The Skinner-Wiles trick for $GSp(4)$:

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The Skinner-Wiles trick for GSp_4: (level lowering)

§ Introduction:

- Fundamental problem: Given an irred. cont

$$\rho : Gal(\overline{\mathbb{Q}}/F) \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell).$$

Suppose ρ is "geometric", i.e., ρ occurs in $H^{\cdot}(X)$ for some smooth proj. variety X/F. Is ρ then "automorphic"? That is, $\exists$ cuspidal π on $GL_n(\mathbb{A}_F)$ with the same L-fctns?

Example: $X = E_{/\mathbb{Q}}$ elliptic

$$\rho = \rho_{E,\ell} \quad \text{on} \quad V_\ell(E) = H^1(E, \mathbb{Q}_\ell)^*$$

Then by Wiles, et. al. $\rho = \rho_{f,\ell}$ for some eigenform $f \in S_2(\Gamma_0(N))$.

($\Rightarrow$ FLT).

Assuming $\bar{\rho}$ is irreducible, one key ingredient is "modularity lifting.":

$$\bar{\rho} \text{ automorphic} \Rightarrow \rho \text{ automorphic}$$

- Can rephrase this:

$$R_\Sigma(\bar{\rho}) \xrightarrow{\sim} \mathbb{T}_\Sigma(\bar{\rho})$$

universal def. ring. localization of some Hecke algebra

- One reduces to the minimal case where $\Sigma = \phi$. (Taylor-Wiles systems)

- However, $\Pi_\phi(\bar\rho)$ is well-defined? To see this, one needs level-lowering congruences.

Thm (Mazur-Ribet): $\rho := \rho_{f,\ell}$ (ℓ odd) for some eigenform $f \in S_2(\Gamma_0(N))$, assume $\bar\rho$ is irred, and unramified at $p \| N$, $p \neq \ell$. Then, if either $p \not\equiv 1 \pmod{\ell}$ or $\ell \nmid N \Rightarrow \bar\rho$ is modular of level $\frac{N}{p}$ and weight 2.

The proof is a delicate geometric analysis of (various) Shimura curves mod. p. (Cerednik-Drinfeld). Generalize??

- Hilbert modular forms some progress Jarvis, Rajaei.
- $U(2,1)$: Helm.

A result much more amenable to generalization is:

Thm (Skinner-Wiles): $\ell > 2$, $\rho = \rho_{f,\ell}$ for some eigenform $f \in S_{(2)}(\Gamma_0(N))$ Hilbert mod. form over some totally real field F. Assume $\bar\rho$ is irreducible. Then $\exists$ finite solvable totally real E/F in which the primes of F above ℓ split, s.t.

- $\bar\rho|_{Gal(\bar{\mathbb{Q}}/F)}$ irreducible
- $\bar\rho|_{Gal(\bar{\mathbb{Q}}/E)}$ admits a "minimal" modular deformation

(more precisely, is modular over E of level $M \mid N_\ell \cdot \prod \wp$
$\uparrow$
$\bar\rho_E$ is ram.)

§2 Main Results:

The goal is to get an analogue of Skinner-Wiles Thm for $GSp(4)/_F$.

(Genestier-Tilouine: Taylor-Wiles systems for $F = \mathbb{Q}$.

Notation: · $\ell \geq 5$ prime

· F totally real, $d = \deg(F/\mathbb{Q})$ even

· $G_0 = GSp(4)/_F$

· fix $\bar{\mathbb{Q}} \subset \mathbb{C}$ and $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_\ell$.

We admit:

Conjecture: Π cuspidal on G_0 s.t. Π_∞ is any discrete series,

Assume $\Pi \otimes |\det|^{w/2}$ is "algebraic" for some integer $w \in \mathbb{Z}$.

($k_1(v) + k_2(v) \pmod 2$ indep of $v|\infty$)

Then $\exists$

$$\rho_{\Pi,\ell} : Gal(\bar{\mathbb{Q}}/F) \to GL_4(\bar{\mathbb{Q}}_\ell)$$

unramified at $v \notin \underbrace{S_\Pi}_{\text{places where } \Pi \text{ unram}} \cup \underbrace{S_\ell(F)}_{\text{places of } F \text{ over } \ell}$ and

$$L_v(s - \tfrac{w}{2}, \Pi, spin) = \det(1 - \rho_{\Pi,\ell}(Frob_v) N(\wp_v)^{-s})^{-1}.$$

Moreover, if Π is non-CAP, ρ_Π is pure of weight w.

If $F = \mathbb{Q}$, this is known via work of Laumon & Weissauer. Thus, this is ok if Π is a base change from $\mathbb{Q}$!

(Labesse: weak cyclic BC (for any grp) assuming $\exists$ 2 Steinberg components.)

<u>Theorem A:</u> Π cuspidal on G_0, Π_∞ any d.s. Assume $\rho_{\Pi,\ell}$ exists and $\bar{\rho}_{\Pi,\ell}$ is irreducible where $\ell \geq 5$. Let w be a finite place of F s.t.

- $\Pi_w^{I_w} \neq 0$ where $I_w = \{g \in G_0(\mathcal{O}_w) : g \equiv \begin{pmatrix} \ulcorner & * \\ 0 & \lrcorner \end{pmatrix} \pmod{\mathfrak{p}_w}\}$
  Iwahori subgroup.

  (equivalently, Π_w is a component of an unram. prin. series such as Steinberg)

- $N(\mathfrak{p}_w) \equiv 1 \pmod{\ell}$

For each finite $v \neq w$, fix K_v s.t. $\pi_v^{K_v} \neq 0$.

Then, $\exists$ cuspidal $\tilde{\Pi}$ on G_0 with $\omega_{\tilde{\Pi}} = \omega_\Pi$ and $\tilde{\Pi}_\infty = \Pi_\infty$ s.t.

- $\tilde{\Pi} \equiv \Pi \pmod{\ell}$
- $\tilde{\Pi}_w =$ (at most) tamely ramified principal series
- $\tilde{\Pi}_v^{K_v} \neq 0$ for all $v \neq w$.

<u>Remark:</u> If <u>strong</u> cyclic BC is available for GSp_4, then Thm A $\Rightarrow$ precise analogue of the SW thm.

We prove Thm A by going to an inner form of G_0. $\deg(F/\mathbb{Q})$ is even, so $\exists!$ $D_{/F}$ quat. alg. s.t.

$D_\infty = \mathbb{H}^d$ and $D_f = M_2(\mathbb{A}_F^f)$.

Take $G_{/F}$ to be the unitary similitude group:

$$G(R) = \{ g \in GL_2(D \otimes_F R) \ : \ {}^t\bar{g}g = \mu(g) I \}$$

G_∞^{der} is compact and split at all finite places. This allows us to go from G_0 to G via the stable trace formula. Get the analogue of Jacquet-Langlands:

<u>Theorem B:</u> Fix an irred. $\xi : G_\infty \to GL(V_\xi)$, and fix some discrete series Π_ξ^+ with the same infinitesimal (and central) character. Then $\exists$ natural multiplicity-preserving bijection:

$$\{ \text{stable temp. auto } \pi \text{ of } G, \ \omega_\pi = \omega, \ \pi_\infty = \xi \}$$
$$\updownarrow 1\text{-}1$$
$$\{ \text{stable temp. cusp. } \Pi \text{ on } G_0, \ \omega_\Pi = \omega, \ \Pi_\infty = \Pi_\xi^+ \}$$

(takes $\pi \mapsto \Pi_\xi^+ \otimes \pi_f$ and $\Pi \mapsto \xi \otimes \Pi_f$)

"Stable": non-CAP, not a lift from $H = (GL_2 \times GL_2)/GL_1$ (i.e, endoscopic)

§3 <u>Congruences:</u>

G_∞^{der} is compact, so modular forms for G are all "combinatorial" objects:

$$\mathcal{A}_\xi(K, \omega) := \mathrm{Hom}_{G_\infty}(\xi, L^2(G(F)\backslash G(\mathbb{A}_F), \omega)^K).$$