Quantum Unique Ergodicity and Number Theory:

Sound
12-4-09
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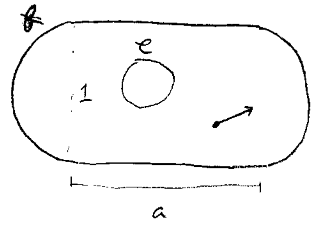


Bunimovich stadium

If you shoot a billiard randomly, it will fill up the stadium.

A small change in the angle makes a large change, so it is chaotic.

The classical dynamics is chaotic here.

Interested in finding solutions to $\Delta\varphi = \lambda\varphi$ in the quantum mechanics situation where $\Delta = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)$; $\lambda \geq 0$, $\varphi = 0$ on boundary.

$$\int_{\mathcal{R}} |\varphi|^2 = 1.$$

$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, $\varphi_1, \varphi_2, \ldots$

As $\lambda \to \infty$, what can we say about the eigenfunctions φ?

For example, given a compact $\mathcal{C} \subseteq \mathcal{R}$, what can we say about

$$\int_{\mathcal{C}} |\varphi|^2 \ ?$$

The conjecture would be that this integral should go to $\frac{\text{Area}(\mathcal{C})}{\text{Area}(\mathcal{R})}$.

Schnirelman Theorem (Colin de Verdière, Zelditch) Quantum Ergodicity:

For a density 1 subsequence of eigenfunctions, this conj. is true.

Quantum Unique Ergodicity: is it true always?

Andrew Hassell ('08): For the stadium the answer is no! (called scarring by physicists)

(at least for a generic stadium, as you vary a)

Eric Heller: (numerics, has an art gallery of pictures that you get)

Rudnick - Sarnak: Conjectured that for surfaces with strictly negative curvature QUE holds.

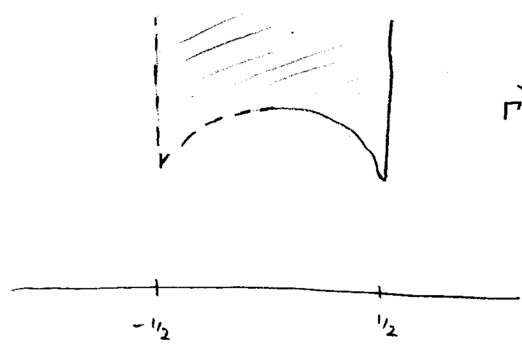
Ananthararaman ('05): Limiting measures have positive entropy.

$\mathbb{H} = \{ z = x+iy : y>0 \}$

$\Gamma \backslash \mathbb{H}$, Γ = discrete subgroup of $SL_2(\mathbb{R})$ with $\Gamma \backslash \mathbb{H}$ having finite volume

area on $\mathbb{H}$, $\frac{dx\,dy}{y^2}$; $\Delta = -y^2\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)$

Γ = "arithmetic group". For example, $\Gamma = SL_2(\mathbb{Z})$.



$\Gamma \backslash \mathbb{H}$

We are interested in looking at functions on $\Gamma\backslash\mathbb{H}$, i.e.,

functions on $\mathbb{H}$ w/ $f(\gamma z) = f(z)$ $\forall \gamma \in SL_2(\mathbb{Z})$. Also

want

$$\Delta f = \lambda f$$

normalized so that

$$\int_{\Gamma\backslash\mathbb{H}} |f(z)|^2 \frac{dx\,dy}{y^2} = 1.$$

Question: As $\lambda \to \infty$, does $\int_{\mathcal{C}} |f(z)|^2 \frac{dx\,dy}{y^2} \longrightarrow \frac{\text{Area}(\mathcal{C})}{\pi/3}$?

The existence of these eigenfunctions is already due to Selberg and is a deep result.

Such f are called Maass cuspforms.

Theorem: QUE holds for these Hecke-Maass forms.

(Lindenstrauss ('03): Any limiting measure is of the form $c\,\frac{dx\,dy}{y^2}\left(\frac{3}{\pi}\right)$ for some $0 < c < 1$.

Sound ('09): In fact, $c = 1$ so there is no escape of mass)

Holomorphic version: Modular forms of weight k is a holomorphic function f w/ $f(\gamma z) = (cz+d)^k f(z)$ $\quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

Cusp forms (decrease rapidly at ∞)

$$\langle f, g \rangle = \int_{\Gamma\backslash\mathbb{H}} f(z)\,\overline{g(z)}\, y^k \frac{dx\,dy}{y^2}$$

Normalize it so that

$$\int_{\Gamma\backslash\mathbb{H}} |f(z)|^2 \frac{dx\,dy}{y^2} = 1.$$

What happens as $k \to \infty$?

<u>Ramanujan's Δ-function:</u> $\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$; $q = e^{2\pi i z}$

$$= \sum_{n=1}^{\infty} \tau(n) q^n \qquad \tau(1) = 1,\ \tau(2) = -24, \ldots$$

Δ is a cuspform of wt 12, Δ^k = mod. form of wt. $12k$.

For a large even k, the space of modular forms of wt k is a v.s of dim $\approx \frac{k}{12}$.

Ramanujan noticed two important features of $\tau(n)$:

(1) $\tau(m)\tau(n) = \tau(mn)$ if gcd$(m,n) = 1$ (Mordell & generalized by Hecke)

(2) $|\tau(p)| \leq 2p^{11/2}$. (Deligne)

Large family of self-adjoint commuting operators (Hecke operators)

$$T_p f(z) = \frac{1}{\sqrt{p}}\left(f(pz) + \sum_{a=0}^{p-1} f\left(\frac{z+a}{p}\right)\right) \quad \text{for Maass forms}$$

One can simultaneously diagonalize all Hecke operators.

This suggests looking at Hecke eigenforms.

<u>Thm ('08) (Holowinsky & S.)</u>: For large k, a Hecke eigenform gets equidistributed on $\Gamma\backslash\mathbb{H}$.

Cor: f of wt k has $\approx k/12$ zeros in a fund. domain. These zeros get equidistributed wrt $\frac{3}{\pi}\frac{dx\,dy}{y^2}$.

Spectral expansion wrt eigenfunctions of Δ & λ_e:

① Constant fctn $\sqrt{\frac{3}{\pi}}$.

② Maass forms f (discrete spectrum)

③ Eisenstein series (continuous spectrum)

WTS: Take a Hecke e.f. f, $\iint_{\Gamma\backslash\mathbb{H}} \varphi(z)\,|f|^2 y^k \frac{dx\,dy}{y^2} \to 0$.

$$\left| \iint_{\Gamma\backslash\mathbb{H}} \varphi |f|^2 \right|^2 \to \frac{L(1/2, f\times f\times\varphi)}{L(1, \mathrm{Sym}^2 f)\,L(1,\mathrm{Sym}^2\varphi)} \qquad \text{(Watson '01)}$$

From here we see why the theorem is true, it is implied by GRH.

Since we don't know GRH, we need something else.

Convexity bounds are barely not enough.

Weak-subconvexity gives almost every case. (Sound developed this)

Poincare series and inner products w/ them was developed by Holowinsky. This catches all the ones Sound's method misses.