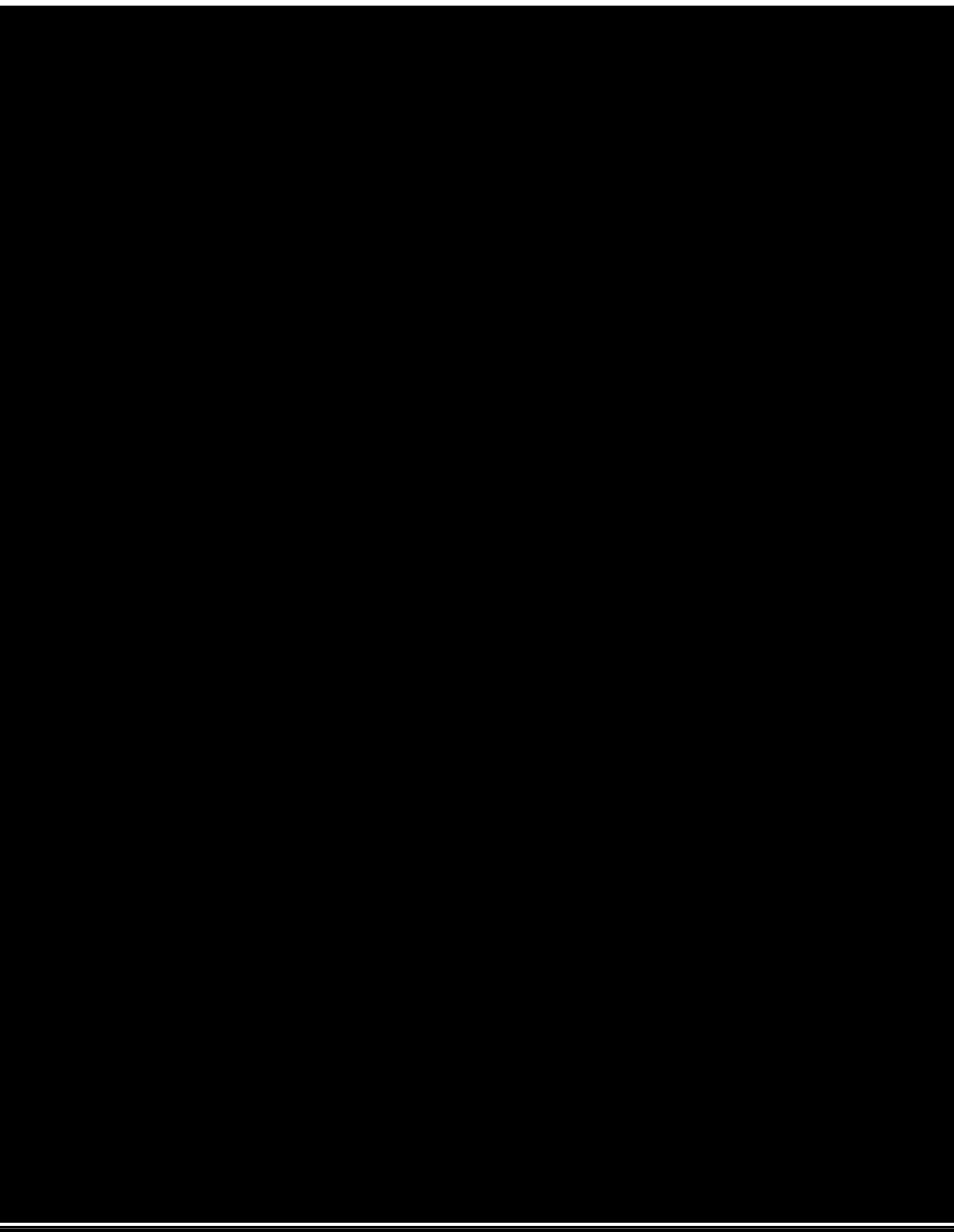


Standard rep for genus 2 Argyel families.



$$p_\sigma : G_{\mathbb{Q}} \rightarrow G_{Sp_4(A_2)}$$





Conj:  $(L_p(st_p)) \stackrel{?}{=} \sigma([\theta]).$

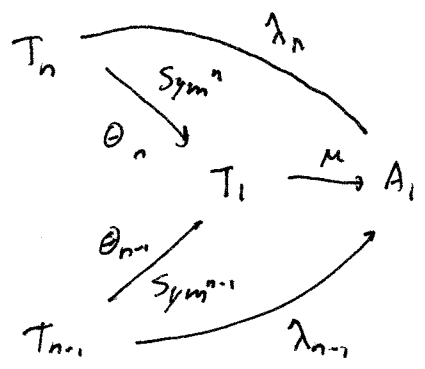
Other  $Sym^n$

$C_n = 2k(n-1) \quad p > 2n$   
 $|S_k \leq 2n$

$\alpha = \mu(U_p)$

$\alpha^{C_n} \neq 1 \text{ (mod } A_{cl})$

$Ad_{SL_n}(Sym^n) = Ad_{SL_n}(Sym^{n-1}) \oplus \alpha^n$



$\frac{\mu([\theta_n])}{\mu([\theta_{n-1}])} = \left( \prod Ad(a_i)^* \right)$

Assuming  $Sym^{n-1}, Sym^n$  exist.