Congruence ideals and Selmer groups for symmetric powers:

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PSI

joint with H. Hida, in progress.

① $N \geq 1$, $p \nmid N$ $\quad \Lambda_1 = \mathbb{Z}_p[[X]]$ $\qquad p > 2$

$$h_1 = \varprojlim_m e h_2(\Gamma_0(N) \cap \Gamma_1(p^m), \mathbb{Z}_p).$$

generated by T_ℓ, U_p, $\ell \nmid Np$.

finite flat over Λ_1, reduced.

A_1 = domain, finite torsion free over Λ_1

$$\mu : h_1 \twoheadrightarrow A_1 \qquad \Lambda_1\text{- alg. homom.}$$

μ = fixed Hida family

Assume $\bar{\rho}_\mu$ irred. $\rho_\mu : G_{\mathbb{Q}} \to GL_2(A_1)$ rep. assoc. to μ.

$j \geq 1 \qquad a^j = \mathrm{Sym}^{2j} St_2 \otimes \det^{-j} St_2 \circlearrowright GL_2/\mathbb{Z}_p \qquad p \geq 2j$

$a^j_\mu = a^j \circ \rho_\mu$ Galois rep. of dim $2j+1$.

$$\text{p-ord: } \quad G_{\mathbb{Q}_p} \circlearrowright \{F^k a^j_\mu\}_k$$

$$gr^k a^j_\mu \circlearrowleft I_p = \mathcal{X}^k \qquad (A_1\text{- free})$$

$$\begin{array}{ccc} \mathcal{X}^k : G_{\mathbb{Q}_p} & \longrightarrow & \Lambda_1^\times \\ & \searrow^{\chi} & \downarrow \\ & & \mathbb{Z}_p^\times \end{array} \qquad \chi(\sigma) = \omega(\sigma) \cdot u^{\ell(\sigma)}$$

$u = 1+p$ top. gen. of $1 + p\mathbb{Z}_p$

$$\mathcal{X}(\sigma) = \omega(\sigma)(1+X)^{\ell(\sigma)}$$

$$\mathrm{Sel}(a^j_\mu) = H^1_{\text{min ord}}(\mathbb{Q}, a^j_\mu \otimes_{A_1} \tilde{A}_1^*)$$

$$(M^* = \mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p))$$

$$= \ker\left(H^1 \longrightarrow \bigoplus_{\ell \neq p} H^1(I_\ell, -) \oplus H^1\left(I_p, \frac{F^- a_\mu^j}{F^+ a_\mu^j} \otimes \tilde{A}_1^\wedge\right)\right)$$

$\tilde{A}_1$ = normal closure of A_1, flat over Λ_1

A_1-mod. $Sel(a_\mu^j)$ is f.g. $\tilde{A}_1$.

Conj. (Iwasawa-Greenberg MC): $Sel(a_\mu^j)^*$ is torsion over $\tilde{A}_1$

and

$$\chi_{Sel(a_\mu^j)^*} \doteq \underbrace{L_p(a_\mu^j)}_{\text{interpolates } L^{int}(a^j_{f_\kappa}, 1) \quad f_\kappa \in \mu.}$$

② Congruence ideals

$$\begin{tikzcd} h_1 \arrow[r, "\mu"] \arrow[d] & A_1 \\ T_1 \arrow[ur, "\mu"] & \end{tikzcd}$$

T_1 — loc. of h_1 at the max. ideal assoc. to $\bar{\rho}_\mu$.

$$\underbrace{\tilde{T}_1 = T_1 \otimes_{\Lambda_1} \tilde{A}_1}_{\text{reduced}} \xrightarrow{\tilde{\mu}} \tilde{A}_1$$

$\mathcal{L}_1 = \mathrm{Frac}(\Lambda_1)$

$$\tilde{T}_1 \otimes_{\Lambda_1} \mathcal{L}_1 = (\tilde{A}_1 \otimes_{\Lambda_1} \mathcal{L}_1) \times \tilde{T}_\mu' \otimes_{\Lambda_1} \mathcal{L}_1$$

$\tilde{T}_1 \hookrightarrow \tilde{A}_1 \times \tilde{T}'_{\tilde{\mu}}$ torsion cokernel

$pr_1 = \tilde{\mu}$

$I_{\tilde{\mu}} = \tilde{T}_1 \cap (\tilde{A}_1 \times 0)$

$I_{\tilde{\mu}} \subset \tilde{A}_1$ ideal : congruence ideal of $\tilde{\mu}$.

<u>Thm (Wiles + Hida):</u> Assume N is sq. free, $\mathrm{Im}\,\bar{\rho}_\mu \supseteq SL_2(\mathbb{F}_p)$, $\alpha = \mu(U_p)$, $\alpha^2 \not\equiv 1 \pmod{m_{A_1}}$, $\forall \ell \mid N$, $\bar{\rho}_\mu|_{I_\ell} \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, $* \neq 0$ $\Big\}$ $(*)_1$

1) $R_1 = T_1$ is recl complete int. over Λ.
R_1 = univ. mi ord def. ring of $\bar{\rho}_\mu$

2) $\mathrm{Sel}(a_\mu^1)^*$ is torsion, $I_{\tilde{\mu}}$ is principal and

(a) $I_{\tilde{\mu}} = (\chi_{\mathrm{Sel}(a_\mu^1)^*})$

(b) $I_{\tilde{\mu}} = (L_p(a_\mu^1))$ $\qquad a_\mu^1 = \mathrm{Ad}_{SL_2}(\rho_\mu)$.

<u>Thm 1:</u> $p \geq 13$

$(*)_4 = (*)_1 + \alpha^{12} \not\equiv 1 \pmod{m_{A_1}}$

Assume $(*)_4$ then (a) is true for a_μ^j $j = 3, 2, 4$.

for some congruence ideals.

③ Precise statements and sketch of proof.

$j = 3, 2$

Consider $\mathrm{Sym}^3 \rho_\mu : G_{\mathbb{Q}} \to GSp_4(A_1)$

$GL_{2/\mathbb{Q}} \overset{\mathrm{Sym}^3 \mathrm{BC}}{\rightsquigarrow} GSp_{4/\mathbb{Q}}$ established by H. Kim and

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Ramakrishnan - Shahidi.

$h_2 = h_2^s$: Hida Hecke algebra for $GSp_4(\mathbb{Q})$

finite torsion free over $\Lambda_2 = \mathbb{Z}_p[[X_1, X_2]]$

$$\begin{tikzcd} T_{\ell,1},\ T_{\ell,2} \ \ \ell \nmid Np \\ U_{p,1},\ U_{p,2} \end{tikzcd}$$

$$\begin{tikzcd} h_2 \arrow[r, "\Theta"] & h_1 \arrow[r, "\mu"] & A_1 \\ \Lambda_2 \arrow[u, dash] \arrow[r] & \Lambda_1 \arrow[u, dash] \arrow[ur] & \end{tikzcd}$$

$1 + X_1 \mapsto (1+X)^2$

$1 + X_2 \mapsto k(1+X)$

$$\begin{tikzcd} T_2 \arrow[r, "\Theta"] \arrow[rr, bend left, "\lambda"] & T_1 \arrow[r, "\mu"] & A_1 \\ \Lambda_2 \arrow[u, dash] \arrow[r] & \Lambda_1 \arrow[u, dash] \arrow[ur, dash] & \end{tikzcd}$$

$$\begin{tikzcd} \tilde{T}_2' \arrow[r, "\tilde{\Theta}"] \arrow[dr, dash] & \tilde{T}_1 \arrow[r, "\tilde{\mu}"] \arrow[d, dash] & \tilde{A}_1 \arrow[dl, dash] \\ & \tilde{A}_1 & \end{tikzcd}$$

$\tilde{T}_2 = T_2 \otimes_{\Lambda_2} \tilde{A}_1$

$\tilde{T}_1 = T_1 \otimes_{\Lambda_1} \tilde{A}_1$

$\tilde{T}_2 \hookrightarrow \tilde{A}_1 \times \tilde{T}_\mu'$

$\tilde{T}_2 \hookrightarrow \tilde{T}_1 \times \tilde{T}_\Theta'$

$\tilde{T}_1 \hookrightarrow \tilde{A}_1 \times \tilde{T}_\mu'$

$C_{\tilde{\lambda}} = \tilde{T}_2 \cap (\tilde{A}_1 \times 0)$

$C_{\tilde{\Theta}} = \tilde{T}_2 \cap (\tilde{T}_1 \times 0)$

$L_{\tilde{\mu}}$

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<u>Thm (Pollack, Coates):</u> Assume $(*)_4$ then $R_2 = T_2$ rel c.i. $/\Lambda_2$

univ. def. ring and def ring of $\mathrm{Sym}^3 \bar{\rho}_\mu$

<u>Cor 1:</u> $\mathrm{Ad}(\mathrm{Ad}_{\check{Sp}_4}^{\mathrm{Sym}^3} \rho_\mu)^*$ torsion and $\left(\chi_{\mathrm{Ad}(\mathrm{Ad}_{\check{Sp}_4}^{\mathrm{Sym}^3}\rho_\mu)^*}\right) = L_{\tilde{\lambda}}$.

<u>Cor:</u> $L_{\tilde{\lambda}} = \tilde{\lambda}(L_{\tilde{\theta}}) L_{\tilde{\mu}}$ and these ideals are all principal.

$p > 2n$

$$\mathrm{Ad}_{\check{Sp}_{2n}}(\mathrm{Sym}^{2n-1} St_2) = \bigoplus_{j=0}^{n-1} a^{2j+1} \circlearrowleft GL_2/\mathbb{Z}_p$$

$n = 2$

$$(*)_4 \quad \mathrm{Ad}_{\check{Sp}_4}(\mathrm{Sym}^3 \rho_\mu) = a_\mu^1 \oplus a_\mu^3$$

$$\begin{array}{ccccc} \mathrm{Ad}(\quad) & = & \mathrm{Ad}(a_\mu^1) & \oplus & \mathrm{Ad}(a_\mu^3) \\ \updownarrow & & & & \\ L_{\tilde{\lambda}} & & L_{\tilde{\mu}} & & \tilde{\lambda}(L_{\tilde{\theta}}) \end{array}$$

$$\tilde{\lambda}(L_{\tilde{\theta}}) = \left(\chi_{\mathrm{Ad}(a_\mu^3)^*}\right)$$

$$T_2 \xrightarrow{\theta} T_1 \xrightarrow{\mu} A_1, \quad \lambda = \mu \circ \theta$$

j = 2 $\quad Ad_{Sl_{2n}}(Sym^{2n-1}) = \bigoplus_{i=1}^{2n-1} a^i$

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$n = 2$

$$a^1 \oplus a^3 \oplus a^2 \qquad (*)$$

K = imag. quad field in which p split

$q_1 \neq q_2 \mid N$ are inert in K

$U(4)_{/\mathbb{Z}}$ compact at ∞

q. split at all inert primes except q_1, q_2.

Geraghty

$$\begin{tikzcd}
h_3^u \arrow[r, "U(4)"] \arrow[r, swap, "Mor"] \arrow[d, dash, swap, "\text{Finite torsion free}"] \arrow[rr, bend left, "\lambda'"] & h_2^s \arrow[r, "\lambda"] \arrow[d, dash] & A_1 \\
\Lambda_3 \arrow[r] \arrow[ur, phantom, "GSp_4 \to U(4)"] & \Lambda_2 & (a+b, a, b) \mapsfrom (a, b)
\end{tikzcd}$$

$$\begin{tikzcd}
T_3^u \arrow[r, "\Theta'"] \arrow[rr, bend left, "\lambda'"] & T_2^s \arrow[r, "\lambda"] & A_1
\end{tikzcd}$$

<u>Thm (Hida-T.):</u> Under $(*)_q$ we have $R_3 = T_3$ complete intersection $/\Lambda_3$.

<u>Cor:</u> $\Gamma_{\tilde{\lambda}'} = \Gamma_{\tilde{\lambda}} \quad \tilde{\lambda}(\Gamma_{\tilde{\Theta}'})$.

(*) says $\quad Ad_{Sl_4} = Ad_{Sp_4} \oplus a^2$

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Thus,

$$\begin{array}{ccccc} L_{\tilde{\lambda}'} & = & L_{\tilde{\lambda}} & \tilde{\lambda}(L_{\tilde{\theta}'}) & \\ \| & & \| & \Rightarrow \| & \leftarrow \text{Second thm} \\ \chi_{\mathrm{Sel\,Ad}_{SL_4}} & & \chi_{\mathrm{Sel\,Ad}_{Sp}} & \chi_{\mathrm{Sel}(a_p^2)^*} & \end{array}$$

$\tilde{\lambda}(L_{\tilde{\theta}'}) \leftarrow$ congruences between $\mathrm{Sym}^3 \rho$ and families in $U(4)$ that are not symplectic.

$j=4$:

$$\begin{tikzcd} & GL_2 \arrow[dl, "\mathrm{Sym}^3"'] \arrow[dr, "\mathrm{Sym}^4"] & \\ GSp_4 \arrow[rr, "2:1"'] & & GSO_5 \end{tikzcd}$$

$$\mathrm{Ad\,Sym}^3 \circlearrowright GL_2 \qquad \circlearrowright \mathrm{Ad\,Sym}^4$$

$$Sp_4 = SO_5$$

$$Sl_5 = SO_5 \oplus a^2 \oplus a^4$$

$$\begin{tikzcd} R_4 \arrow[r] \arrow[d, equal] & R_2 \arrow[d, equal] & \\ T_4 \arrow[r, "\theta''"] \arrow[rr, bend right, "\lambda''"'] & T_2 \arrow[r, "\lambda"] & A_1 \end{tikzcd}$$

$$\begin{tikzcd} & G_{\mathbb{Q}} \arrow[dl, "\rho"'] \arrow[dr] & \\ GSp_4(B) \arrow[rr] & & GSO_5(B) \end{tikzcd}$$

$$GSO_5(B) = G_m \times SO_5$$

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$$C_{\tilde\lambda''} = C_{\tilde\lambda}\ \tilde\lambda(C_{\tilde\Theta'})$$

$$\| \qquad\qquad \| \qquad\qquad \|$$

$$\chi_{\mathrm{Sel}(\mathrm{Ad}\,\mathrm{Sym}^4)_{Sl_5}} \qquad \chi_{\mathrm{Sel}\,\mathrm{Ad}\,\mathrm{Sym}^3_{Sp_4}} \qquad \chi_{\mathrm{Sel}(a^2_\mu)^*}\ \chi_{\mathrm{Sel}(a^4_\mu)^*}$$

$$\| \quad \lambda(C_{\tilde\Theta'})$$

$$\chi_{\mathrm{Sel}(a^4_\mu)^*} = \frac{\tilde\lambda(C_{\tilde\Theta''})}{\hat\lambda(\check C_{\theta'})}$$

$$\begin{array}{c} \mathrm{Sym}^3_\mu \in F(\beta) \\ \qquad\qquad | \\ \qquad\qquad U(4) \\ \Downarrow \\ \mathrm{Sym}^4_\mu \equiv G\ (\beta) \\ | \\ U(5) \end{array}$$

Standard rep for genus 2 Siegel families.

$$\begin{array}{ccc} h^S_2 & \xrightarrow{\sigma} & A_2 \\ | & / & \\ \Lambda_2 & & \end{array} \qquad \bar\rho_\sigma \quad \text{big} \quad \begin{array}{l} \supset Sp_4(k) \\ \supset \mathrm{Sym}^3(Sl_2(k)) \\ \supset N_{(GL_2 \times GL_2)^0}(k) \end{array}$$

$$\rho_\sigma : G_{\mathbb{Q}} \to GSp_4(A_2)$$

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<u>Conj:</u> $(L_p(St_\sigma)) \overset{?}{\doteq} \sigma(\Gamma_\theta)$.

<u>Other Sym^n</u>

$$C_n = 2\,\underset{1\le k\le 2n}{\mathrm{lcm}}(2n-k) \qquad p > 2n$$

$$a = \mu(U_p)$$

$$a^{c_n} \not\equiv 1 \ (\mathrm{mod}\ Ad$$

$$Ad_{sl_{n+1}}(Sym^n) = Ad_{sl_n}(Sym^{n-1}) \oplus a^n$$

$$\begin{tikzcd}
T_n \arrow[dr, "Sym^n", "\theta_n"'] \arrow[drr, bend left, "\lambda_n"] & & \\
& T_1 \arrow[r, "\mu"] & A_1 \\
T_{n-1} \arrow[ur, "\theta_{n-1}", "Sym^{n-1}"'] \arrow[urr, bend right, "\lambda_{n-1}"'] & &
\end{tikzcd}$$

$$\frac{\mu(\Gamma_{\theta_n})}{\mu(\Gamma_{\theta_{n-1}})} = \left(\chi_{Ad(a_p^n)^*}\right) \qquad \text{assuming } Sym^{n-1}, Sym^n \text{ exist.}$$