Theory Sem
9-16-2
PSI
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$E/\mathbb{Q}$ elliptic curve, $E(\mathbb{Q})$ = Mordell-Weil group
$\simeq \mathbb{Z}^r \oplus$ (finite grp)

ℓ = prime, $N_\ell = \# E(\mathbb{F}_\ell)$, $\ell \nmid N_E \leftarrow$ conductor

$a_\ell = 1 + \ell - N_\ell$

$$L_{(N_E)}(E,s) = \prod_{\ell \nmid N_E} (1 - a_\ell \ell^{-s} + \ell^{1-2s})^{-1}$$

- converges for $\mathrm{Re}(s) > 3/2$
- analytic cont to all of $\mathbb{C}$

Abuse notation and put $L(E,s) = L_{(N_E)}(E,s)$.

at $s=1$: $L(E,s)$ "looks like" $\prod_{\ell \nmid N_E} \left(\frac{N_\ell}{\ell}\right)^{-1}$

Look at $\prod_{\ell \le P} \left(\frac{N_\ell}{\ell}\right)^{-1}$ for some finite P.

Birch + Swinnerton-Dyer: Observed this quantity behaves like $(\log P)^r$ by looking at empirical evidence.

They conjectured:

$$\mathrm{ord}_{s=1} L(E,s) = r = \text{rank of } E(\mathbb{Q})$$

Evidence:

① Known to be true if $\mathrm{ord}_{s=1} L(E,s) \le 1$
(Gross-Zagier, Kolyvagin, Coates-Wiles)

② Known for constant abelian varieties over fctn fields
(Tate-Artin, Milne)

③ Various calculations
(look at Stein at Harvard)

From Mordell-Weil to Selmer:

p = prime

$A = E[p^\infty] \simeq \mathbb{Q}_p/\mathbb{Z}_p \times \mathbb{Q}_p/\mathbb{Z}_p$

$T = \varprojlim_n E[p^n] \simeq \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, A) \simeq \mathbb{Z}_p^2$

$G_{\mathbb{Q}}$ always have action here

$V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathbb{Q}_p^2$

$S(E) = S_p(E) = \ker\{ H^1(\mathbb{Q}, A) \longrightarrow \bigoplus_\ell H^1(\mathbb{Q}_\ell, E(\bar{\mathbb{Q}}_\ell)) \}$ is

the p-adic Selmer group.

$Ш(E) = Ш(E) = \ker\{ H^1(\mathbb{Q}, E(\bar{\mathbb{Q}})) \to \bigoplus_\ell H^1(\mathbb{Q}_\ell, E(\bar{\mathbb{Q}}_\ell)) \}$ is

the Tate-Shafarevich grp.

$0 \to E[p^n] \to E(\bar{\mathbb{Q}}) \xrightarrow{p^n} E(\bar{\mathbb{Q}}) \to 0 \quad \Rightarrow$ by Galois cohom.

$0 \to E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to S_p(E) \to Ш_p(E) \to 0$

Conjecture: $\# Ш(E) < \infty$

This implies the p-divisible rank of $E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p = \mathrm{rk}\, E(\mathbb{Q})$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad$ p-div rk $S_p(E)$

"Kummer seq for elliptic curves"

$0 \to \varprojlim E(\mathbb{Q}_\ell)/p^n E(\mathbb{Q}_\ell) \to H^1(\mathbb{Q}_\ell, T)$

$\otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ gives

$0 \to E(\mathbb{Q}_\ell) \otimes \mathbb{Q}_p \xrightarrow{\partial_\ell} H^1(\mathbb{Q}_\ell, V)$

$\ell \neq p$:

$\mathrm{im}(\partial_\ell)$ = unramified cocycles

$= \ker\{ H^1(\mathbb{Q}_\ell, V) \to H^1(I_\ell, V) \}$

(full Galois grp)

$= H^1(\mathrm{Gal}(\mathbb{Q}_\ell^{un}/\mathbb{Q}_\ell), V^{I_\ell})$.

$\underline{l=p:}$

Fontaine's ring of p-adic periods

less nice description

$$\mathrm{im}(\partial_p) = \mathrm{Ker}\left\{ H^1(\mathbb{Q}_p, V) \to H^1(I_p, B_{cris} \otimes_{\mathbb{Q}_\ell} V) \right\}$$

$H^1_f(\mathbb{Q}_\ell, V) = \mathrm{im}(\partial_\ell)$ for all l.

$$S(A) = \left\{ c \in H^1(\mathbb{Q}, A) : \mathrm{res}_\ell c \in \mathrm{im}\left\{ H^1_f(\mathbb{Q}_\ell, V) \to H^1(\mathbb{Q}_\ell, A) \right\} \right\}$$

It turns out that $S(A) = S_p(E)$.

$$S(V) = \left\{ c \in H^1(\mathbb{Q}, V) : \mathrm{res}_\ell c \in H^1_f(\mathbb{Q}_\ell, V) \right\}$$

$$= \mathrm{Ker}\left\{ H^1(\mathbb{Q}, V) \to \bigoplus_{\ell \neq p} H^1(I_\ell, V) \oplus H^1(I_p, B_{cris} \otimes V) \right\}$$

$$\dim_{\mathbb{Q}_p} S(V) = \mathrm{div\ rk}\ S(A) = \mathrm{div\ rk}\ S_p(E) = \mathrm{rk}(E) = r$$

<u>BSD-Conjecture revisited:</u> $\underbrace{\mathrm{ord}_{s=1} L(E,s)}$ $= \dim_{\mathbb{Q}_p} S(V)$

V comes from a motive
and $L(E,s)$ is just the L-fctn
of that motive.

As we have "freed" ourself from geometry and the world of elliptic curves. We can now generalize this conjecture:

Bloch - Kato
Fontaine - P.R.
(Deligne, Beilinson)

$E \longrightarrow$ wt 2 modular form

$$F = \sum_{n=1}^{\infty} a_n q^n \qquad q = e^{2\pi i z}$$

normalized new form
$(a_1 = 1)$

$$L(E,s) = L(F,s) = \sum a_n n^{-s}$$

$$\begin{array}{ccc} E & & F \\ \downarrow & & \downarrow \\ V & & V_F \end{array}$$

V_F - dual of usual representation

← means dual

$$(V \simeq V_F(1))$$

$$S(V_F(1)) = S(V)$$

$$\operatorname{ord}_{s=1} L(F,s) = \dim_{\mathbb{Q}_p} S(V_F(1))$$

No reason to stick to wt 2.

$F = \sum_{n=1}^{\infty} a_n q^n$ a normalized newform of wt $2k$.

V_F - p-adic representation - 2-dim over some finite ext K of $\mathbb{Q}_p$ - dual of usual one.

Selmer group for $V_F(n) = V_F \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)$

↖ $G_{\mathbb{Q}}$ acts through the n^{th} power of the cyclo. character

Frob_ℓ acts as ℓ^n $(\ell \neq p)$

$S(V_F(n))$

$$H^1_f(\mathbb{Q}, V_F(n)) = \operatorname{Ker}\left\{ H^1(\mathbb{Q}, V_F(n)) \to \bigoplus_{\ell \neq p} H^1(I_\ell, V_F(n)) \oplus H^1(I_p, B_{cris} \otimes V_F(n)) \right\}$$

<u>Conjecture:</u>

$$\operatorname{ord}_{s=n} L(F,s) = \dim_{\mathbb{Q}_p} S(V_F(n))$$

Interesting when $s=k$ since $L(F,s)$ can vanish at $s=k$.

<u>Thm (G-Z, Nekovar, Skinner-Urban):</u> Suppose F is a normalized newform of wt $2k$ on $\Gamma_0(N)$ (trivial character). If $\mathrm{ord}_{s=k}(L(F,s))$ is odd then $H^1_f(\mathbb{Q}, V_F(k)) \neq 0$ ($\dim \geq 1$) for all primes p that are ordinary for F. (ie., $|a_p|_p = 1$) provided a certain hypothesis holds for Galois reps assoc. to Siegel modular forms (true if $N=1$)

<u>Note:</u> If $N=1$, k odd, then the sign of the functional eq. of $L(F,s)$ is $i^{2k}=-1$, so have odd order of vanishing.

<u>How to prove such a result:</u>

$H^1(\mathbb{Q}, V_F(n))$ classifies $G_{\mathbb{Q}}$-extensions

$$0 \to K(n) \to \underset{\substack{\uparrow \\ \dim 3}}{M} \to \underset{\substack{\wr\wr \\ \text{dual of } V_F}}{V_F^\vee} \to 0$$

$H^1(\mathbb{Q}, V_F(n))$

$\cup|$

$H^1_f(\mathbb{Q}, V_F(n))$ extensions that are

(i) split as I_ℓ-ext for all $\ell \neq p$

(ii) "crystalline at p"

Where to find them? Try automorphic forms.

Since $\dim_K M = 3$, it suggests trying $U(3)$ – have to deal w/ endoscopic grps

GL_3 – in general don't know that we have Gal. reps.

We look at $GSp(4)$ – 4 dimensional Galois reps.

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Siegel Modular Forms:

$$\mathbb{H}_2 = \{ \underset{Z}{\overset{\shortparallel}{X+iY}},\ \text{symm},\ Y>0 \}.$$

$$f : \mathbb{H}_2 \to \mathbb{C}$$

$$f((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k f(Z), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \subseteq Sp_4(\mathbb{Z})$$

$= (k,k)$

Can generalize this to other automorphy factors (k_1, k_2)

Have notion of Hecke operators and eigenforms

$f \to$ two L-series $\to$ Standard (degree 5), $\to$ Spinor (degree 4)

- Siegel E.f. $f \longrightarrow V_f$ 4-dim Galois rep s.t. $L_{\text{Spinor}}(f,s) = L(V_f^\vee, s)$ (up to places of bad reduction)

- F wt $2k-2$ modular form on $\Gamma_0(N)$. s.t. $L(F, k-1) = 0$

$\longrightarrow$ Siegel eigenform $\mathcal{F}$, of wt k s.t.

[(Saito Kurokawa lift.) (Zagier, Eichler, P-S)]

$$L_{\text{Spinor}}(\mathcal{F}, s) = L(F,s)\,\zeta(s-k+1)\,\zeta(s-k+2)$$

$$V_{\mathcal{F}}^{ss.} = \begin{pmatrix} \varepsilon^{2-k} & & \\ & \varepsilon^{1-k} & \\ & & V_F \end{pmatrix}$$

Constructing Extension:

- using ordinarity of F at p we can put $\mathcal{F}$ into a 2-variable p-adic family $\mathcal{G}$ of Siegel eigenforms
- if $\mathcal{F}$ is "minimally ramified" then the specializations of $\mathcal{G}$ are generically irreducible.

minimally ramified just means that the level $\mathcal{F}$ is as minimally ramified as can expect:

$$F = \pi_F = \otimes \pi_{F,\ell}$$
$$\mathcal{F} = \sigma_{\mathcal{F}} = \otimes \sigma_{\mathcal{F},\ell}$$

$\sigma_{\mathcal{F},\ell}$ = Langland's subgt. of $\mathrm{Ind}_{\text{Siegel parabolic}}^{GSp(4)} \pi_{F,\ell}$

(true for all unramified primes)

minimally ramified things exist only if $\mathrm{ord}_{s=1} L(F,s)$ is odd.

Upshot is that we have a Galois rep.

$$\rho_{\mathcal{G}} : G_{\mathbb{Q}} \to GL_4(R)$$

(R = valuation ring of some étale cover of $\mathbb{Z}_p[[t_1,t_2]]$
$\wp$ = prime of R corresponds to specializing at (k,k))

s.t.

$$\rho_{\mathcal{G}} \bmod \wp = \left(\begin{array}{cc|cc} \varepsilon^{2-k} & * & * & * \\ & \varepsilon^{1-k} & * & * \\ \hline & & \multicolumn{2}{c}{V_F} \end{array}\right)$$

one ext

other ext

non-split. In fact

we have two possible extensions (one must exist)

$$0 \to K(2-k) \to M_1 \to V_F \to 0$$

$$0 \to K(1-k) \to M_2 \to V_F \to 0$$

WANT this one to be non-trivial.

Can show it always exists...

It also needs to have the correct properties at primes ℓ.

$\underline{\ell \neq p}$: hypothesis on Galois reps for Siegel modular forms intervenes

(NOT A prob for level 1)

(close to being verified for F of sq. free level)

$\underline{\ell = p}$: can verify that this is crystalline.