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191

Main Conjectures For Modular Forms:

Motivation:

E/K elliptic curve , $G_K = \mathrm{Gal}(\bar{K}/K)$

$E(K)$ is what we are interested in.

$$\begin{tikzcd}
0 \arrow[r] & E(K)/p^n E(K) \arrow[r] & \mathrm{Sel}(K, E[p^n]) \arrow[r] \arrow[d, phantom, "\cap"] & Ш(E/K)[p^n] \arrow[r] & 0 \\
 & & H^1(G_K, E[p^n]) & &
\end{tikzcd}$$

Take limit over n, one gets

$$\begin{tikzcd}
0 \arrow[r] & E(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{Sel}(K, E[p^\infty]) \arrow[r] & Ш(E)[p^\infty] \arrow[r] & 0 \\
 & (\mathbb{Q}_p/\mathbb{Z}_p)^{rk(E(K))} & & &
\end{tikzcd}$$

So we lose torsion doing this.

LH & RH objects connected: BSD.

BSD: ① $\mathrm{ord}_{s=1} L(E/K, s) = rk\, E(K)$

② $$\lim_{s \to 1} \frac{L(E/K, s)}{(s-1)^{rk(E(K))}} = \frac{\Omega_{E/K} \cdot \# Ш(E/K) \overset{\text{regulator}}{R(E/K)}}{\# E(K)^{tors}} \overbrace{\prod_{\substack{\text{places} \\ v}} c_v}^{\text{Tamagawa factors}}$$

So a way of studying both is to study the intermediate group, the Selmer group.

"cohomological" BSD (at p)

① $\mathrm{corank}\ \mathrm{Sel}(K, E[p^\infty]) = rk\, E(K)$

$= \mathrm{ord}_{s=1} L(E/K, s)$

② same as before, compare p-valuations.

There are some partial results when $K = \mathbb{Q}$:

(a) Gross-Zagier, Kolyvagin $\operatorname{rk} E(\mathbb{Q}) \le 1$ $\left(\operatorname{ord}_{s=1} L(E,s) \le 1 \right)$

(b) if E has ordinary reduction at p (good or mult)

& $\operatorname{ord}_{s=1} L(E,s) = \text{odd}$ then $\operatorname{corank} \mathrm{Sel}(E[p^\infty]) \geq 1$

Nekovar, Skinner-Urban.

Iwasawa Theory: (example)

$$\mathbb{Q}_n = \text{cyclotomic } \mathbb{Z}/p^n\text{-ext.}/\mathbb{Q} \subseteq \mathbb{Q}(\mu_{p^{n+1}})$$

$$\mathbb{Q}_\infty = \bigcup \mathbb{Q}_n \quad \Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \simeq \mathbb{Z}_p$$

$$\overset{\Psi}{\gamma} \text{ topological generator.}$$

$$\mathrm{Sel}_\infty(E) := \varinjlim_n \mathrm{Sel}(\mathbb{Q}_n, E[p^\infty]) \circlearrowleft \mathbb{Z}_p[[\Gamma]] \xrightarrow{\sim} \mathbb{Z}_p[[T]]$$

$$\gamma \longmapsto 1+T$$

$$\cap \qquad H^1(\overline{\mathbb{Q}}/\overline{\mathbb{Q}}_\infty, E[p^\infty])$$

$$S_\infty(E) = \mathrm{Hom}(\mathrm{Sel}_\infty(E), \mathbb{Q}_p/\mathbb{Z}_p)$$

compact $\mathbb{Z}_p$-module (even $\mathbb{Z}_p[[\Gamma]]$-module)

Assume E has ordinary reduction at p.

Main Conjecture: ① $S_\infty(E)$ is a torsion Λ-module

② $\mathrm{char}_\Lambda S_\infty(E) = (\mathcal{L}_E)$

$\mathcal{L}_E \in \Lambda$ s.t. $\mathcal{L}_E \bmod \gamma - \zeta_{p^m}$ „ $L^{alg}(E,\psi,1)$

$$= (*) \; \frac{L(E, \psi_{\zeta_{p^m}}, 1)}{\Omega_E}$$

$$\psi_{\zeta_{p^m}} : G_{\mathbb{Q}} \longrightarrow \Gamma \longrightarrow \mathbb{C}^\times$$

$$\gamma \longmapsto \zeta_{p^m}$$

One can even go back and recover some relations for $L(E,1)$ from this.

$$\mathrm{Sel}_\infty(E)^{\gamma=1} = \mathrm{Sel}(E[p^\infty])$$

At least in "good" situations this is true.

Then

$$\begin{array}{ccccc} \#\left(\mathrm{Sel}_\infty(E)^{\gamma=2}\right) & & & & \\ \| & \geq & & & \\ \# \, S_\infty(E) \big/ (\gamma-1) S_\infty(E) & \overset{\text{by }(1)}{=} & \# \, \Lambda \big/ (\mathcal{L}_E, \gamma-1) & \overset{\text{by }(2)}{=} & \# \, \mathbb{Z}_p \big/ L^{\text{alg}}(E,1) \end{array}$$

Selmer groups (a la Greenberg)

K = # field

K_∞/K = $\mathbb{Z}_p^d$-extension

$\Gamma = \mathrm{Gal}(K_\infty/K)$

F = finite extension of $\mathbb{Q}_p$

$\rotatebox{90}{$\subseteq$}$

$\mathcal{O}$ = ring of integers

$\rotatebox{90}{$\in$}$

π - uniformizer

V = finite dimensional F-space $\circlearrowleft$ G_K acts cont. ρ unram outside finitely many places

$\rotatebox{90}{$\subseteq$}$

T = $\mathcal{O}$ lattice stable under ρ

ε = cyclotomic char

$$\rho : G_K \longrightarrow GL_{\mathcal{O}}(T) \cong GL_{\dim(V)}(\mathcal{O})$$

<u>Assume</u>: V is Hodge-Tate at each $w \mid p$

<u>ex</u>: $$\rho\big|_{I_w} \cong \begin{pmatrix} \varepsilon^{k_1} & & * \\ & \varepsilon^{k_2} \ddots & \\ 0 & & \varepsilon^{k_d} \end{pmatrix} \qquad k_1 > k_2 > \cdots > k_d$$

C. Skinner
3-29-04
pg4

$\forall\ w|p$ $\quad \begin{array}{c} V \\ \cup\!\!\shortmid \\ V_w^+ \end{array} \leftarrow D_w\text{-subspace}$ s.t. HT-weights are >0 and HT-weights of V/V_w^+ are ≤ 0.

$$T_w^+ = T \cap V_w^+$$

$$R = \mathcal{O}[\![\Gamma]\!] \simeq \mathcal{O}[\![T_1, \dots, T_d]\!]$$

$$\begin{array}{ccccc} \Psi : G_K & \twoheadrightarrow & \Gamma & \xrightarrow[\text{can.}]{} & R^\times \\ & & \psi & & \psi \\ & & g & \longmapsto & g \end{array}$$

$$R^* = \mathrm{Hom}_{\mathrm{cts.}}(R, \mathbb{Q}_p/\mathbb{Z}_p) = \text{discrete } R\text{-module}$$

$\circlearrowright$ G_K acts via Ψ $\qquad (r.f)(x) = f(rx)$

$M = T \otimes_{\mathcal{O}} R^*$ discrete $R[G_K]$-module

$\circlearrowright$ G_K acts as $\rho \otimes \Psi$

$\cup$

$$M_w^+ = T_w^+ \otimes_{\mathcal{O}} R^*$$

$$\left(M^{\gamma - \zeta_{p^m}} \longleftrightarrow V \otimes \psi_{\zeta_{p^m}} \right)$$

$$Sel(M) = \ker \left\{ H^1(G_K, M) \xrightarrow{\text{res}} \bigoplus_{w \nmid p} H^1(I_w, M) \bigoplus_{w|p} H^1(I_w, M/M_w^+) \right\}$$

discrete R-module

$$S(M) = \mathrm{Hom}(Sel(M), \mathbb{Q}_p/\mathbb{Z}_p) \quad \text{compact } R\text{-module}$$

Fact: $\uparrow$ f.g. R-module

"Main Conjecture":

① $S(M)$ = torsion R-module

② for a height 1 prime $\mathfrak{p}$ of R.

$$\text{length}_{R_{\mathfrak{p}}}(S(M)_{\mathfrak{p}}) = \text{ord}_{\mathfrak{p}}(\mathcal{L}_V)$$

$\mathcal{L}_V$ - multi variable p-adic L-function interpolating

$$L(V \otimes \psi, 1)$$

Suppose $V = V_p(E)$, p = ordinary red. $K = \mathbb{Q}$, $K_\infty = \mathbb{Q}_\infty$

Show that this conjecture is the same as earlier M.C. with

$\mathcal{L}_V = \mathcal{L}_E$.

Example 1:

$K = \mathbb{Q}$, $K_\infty = \mathbb{Q}_\infty$

$V = \mathbb{Q}_p(n)$, $n > 0$ even (action is ε^n)

$\|$

V_p^+

$$\text{Sel}(M) = \ker\{H^1(G_{\mathbb{Q}}, M) \longrightarrow \bigoplus_{\ell \neq p} H^1(I_\ell, M)\}$$

i.e., classes unramified away from p.

$$L(V,s) = \prod_{\ell \neq p} (1 - \ell^{-s}\,\text{Frob}_\ell|_V)^{-1}$$

$$= \prod_{\ell \neq p} (1 - \ell^{n-s})^{-1}$$

$$\left.\begin{array}{l} \zeta(s-n)\,(1-p^{n-1}) \\ \zeta(1-n) \ \text{ or } \ L(\psi_{\zeta_{p^n}}, 1-n) \end{array}\right\} \mathcal{L}_V.$$

Thm (Mazur-Wiles): $\text{char}_R\, S(M) = (\mathcal{L}_V)$

Example: $K = \mathbb{Q}$, $K_\infty = \mathbb{Q}_\infty$.

$f \in S_k(N, \chi)$ eigenform

$$f = \sum_{n=1}^{\infty} a_f(n) q^n$$

$$F \supset \{a_f(n)\}$$

(ord) $|a_f(p)|_p = 1$

$\overset{\circlearrowright \rho_f}{V_f}$ = usual 2-dim. F-rep of $G_{\mathbb{Q}}$ assoc. to f.

$$\mathrm{trace}\, \rho_f(\mathrm{Frob}_\ell) = a_f(\ell) \qquad \ell \nmid Np.$$

$$\rho_f|_{D_p} \cong \begin{pmatrix} \alpha\varepsilon^{k-1} & * \\ & \beta \end{pmatrix} \qquad \begin{array}{l} \beta|_{I_p} = 1 \\ \alpha|_{I_p} = \text{finite} \end{array}$$

$V_{f,p}^{+}$ = subspace on which D_p acts as $\alpha\varepsilon^{k-1}$.

$V = V_f(1-m) \qquad 1 \le m \le k-1$

$(\mathcal{L}_V \longrightarrow L^{alg}(f, \Psi, m))$

$V_p^{+} = V_f^{+}(1-m)$

assoc. to V.

$\mathrm{Sel}(f, m) := \mathrm{Sel}(m)$

$S(f, m) := S(m)$.

Main Conjecture: ① $S(f,m)$ is a torsion R-module

② $\mathrm{char}_R\, S(f,m) = (\mathcal{L}_{f,m})$

$$\mathcal{L}_{f,m} \bmod \gamma - \zeta_{p^r} = L^{alg}(f, \Psi_{\zeta_{p^r}}, m)$$

$k=2$ case includes Elliptic curves.

What's known: (still assuming ordinary)

K. Kato: ① $S(f,m)$ is torsion (NOT in print)

② $\mathrm{char}_R S(f,m) \mid \mathcal{L}_{f,m}$ in $R \otimes \mathbb{Q}_p$

(even in R under some special circumstances)

C.S - Urban: Assume: ⓐ $\bar{\rho}_f = \rho_f \pmod{\pi}$ irred & D_p distinguished

ⓑ $\exists\, \ell \,\|\, N$ s.t. $\bar{\rho}_f|_{I_\ell} = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ non-split $\ell \neq p$.

ⓒ existence of Certain Galois reps. assoc. to cuspforms on $U(2,2)$.

$$\text{Then } \mathcal{L}_{f,m} \overset{=}{|} \mathrm{char}_R S(f,m) \text{ in } R \otimes \mathbb{Q}_p \text{ even in } R$$

when Kato holds in R.

Actually, C.S - Urban prove that there is an $K/\mathbb{Q}$ imag. quad. ext. in which p- splits s.t.

$$\mathcal{L}_{f,m}\, \mathcal{L}_{f\otimes\chi_K,m} \mid \mathrm{char}_R S(f,m) \times \mathrm{char}\, S(f \otimes \chi_K, m)$$

in R.

Example 3: K = imag. quad. field.

K_∞ = "anticyclotomic $\mathbb{Z}_p$ - extension", p split $= w_1 w_2$.

χ = Hecke character of K s.t. $\chi(z) = z^{k-1}$

f as before but not necessarily ordinary.

$V = (V_f \otimes \chi)(-k)$ rep. of G_K

$V_{w_1}^+ , V_{w_2}^+ = ?$

H-T weights	w_1	w_2
V_f	$k-1, 0$	$k-1, 0$
χ	$k+1$	0
V	$k, 1$	$-1, -k$

$V_{w_1}^+ = V$, $V_{w_2}^+ = 0$

↑ allowing ram.   ↑ not allowing any ram.

$$\mathcal{L}_V \longrightarrow L^{alg}(f \otimes \chi\chi_{\zeta_{p^r}}, k+1)$$

$$\chi_{\zeta_{p^r}} : G_K \twoheadrightarrow \Gamma \to \overline{\mathbb{Q}}^\times$$
$$\gamma \longmapsto \zeta_{p^r}$$

wt k+2 -cm form

$$\frac{L(f \times g_{\chi\chi_{\zeta_{p^r}}}, k+1)}{\text{CM-period}_{\text{ assoc. to } K.}}$$

C-S. & M. Harris & J.S. Li: expect to prove

$$\mathcal{L}_V \mid \mathrm{char}_R S(M)$$

in $R \otimes \mathbb{Q}_p$.