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Galois Representations Arising from some Compact Shimura Varieties.

O. Langland's Correspondence:

Conj: $F/\mathbb{Q}$ # field. $\exists$ a bijection between

$$\left\{\begin{matrix}\text{cusp. auto. rep.}\\ \text{of } GL_n(\mathbb{A}_F)\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{irred. cont.}\\ Gal(\bar{F}/F) \to GL_n(\bar{\mathbb{Q}}_\ell)\end{matrix}\right\} \qquad \pi \longmapsto R_\ell(\pi)$$

"algebraic" "geometric"

requiring that $\forall$ w

$$\pi_w \longleftrightarrow R_\ell(\pi)\big|_{Gal(\bar{F}_w/F_w)} \qquad G_{F_w} = Gal(\bar{F}_w/F_w)$$

local Langlands.

Goal: Prove some cases of "$\longrightarrow$".

$n=1$ is completely understood via CFT.

$n=2$, F totally real a lot is known.

I Statement of Theorem:

Thm: $n \geq 3$ odd, F CM field. π cuspidal auto. rep. of $GL_n(\mathbb{A}_F)$ s.t.

- $\pi^\vee \simeq \pi \cdot c$ $\qquad$ c = complex conjugation.
- π regular alg. (= cohomological)

For each prime ℓ, $\exists$ $R_\ell(\pi)$ s.t. $\forall$ $w \nmid \ell$

$$R_\ell(\pi)\big|_{G_{F_w}} \longleftrightarrow \pi_w.$$

Remarks: ① In case $\exists v$ s.t. π_v is square integrable, then the Thm was already known due to (Clozel, Kottwitz, Harris-Taylor, Taylor-Yoshida), but they get for all n.

② Recently Morel proved the thm

- up to multiplicity
- at unram. places.

Also gets for $n \equiv 2 \pmod 4$, $[F:\mathbb{Q}]/2$ odd.

③ n odd is assumed (at least one reason why) is we need a unitary group $U/_{F^+}$ s.t. U_v is quasi-split for all $v \nmid \infty$ and $U(\mathbb{R}) \simeq U(1,n-1) \times U(0,n)^{\text{power}}$

II Warm-up:

Π cuspidal auto. rep. $\qquad U(\mathbb{A}_F) \simeq GL_n(\mathbb{A}_F)$.

$\Big\downarrow$ "descent"

π auto. rep. of $U(\mathbb{A}_{F^+})$ with U as in Remark ③.

Consider Shimura varieties for G where G is a connected reductive group over $\mathbb{Q}$ s.t. $G(\mathbb{Q})$ "=" $U(F^+)$. Call the Shimura variety Sh. Sh is a smooth projective variety over F.

$$\begin{matrix} G(\mathbb{A}^\infty) \\ \times \\ \mathrm{Gal}(\bar{F}/F) \end{matrix} \circlearrowright H(Sh, \mathcal{L}) = \bigoplus \pi^\infty \otimes R_\ell(\pi^\infty)$$

really $Sh \otimes \bar{F}$

π^∞ rep of $G(\mathbb{A}^\infty)$ ← finite adeles.

Idea: Expect $R_\ell(\pi^\infty)$ is almost $R_\ell(\Pi)$ (up to multiplicity, twist, ...)

III Setting:

F CM field, $F = EF^+$, E imaginary quadratic, $F^+ \neq \mathbb{Q}$, c = complex conj.

$V \cong F^n$ F-v.s.

$\langle\,,\,\rangle : V \times V \to \mathbb{Q}$ Hermitian pairing

$h : \mathbb{C} \longrightarrow \mathrm{End}_F(V)$.

$\rightsquigarrow$ $G = GU(V, \langle\,,\,\rangle) \subseteq \mathrm{End}_F(V)$, G connected reductive group over $\mathbb{Q}$,

$G(\mathbb{R}) \text{ "=" } U(1, n-1) \times U(0,n)$?

$Sh = \{Sh_U\}_{U \subseteq G(\mathbb{A}^\infty)}$ ← open compact

↖ smth. proj/F.

$H(Sh, \mathcal{L}) = \sum (-1)^i H^i(Sh, \mathcal{L})$ étale cohom.

$\circlearrowleft$ virtual $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\bar{F}/F)$.

Fix w a place of F, $w | p$. We assume p split in E.

$\overline{Sh} := Sh \times_{\mathcal{O}_{F_w}} k(w)$.

$\overset{\text{set}}{=} \coprod_b \overline{Sh}^{(b)}$ b : isogeny class of p-divisible group/$\bar{\mathbb{F}}_p$. $\ni \Sigma_b$

$\updownarrow$

certain Newton polygon

One can construct :

- Igusa variety Ig_{Σ_b} : smth. var / $\bar{\mathbb{F}}_p$
- Rappaport - Zink spaces RZ_b : formal scheme / $\mathrm{Spf}\, \mathcal{O}_{\hat{F}_w^{ur}}$.

RZ_b $J_b(\mathbb{Q}_p) = \mathrm{Aut}^\circ(\Sigma_b) = D^\times_{1/n-h} \times GL_n(F_w)$.

$\rightsquigarrow$

$\mathrm{Mant}_b : \mathrm{Groth}(J_b(\mathbb{Q}_p)) \longrightarrow \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{F_w})$

<u>Thm: (Harris-Taylor, Mantovan):</u> (1st identity)

$$H(Sh, \mathcal{L}_\xi)\Big|_{W_{F_w}} = \sum_b \mathrm{Mant}_b(H_c(Ig_{\Sigma_b}, \mathcal{L}_\xi)).$$

$\circlearrowleft$

$G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$.

in $\mathrm{Groth}(G(\mathbb{A}^\infty) \times W_{F_w})$

(ξ = irred. rep. of G)