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4-14-16

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Distribution of rational points on some homogeneous varieties

(joint work with Sho Tanimoto and Daniel Loughran.)

Suppose we have

$$X \begin{cases} F_1(x_1, \dots, x_n) = 0 & \deg d_1 \\ \vdots & \vdots \\ F_k(x_1, \dots, x_n) = 0 & \deg d_k \end{cases}$$

want solutions w/ $|x_i| \leq B$

One expects about $B^{n+1 - \Sigma d_i}$ solutions.

$-K_X = O(n+1 - \Sigma d_i)$ ($-K_X$ anti canonical class)

- $n+1 - \Sigma d_i > 0$ you expect many solutions
- $n+1 - \Sigma d_i \leq 0$ you expect sparse solutions

Conj. (Manin): 1) Let X Fano variety (smooth proj.), $-K_X$ ample, defined over F. There is a finite extension E/F s.t. $X(E)$ is Zariski dense.

2) $U \subseteq X$ $\#\{p \in U(E) \mid H_{\mathcal{K}}(P) < B\} \gg B^a$ w/ $a > 0$.

2') $(-K_X)^{\otimes m} \rightsquigarrow \phi : X \longrightarrow \mathbb{P}^N$

$p \in X(E) \longrightarrow \mathbb{R}$

$H(\phi(p))^{\frac{1}{m}} \longleftrightarrow H_{-K_X}$

$\#\{P \in U(E) \mid H_{-K_X}(P) \leq B\} \sim C B^1 (\log B)^{r-1}$ $r = rk\ Pic(X)$

3) (B-M) can't read this ☹

1. Complete intersection in small degree (circle method)
2. Flag variety G/P
3. Large group of automorphisms $\mathbb{C}^n \circlearrowright X$
4. toric
5. Bi-equiv compactifications of semi-simple groups S-T-T, G-M-O
6. Dynamical systems proofs G/U
7. $G^n/\Delta(G)$ G-T-T.

The result is known in the above cases.

Dynamical systems seems to give easier proofs of all these results than spectral analysis and automorphic forms. 8. However, if one considers a line in $\mathbb{P}^3$ and blows up along this and get a variety X, it is not clear dynamical systems can deal w/ this though the spectral methods can

9. (with L. + T.) X non-singular variety $/k$

$$Br(X) = H^2(X, \mathbb{G}_m) \quad \text{Brauer group.}$$

$$Br(X) \times X(k) \to Br\, k$$

$$\operatorname{Spec} k \to X.$$

$\mathcal{B} \subseteq Br\, X$

$X(k)_{\mathcal{B}} = \{ x \in X(k) : b(x) = 0 \ \forall \ b \in \mathcal{B} \}.$

Asymp $X(k)_{\mathcal{B}}$.

$\sim \# \{ (\alpha, \beta) \in \mathbb{Q}^\times \times \mathbb{Q}^\times : H(\alpha) \leq A,\ H(\beta) \leq B,\ (\alpha, \beta) \text{ split} \}.$

studied by analytic number theory.

Thm: X smooth compactification of a semi simple group G

$$\mathcal{B} \subseteq \mathrm{Br}\, X$$

$$\#\{x \in X(k)_{\mathcal{B}} : H_{\mathcal{L}}(x) \le B\} \sim C \cdot \frac{B^{a(\mathcal{L})}(\log B)^{b(\mathcal{L})-1}}{(\log B)^{\sum_{b \in \mathcal{B}} (*)}}$$

$\leftarrow$ too technical to describe here.

$G \circlearrowleft X$ bi-equivariant action

$U = G$

$$\#\{\gamma \in G(\mathbb{Q}) : H(\gamma) \le B\}$$

$$Z_X(s) = \sum_{\gamma \in G(\mathbb{Q})} \frac{1}{H(\gamma; \underline{s})} \qquad \mathbb{C}^{n-1} \sim \mathbb{C}$$

dominant weight ρ $\quad \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto |t_1|^{s_1} \cdots |t_n|^{s_n} \qquad s_i \in \mathbb{C}$

$$H = \prod_v H_v$$

$g \in PGL_n(\mathbb{Q}_p)$

$g = k_1 \vec{t} k_2 \quad \vec{t} \in T^{++}$

$t = t_1^{k_1} t_2^{k_2} \cdots t_{n-1}^{k_{n-1}}$

$$t_i = \begin{pmatrix} \underbrace{1 \ \cdots \ 1}_{i} & & \\ & \underbrace{\varpi \ \cdots \ \varpi}_{n-i} \end{pmatrix}$$

$$H_v(\underline{s}; g) = \prod_{i=1}^{n-1} q^{k_i s_i}$$

$g \in G(\mathbb{A})$

$$Z_X(\underline{s}; g) = \sum_{\gamma \in G(\mathbb{Q})} \frac{1}{H(\gamma g; \underline{s})}.$$

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This is bounded for $g \in G(\mathbb{A})$.

$$Z_X(\underline{s};\cdot) \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A})).$$

(Need $G(\mathbb{Q})\backslash G(\mathbb{A})$ finite volume here).
assume G semi-simple w/ trivial center to get this.

<u>Eg:</u> G anisotropic

$$\text{1-dim} + \sum_{\pi} \sum_{\phi \in \mathcal{B}_\pi} \langle Z_X(\underline{s};\cdot), \phi \rangle \, \phi(g) = Z_X(\underline{s};g).$$