Shankar
11.4-19
PSI

Geometry of Numbers Methods over Number Fields

Joint work w/ M. Bhargava and X. Wang

Fixed number field F, totally real (though results work for not totally real fields)

$E/_F$

$$E: \; y^2 = x^3 + Ax + B, \quad A, B \in F.$$

<u>Thm (Mordell-Weil):</u> $E(F) \cong \mathbb{Z}^r \oplus T$

r = rank of E.

In fact, we can assume $A, B \in \mathcal{O}$.

$$\{E/_F\} \longleftrightarrow \{\mathbb{G}_m(F) \backslash F^2\} \setminus \{\Delta(A,B) = 0\}$$

$\{(A,B) \in \mathcal{O}^2 : \alpha^4 | A \text{ and } \alpha^6 | B, \text{ then } \alpha \in \mathcal{O}^\times\}$ is not exactly the set, but it is close to this.

Height:

$$H(E_{A,B}) = \prod_v \max\{|A|_v^3, |B|_v^2\}$$

v|∞ in F.

We order the elliptic curves wrt this height.

$$Av(f) = \lim_{X \to \infty} \frac{\sum_{H(E) < X} f(E)}{\sum_{H(E) < X} 1}.$$

Shankar
11-4-19
ps 2

<u>Thm:</u> $\mathrm{Av}(\text{rank}) \leq 1.5$, $\mathrm{Av}(\#\mathrm{Sel}_2) = 3$

This talk will concern proving this Theorem.

We have the following exact sequence:

$$0 \to E(F)/2E(F) \to \mathrm{Sel}_2(E) \to Ш_E[2] \to 0.$$

We have $2^r \leq \#\mathrm{Sel}_2(E)$, so it is enough to prove $\mathrm{Av}(\#\mathrm{Sel}_2) = 3$.

The conjecture is that $\mathrm{Av}(\text{rank}) = 1/2$, but since one is using the 2-Selmer group one cannot get near this b/c of the Shaf-Tate group.

$\sigma \in \mathrm{Sel}_2(E) \longleftrightarrow$ locally soluble 2-coverings of E.

$$\begin{tikzcd} C \arrow[d, "\text{over } \bar{F}", "S"'] \arrow[dr, "\text{over } F"] & \\ E \arrow[r, "{[2]}"'] & E \end{tikzcd}$$

C is said to be ~~Standa~~ soluble if $C(F) \neq \emptyset$.

C is said to be loc. soluble if $C(F_v) \neq \emptyset \ \forall v$.

<u>Thm (Cassels):</u> If C is a loc. soluble 2-covering, C has a degree 2 line bundle def. over F.

$C \to \mathbb{P}^1 \longleftrightarrow$ binary ~~quadratic~~ forms.
quartic

Shankar

11-4-14

pg 3

Binary quartic forms:

$V =$ binary quartic forms

$V \circlearrowleft PGL_2$

$$v \cdot f(x,y) = \frac{1}{(\det v)^2} f((x\,y)\cdot v).$$

invariants

Ring of invariants for this action is freely generated I, J.

loc. soluble

Thm (Cassels, Birch-Swinnerton-Dyer):

$$Sel_2(E_{A,B}) \longleftrightarrow \{ PGL_2(F) \backslash V(F)^{L.S.}_{A,B} \}.$$

Thm (Cremona, Fisher, Stoll):

$F_\wp$, $f \in V(F_\wp)$, $I(f), J(f) \in \mathcal{O}_\wp$. Then

$\exists\, v_\wp \in PGL_2(F_\wp) : v_\wp f \in V(\mathcal{O}_\wp)$.

$$PGL_2(\mathbb{A}_F) = \coprod_{i=1}^{cl} \Big(\prod_\wp PGL_2(\mathcal{O}_\wp) \Big) \beta_i\, PGL_2(F) \qquad \text{(Strong approx)}$$

$$(v_\wp) = (\delta_\wp)\cdot \beta_i \cdot \gamma \qquad \gamma \in PGL_2(F).$$

$$\Rightarrow \quad (\beta_\wp \cdot \gamma)\cdot f \in V(\mathcal{O}_\wp).$$

$$\mathcal{L}_\beta = \{ f \in V(F) : f \in \beta_\wp^{-1} V(\mathcal{O}_\wp) \}.$$

Thm: If $f \in V(F)$, $I(f), J(f) \in \mathcal{O}$, $\exists\ \gamma \in PGL_2(F)$ s.t.

$\gamma \cdot f \in \mathcal{L}_\beta$ for some β.

$$\Gamma_\beta = \{ \gamma \in PGL_2(F) : \gamma \in \beta_\wp^{-1} PGL_2(\mathcal{O}_\wp) \beta_\wp \}.$$

$\Gamma_\beta \circlearrowleft \mathcal{L}_\beta$.

Shankar
11-9-14
pg 4

<u>Next Goal:</u> Estimate $\#(\Gamma \backslash \mathcal{L}^{irr})_{H<X}$.

$$\mathcal{L}_\beta \hookrightarrow V(F_\infty) \qquad F_\infty = \prod_{v\mid\infty} F_v.$$

$$\Gamma_\beta \hookrightarrow PGL_2(F_\infty)$$

We want to construct a fundamental domain $\Gamma_\beta \backslash V(F_\infty)$.

$$PGL_2(F_\infty)\backslash V(F_\infty) \cong \coprod F_\infty^2$$

$$PGL_2(\mathbb{R})\backslash V(\mathbb{R}) \cong \coprod \mathbb{R}^2$$

Let R be the image of a section $F_\infty^2 \to V(F_\infty)$.

$\{X^3y + Ixy^3 + Jy^4\}$ is R when $F=\mathbb{Q}$. $\left(\begin{matrix} I = A \\ J = B\end{matrix} \text{ in this case.}\right)$

$$\Gamma\backslash PGL_2(F_\infty) \cdot PGL_2(F_\infty)\backslash V(F_\infty)$$
$$\overset{\shortparallel}{\mathcal{F}}.$$

It turns out $\mathcal{F}$ can be put in a Siegel domain.

Siegel domain D:

$D = N'A'K \subseteq PGL_2(F_\infty)$

K: maximal compact.

N': bounded unipotent set

$A' = \left\{\begin{pmatrix} t_1^{-1} & \\ & t_1\end{pmatrix}, \begin{pmatrix} t_2^{-1} & \\ & t_2\end{pmatrix}, \ldots, \begin{pmatrix} t_n^{-1} & \\ & t_n\end{pmatrix}\right\}$

where each $t_i \geq c > 0$.

$$\left|t_i/t_j\right| < c' \quad \forall i,j.$$

Thm: $\Gamma \backslash PGL_2(F_\infty) \leq \coprod_{g_i} g_i D$, $g_i \in PGL_2(F)$.

Thm: $\mathcal{F} \cdot R$ is an n-fold cover of a fund. domain for the action of Γ on $V(F_\infty)$
$n = \# \mathrm{Stab}_{PGL_2(F_\infty)}(f)$ for $f \in R$.

Now we want to count $\#(\mathcal{F} \cdot R_{H<X} \cap \mathcal{L}^{irr})$

Thm: $\#(\mathcal{F} \cdot R_{H<X} \cap \mathcal{L}^{irr}) = \mathrm{Vol}(\mathcal{F} \cdot R_{H<X}) + O(E)$.

We now count weighted lattice points.

0 if f is not loc. soluble

$$m(f) := \frac{1}{\# \text{conic red}} \left(\sum \sum \frac{1}{\# \mathrm{Aut}_{\Gamma_\beta}(V_i)} \right)^{-1}$$

$$m(f) = \prod_p m_p(f).$$

$\Rightarrow$

$$\sum_{H(E)<X} (\# Sel_2(E) - 1) \sim \sum_\beta \#_m (\Gamma_\beta \backslash \mathcal{L}_\beta^{irr})$$

orbits weighted by m.

$$= \sum_\beta \frac{\# \text{inf. orbits}}{\# \text{inf. stab}} \mathrm{Vol}(F_\beta \cdot R_{H<X})$$

$$\cdot \prod_p \int_{V(\mathbb{Z}_p)} m_p(f)\, df$$

$$= \sum_{\beta} \frac{\#\text{inf. orbits}}{\#\text{Stab at } \infty} \, |J| \, \mathrm{Vol}(F_\beta) \, \mathrm{Vol}(R_{H<X})$$

$$\cdot \prod_{p} |f|_p \, \mathrm{Vol}(PGL_2(\mathcal{O}_p)) \int_{\mathcal{O}_p^2} \frac{\#\text{orbits}}{\#\text{Stab}} \, dA \, dB$$

$$= \sum_{\beta} \mathrm{Vol}(F_\beta) \prod_{p} \mathrm{Vol}(PGL_2(\mathcal{O}_p)) \; \cancel{dA} \; \mathrm{Vol}(R_{H<X})$$

$$= 2 \, \mathrm{Vol}(R_{H<X}).$$