Moments of Zeta and L-functions:

One would like to understand distribution of ζ or L-functions. For example, $\zeta(\sigma+it)$ as t varies, $t \sim T$ large. Another would be $L(1,\chi_d)$, d fund. disc. up to X, which amounts to asking about class numbers.

$$L(1,\chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n} \longleftrightarrow \sum_{n=1}^{\infty} \frac{X(n)}{n}$$ where $X(n)$ is a random function.

It is natural to take

$$X(p) = \begin{cases} 1 & p/2(p+1) \\ -1 & p/2(p+1) \\ 0 & \frac{1}{p+1} \end{cases} \qquad \text{w/ prob.}$$

and extend multiplicatively. One can calculate such things as

$$\mathrm{Prob}(L(1,X) \geq e^{\gamma}\tau) \longrightarrow \exp(-Ge^{\tau}/\tau).$$

$\downarrow \quad \leq c/\tau$

$L(1,\chi_d)$

$\uparrow$ some distribution that is not Gaussian.

At $\sigma = 1/2$ something else happens and the behavior is very different.

Here, $\zeta(1/2+it)$ is not almost periodic.

Selberg: $\left\{\begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\right\} \log|\zeta(1/2+it)|$, $\quad T \leq t \leq 2T$

is Gaussian with mean 0 and variance $\sim \frac{1}{2}\log\log T$.

$$\mathrm{Prob}\left(\log|\zeta(1/2+it)| \geq \lambda\sqrt{\tfrac{1}{2}\log\log T}\right) \doteq \frac{1}{\sqrt{2\pi}}\int_{\lambda}^{\infty} e^{-x^2/2}\,dx.$$

Question: Are the values of $\zeta(1/2+it)$ dense in $\mathbb{C}$? This is still open.

One way to study these distributions is to study the moments

$$\int_0^T |\zeta(1/2+it)|^{2k}\,dt.$$

If this is $\ll T^{1/2+\varepsilon}$ for all $k \in \mathbb{N}$, this is the same as the Lindeloff hypothesis.

Hardy-Littlewood: $k=1$ $\quad \sim T\log T \qquad a_1 = 1\,;\ g_1 = 1$

Ingham: $k=2$ $\quad \sim 2\left(\frac{1}{4\pi^2}\right) T(\log T)^4 \qquad a_1 = \frac{1}{4\pi^2} \quad g_2 = 2$

Conjecture: (Conrey & Ghosh)

$$\int_0^T |\zeta(1/2+it)|^{2k}\,dt \sim g_k\, a_k\, T(\log T)^{k^2}$$

Why would one guess this?

$$\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}$$

$$\int_0^T \left|\sum_n \frac{d_k(n)}{n^{1/2+it}}\right|^2 dt = \sum_{m,n} \frac{d_k(m)\,d_k(n)}{\sqrt{mn}} \underbrace{\int_0^T \left(\frac{m}{n}\right)^{it} dt}_{\approx T\delta_{m=n}}$$

$$\sim T \sum_{n \le T} \frac{d_k(n)^2}{n}$$

$$\sim a_k T(\log T)^{k^2}$$

Conrey-Ghosh: $g_3 = 42$?

Conrey-Gonek: $g_4 = 24024$?

Keating - Snaith: $\mathcal{G}_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(k-j)!}$?

Idea: $t \sim T$ # zero $\sim \frac{T}{2\pi} \log \frac{T}{2\pi}$

Spacing between consecutive zeros $\sim \frac{2\pi}{\log T}$

$U(N)$ $e^{i\theta_1}, \ldots, e^{i\theta_N}$ average spacing $\frac{2\pi}{N}$

$N \approx \log T$

Model $\log \zeta$ by $\log$ (char. poly. of random matrix)

As as one goes to ∞, one gets a Gaussian, but for each T one has a particular distribution.

Gauss moments: $$\int_{U(N)} |\det(I-g)|^{2k} \, dg \rightsquigarrow \mathcal{G}_k N^{k^2}$$

Selberg's integral

There are hybrid versions of this model: Gonek, Keating, Hughes.

Symplectic: $L(1/2, \chi_d)$ d fund. disc. $\leq X$

$$\sum_{|d| \leq X} L(1/2, \chi_d)^k \sim c_k X (\log X)^{\frac{k(k+1)}{2}}$$

Orthogonal: $L(1/2, f \times \chi_d)$ f = mod form normalized as $s \leftrightarrow 1-s$

$$\sum_{|d| \leq X} L(1/2, f \times \chi_d)^k \sim c_k X (\log X)^{\frac{k(k-1)}{2}}$$

Rudnick & S. (lower bounds) If you can compute 1^{st} moment $+\varepsilon$ then right lower bounds for all moments ($k \in \mathbb{Q}^+$).

Radziwill & S.: Get result for $k \in \mathbb{R}^+$

Idea of proof:

$$\sum_{|d| \le X} L(\tfrac12, \chi_d)\, A(\chi_d)^{k-1} \le \Big(\sum L(\tfrac12,\chi_d)^k\Big)^{1/k} \Big(\sum A(\chi_d)^k\Big)^{\frac{k-1}{k}}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\sum_{n \le X^{\frac12}} \frac{\chi_d(n)}{\sqrt n} \qquad\qquad \text{evaluate}$$

$\uparrow$

truncation

for ζ: $\quad I_k(T) \ge (2+o(1))\, a_k\, T(\log T)^{k^2}$

S. Assuming GRH, the upper bound up to $(\log)^{\varepsilon}$

$$RH \Rightarrow \underbrace{I_k(T)}_{\text{moment}} \ll T(\log T)^{k^2+\varepsilon}$$

Selberg's thm

$$\mathrm{Prob}\big(\log|\zeta(\tfrac12+it)| \ge V\big) \asymp \exp\Big(-\frac{V^2}{\log\log T}\Big)$$

Can pretend this is true no matter what V is.

$V = \lambda\sqrt{\tfrac12 \log\log T}$

$$\frac1T \int_0^T |\zeta(\tfrac12+it)|^{2k}\, dt \longrightarrow \int_{V=-\infty}^{\infty} e^{2kV} \exp\Big(-\frac{V^2}{\log\log T}\Big)\, dV$$

$V = k \log\log T$

$$\rightsquigarrow \quad e^{k^2 \log\log T} = (\log T)^{k^2}$$

Symplectic case (expect):

$$\log(L(1/2, \chi_d)) = \text{Gaussian w/ mean } \tfrac{1}{2}\log\log X$$
$$\text{variance} = \log\log X$$

Orthogonal case (expect):

$$\log(L(1/2, f \times \chi_d)) = \text{Gaussian w/ mean } -\tfrac{1}{2}\log\log X$$
$$\text{variance} = \log\log X$$

Hough: In an example of the orthogonal case, Gaussian is an upper bound.

Assume GRH & Katz-Sarnak 1-level density conjecture $\Rightarrow$ both of these are Gaussian.

$$\log|\zeta(1/2+it)| = \mathrm{Re}\left(\sum_p \frac{1}{p^{1/2+it}}\right)$$
$$\leq \sum_{p\leq x} \frac{1}{p^{1/2+it}} + c\,\frac{\log T}{\log x}.$$

Chandee & S.: Assuming R.H. $\Rightarrow |\zeta(1/2+it)| \leq 2^{\frac{1}{2}\frac{\log T}{\log\log T}}$.

Chandee has a nice explicit version. These can be useful for calculations.