On the Drinfeld moduli problem of p-divisible groups:

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p = fixed prime

$F/\mathbb{Q}_p$ finite degree d, $\bar{k}$ = alg. closure of residue field of F, $\breve{\mathcal{O}}_F$ completion of max unram. ext.

<u>Def</u>: Let $S \in (Sch/\mathcal{O}_F)$. A <u>formal $\mathcal{O}_F$-module</u> on S is a

p-divisible formal group X/S and $\iota : \mathcal{O}_F \to \mathrm{End}(X)$ s.t.

induced action on $\mathrm{Lie}\, X$ is given by structure morphism $\mathcal{O}_F \to \mathcal{O}_S$.

Fix $n \geq 2$.

<u>Lubin/Tate</u>: Consider formal $\mathcal{O}_F$-modules (X, ι) of $ht(X) = dn$, $\dim X = 1$. exist unique such $(\mathbb{X}, \iota_{\mathbb{X}})$ over $\bar{k}$

$\mathrm{Nilp}_{\breve{\mathcal{O}}_F}$ = schemes S over $\breve{\mathcal{O}}_F$ s.t. $p\mathcal{O}_S$ loc. nilp. ideal.

Functor $\mathcal{N}^{LT} : \mathrm{Nilp}_{\breve{\mathcal{O}}_F} \longrightarrow (\text{sets})$

$S \mapsto \{$ iso. classes of $(X, \iota, \rho) \mid \rho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\mathrm{Spec}\, \bar{k}} \bar{S}$

$\mathcal{O}_F$-linear quasi-isog of height $0. \}$

<u>Thm</u>: (LT) This functor is representable by a formal scheme isomorphic to $\mathrm{Spf}\, \breve{\mathcal{O}}_F [\![t_1, \ldots, t_{n-1}]\!]$.

<u>Def</u>: Fix a division algebra D with center F, $\mathrm{inv}(D) = \frac{1}{n}$. Consider

(X, ι_D) where X formal $\mathcal{O}_F$-module of $ht(X) = dn^2$, $\dim X = n$.

and $\iota_D : \mathcal{O}_D \to \mathrm{End}(X)$ s.t.

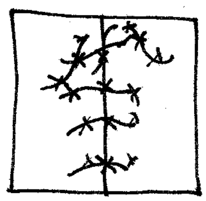
$$\underset{\mathcal{O}_S[T]}{\mathrm{char}(T; \iota(a) \mid \mathrm{Lie}(X))} = \underset{\mathcal{O}_F[T]}{\mathrm{char}\, d(a)(T)} \quad \forall a \in \mathcal{O}_D.$$

There exist unique such $(\mathbb{X}, \iota_D)$ over $\bar{k}$.

$\mathcal{N}^D : \mathrm{Nilp}_{\mathcal{O}_{\breve{F}}} \to (\text{sets})$:

$S \longmapsto \{$ isom classes of (X, ι_D, φ): $\varphi : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\,\bar{k}} \bar{S}$ $\mathcal{O}_D$-linear quasi-isog of height $0 \}$.

<u>Thm (D):</u> The functor $\mathcal{N}^D$ is representable by $\hat{\Omega}^{\,n}_F \hat{\otimes}_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}$.



$\mathbb{P}^1/\mathcal{O}_F$

at each rational pt, blow it up and replace with $\mathbb{P}^1$.. continue this process.

<u>Note:</u> formal $\mathcal{O}_F$-modules are rare in nature! Assume F is the p-adic completion of tot. real field $\mathbb{F}$ in (p), where p ~~is~~ inert? in $\mathbb{F}$. Let $A/\bar{k}$ abelian abelian variety with action of $\mathcal{O}_{\mathbb{F}}$. If (A, ι) lifts to char 0, then Lie A is free $\mathcal{O}_{\mathbb{F}} \otimes \bar{k}$-module. Then
$X = A(p^\infty)$ p-div group with action of $\mathcal{O}_F$ never formal $\mathcal{O}_F$-module unless $F = \mathbb{Q}_p$.

Fix alg. closure of $\mathbb{Q}_p$, $\bar{\mathbb{Q}}_p$. Let $\Phi = \mathrm{Hom}(F, \bar{\mathbb{Q}}_p)$. Fix $\varphi_0 \in \Phi$.

Fix $r : \Phi \to \mathbb{Z}$ s.t. $r_{\varphi_0} = 1$, $r_\varphi \in \{0, n\}$ $\forall \varphi \neq \varphi_0$.

$\rightsquigarrow E = E_r$ reflex field s.t. $\mathrm{Gal}(\bar{\mathbb{Q}}_p/E) = \{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) : r_{\sigma\varphi} = r_\varphi \ \forall \varphi\}$

$\Rightarrow F \overset{\varphi_0}{\hookrightarrow} E$.

Formulate functors on $\mathrm{Nilp}_{\mathcal{O}_{\breve{E}}}$.

LT (: $(X,\iota)/S$ s.t. $ht(X) = dn$.

Let $S \in (Sch/O_{\breve{E}})$.

$$\mathrm{char}(\iota(a) \mid \mathrm{Lie}(X)) = \prod_{\varphi \in \Phi} (T - \varphi(a))^{r_\varphi} \quad \forall a \in O_F.$$

D: $(X, \iota_D)/S$ " " $ht(X) = dn^2$

$$\mathrm{char}(\iota(a) \mid \mathrm{Lie}(X)) = \prod_{\varphi} \mathrm{chard}(a)^{r_\varphi}, \quad \forall a \in O_D.$$

Again uniqueness $/\bar{\mathbb{E}}$ $\rightsquigarrow$ $\mathcal{N}_r^{LT}$, resp. $\mathcal{N}_r^{D}$ over Spf $O_{\breve{E}}$.

<u>Thm 1 (R./Zink):</u> Let $F/\mathbb{Q}_p$ unramified. Then

(i) $\mathcal{N}_r^{LT} \simeq \mathcal{N}^{LT} \otimes_{O_{\breve{F}}} O_{\breve{E}}$, $\mathcal{N}_r^{D} \simeq \mathcal{N}^{D} \otimes_{O_{\breve{F}}} O_{\breve{E}}$.

(ii) Let r^0 be $r^0_\varphi = \begin{cases} 1 & \varphi = \varphi_0 \\ 0 & \varphi \neq \varphi_0. \end{cases}$

$$\mathcal{N}_{r^0}^{LT} = \mathcal{N}^{LT}, \quad \mathcal{N}_{r^0}^{D} = \mathcal{N}^{D}$$

<u>Remarks:</u> (1) The construction of isoms in (i) are very indirect. (uses Dieudonné theory and display theory)

(2) If $F/\mathbb{Q}_p$ is ramified, then $\mathcal{N}_r^{LT}$ and $\mathcal{N}_r^{D}$ are not flat.

<u>Thm 2 (Scholze):</u> $(\mathcal{N}_r^{D})^{rig} \simeq (\mathcal{N}^{D} \otimes_{O_{\breve{F}}} O_{\breve{E}})^{rig}$

<u>Remark:</u> The isom. is very indirect. It uses 2 ingredients:

- Scholze/Weinstein: $(p\text{-div groups}/O_C) \longleftrightarrow \left(T, W \mid T \text{ free } \mathbb{Z}_p\text{-module of finite rank, } W \subseteq T \otimes_{\mathbb{Z}_p} C \text{ subspace} \right)$

• description of such objects in terms of vector bundles on FF curves.

For integral theory, introduce formal subschemes $(\mathcal{N}_r^{LT})'$ of $\mathcal{N}_r^{LT}$ and $(\mathcal{N}_r^{D})'$ of $\mathcal{N}_r^{D}$. with identical generic fibers.

<u>Further notation:</u> F^t = max. unram. subfield of F.

$$\Psi = \mathrm{Hom}(F^t, \overline{\mathbb{Q}}_p) \ni \psi_0.$$

Fix uniformizer π of F with Eisenstein poly. $Q \in \mathcal{O}_{F^t}[T]$

$$\psi \rightsquigarrow \psi(Q(T)) = \prod_{\varphi \in \Phi_\psi} (T - \varphi(\pi)).$$

For $\psi \neq \psi_0$, have $Q_\psi(T) = Q_{A_\psi}(T)\, Q_{B_\psi}(T)$ $\qquad A_\psi = \{\varphi \in \Phi_\psi : r_\varphi = n\}$

$B_\psi = \{\varphi \in \Phi_\psi : r_\varphi = 0\}$.

$$Q_{\psi_0} = (T - \varphi_0(\pi))\, Q_{A_{\psi_0}}(T)\, Q_{B_{\psi_0}}(T).$$

For $S \in (\mathrm{Sch}/\mathcal{O}_E)$, have

$$\mathcal{O}_{F^t} \otimes_{\mathbb{Z}_p} \mathcal{O}_S = \bigoplus_{\psi \in \Psi} \mathcal{O}_S$$

$$\mathrm{Lie}\, X = \bigoplus \mathrm{Lie}_\psi X,$$

<u>Eisenstein conditions:</u> • for $\psi \neq \psi_0$, $Q_{A_\psi}(\iota(\pi)) \mid \mathrm{Lie}_\psi X = 0$

• for $\psi = \psi_0$: $(\iota(\pi) - \varphi_0(\pi))\, Q_{A_\psi}(\iota(\pi)) \mid \mathrm{Lie}_{\psi_0} X = 0$

$$\mathrm{rk}\,(Q_{A_{\psi_0}}(\iota(\pi)) \mid \mathrm{Lie}_{\psi_0} X) \leq \begin{cases} 1 & LT \\ n & D. \end{cases}$$

Thm 3: i) $(\mathcal{N}_r^{LT})' \simeq \mathcal{N}^{LT} \otimes_{\mathcal{O}_{\breve{F}}} \mathcal{O}_{\breve{E}}$

(ii) $(\mathcal{N}_r^{D})'$ is a flat normal p-adic formal scheme over $\mathcal{O}_{\breve{E}}$ with special fiber $\simeq \mathcal{N}^D$. (independent of π)

Furthermore, if $F/\mathbb{Q}_p$ is unramified, then $(\mathcal{N}_r^{D})' = \mathcal{N}_r^{D}$.

Thm 4 (Scholze): There exists a unique isom. $(\mathcal{N}_r^{D})' \simeq \mathcal{N}^D \otimes_{\mathcal{O}_{\breve{F}}} \mathcal{O}_{\breve{E}}$

s.t.

- in the generic fiber it is the SW-isom.
- in the special fiber it is the RZ-isom.

RZ-isomorphism on $\bar{k}$-points of $(\mathcal{N}_r^{D})'$: assume $F/\mathbb{Q}_p$ totally ramified.

$(X, \iota_D, \rho)/\bar{k} \rightsquigarrow$ Dieudonné module (M, V, F) $M = W(\bar{k})$-module of finite rank. & $\mathcal{O}_D$-action that commutes w/ V and F.

$$\overbrace{pM \subset \underbrace{VM \subset M}_{an^2+n}}^{dn^2}$$

$Q_A(T) \equiv T^a \pmod p$.

$$\overbrace{\pi^{a+1}M \subset \underbrace{VM \subset \pi^a M}_{n}}^{n^2}$$

$\rightsquigarrow$

(M', V', F') relative Dieud.-module.

$M' = M$ but considered over $W_{\mathcal{O}_F}(\bar{k})$-module of rank n^2

$V' = \pi^{-a} V$, $V'F' = \pi$

$\simeq$ Drinfeld formal $\mathcal{O}_D$-module.

<u>Lemma:</u> Let $\mathcal{O}$ = DVR, let $\mathfrak{X}$, $\mathcal{Y}$ be formal schemes loc. formally of f.t./$\mathrm{Spf}(\mathcal{O})$. Assume $\mathfrak{X}$, $\mathcal{Y}$ are flat and normal. Let

$$f_1 : \mathfrak{X}^{rig} \xrightarrow{\simeq} \mathcal{Y}^{rig} \quad \text{and} \quad f_2 : \mathfrak{X}_{red} \xrightarrow{\sim} \mathcal{Y}_{red}.$$

which are compatible wrt specialization $\forall$ points of $\mathfrak{X}^{rig}(K') = \mathfrak{X}(\mathcal{O}_{K'})$

for all finite extensions $K'/\mathrm{Frac}(\mathcal{O})$.

Then (f_1, f_2) is induced by a unique isom. $f : \mathfrak{X} \to \mathcal{Y}$.

Compatibility follows from the commutativity of the following diagram:

$$\begin{array}{ccc}
\{\text{p-div grps}/\mathcal{O}_C\} & \xrightarrow[\text{SW-Fargues}]{\sim} & \{A_{inf}\text{-}\varphi\text{-modules } (M,\varphi) \text{ s.t. } \tilde{\xi} M \subset \varphi(M) \subset M\} \\
\downarrow & & \downarrow \otimes_{A_{inf}} A_{crys} \\
\{\text{p-div grps}/\mathcal{O}_C/p\} & \xrightarrow[\text{Kisin Drinfeld}]{\sim} & \{A_{crys}\text{-}\varphi\text{-modules } (M,\varphi) \text{ s.t. } pM \subset \varphi(\varphi^*(M)) \subset M\} \\
\downarrow & & \downarrow \otimes_{A_{crys}} W(\bar{k}) \\
\{\text{p-div grps}/\bar{k}\} & \xrightarrow[\text{Dieud}]{\simeq} & \{W(\bar{k})\ \varphi\text{-modules } (M,\varphi) \text{ s.t. } pM \subset pM \subset M\}
\end{array}$$

$A_{inf} = W(\mathcal{O}_C^{\flat})$, $\xi = p - [p]$, $\tilde{\xi} = \varphi(\xi)$

A_{crys} = p-adic completion of A_{inf}-submodules of $A_{inf}[\frac{1}{p}]$, generated by $\dfrac{\xi^n}{n!}$ $\forall n$.