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Higher Rank Drinfeld Modules:

$A \subset K$ $\qquad$ A = ring of regular fctns on $X \setminus \{\infty\}$

K = fraction field of A.

L/K extension of fields (not nec. finite)

$$\rho : A \to L[\tau]$$
$$f \longmapsto \rho_f = f + a_1\tau + \cdots + a_s\tau^s$$

$\exists\, r \geq 1$ s.t. $s = (\deg f) r$ $\qquad$ $\deg f = -v_\infty(f)$.

(L, ρ) = Drinfeld A-module of rank r. So given $x \in L$, $f \in A$, we define an A-module action of f on x:

$$f * x := \rho_f(x)$$
$$= fx + a_1 x^q + \cdots + a_s x^{q^s}. \quad \leftarrow \text{seperable poly.}$$

The f-torsion submodule:

$$\rho[f] := \{ x \in \bar{L} : \rho_f(x) = 0 \}$$

Since f is seperable, the size of $\rho[f]$ is $q^s = \deg(f)$. As an A-module,

$$\rho[f] \simeq (A/f)^r.$$

$$\begin{array}{l} L(\rho[f]) \\ \quad\Big| \quad \leftarrow \text{Galois ext, call the group } G. \\ L \quad \leftarrow \text{field of def of } \rho \end{array}$$

So we have a representation

$$G \longrightarrow GL_r(A/f).$$

Exponential Function for ρ: $\quad (L \cong \mathbb{C}_\infty)$

$$\exp_\rho(z) = z + \sum_{n \geq 1} c_n z^{q^n} \qquad \text{(entire, surj., } \mathbb{F}_q\text{-linear)}$$

$$\exp_\rho(fz) = \rho_f(\exp_\rho(z)) \qquad (\forall\, f \in A)$$

$$0 \to \Lambda \longrightarrow \mathbb{C}_\infty \xrightarrow{\exp_\rho} \mathbb{C}_\infty \longrightarrow 0 \qquad \text{exact as } A\text{-modules.}$$

Λ is a discrete A-submodule of proj. rank r.

$\rho(\mathbb{C}_\infty) \cong \mathbb{C}_\infty/\Lambda$ as A-modules where we write $\rho(\mathbb{C}_\infty)$ to denote $\mathbb{C}_\infty$ w/ action via ρ.

Since $\mathbb{C}_\infty$ is not locally compact, you can find lattices in $\mathbb{C}_\infty$ of arbitrary rank.

Given $\Lambda \subseteq \mathbb{C}_\infty$ a lattice of rank r, define

$$e\text{xp}_\Lambda(z) = z \prod_{\substack{w \in \Lambda \\ w \neq 0}} \left(1 - \frac{z}{w}\right).$$

Then there exists a Drinfeld module ρ^Λ so that

$$e_\Lambda(z) = \exp_{\rho^\Lambda}(z).$$

Elliptic Curves:

$$E: \ y^2 = x^3 + ax + b \qquad a, b \in K$$

Papanikolas

4-16-09

ps3

$$\mathcal{E}ll_K = \{\text{isom. classes of elliptic curves } /K\}$$

$$E_1 \cong_{/K} E_2$$

Define

$$\Delta = -16(4a^3 + 27b^2) \neq 0 \quad (\text{no repeated roots})$$

$$j = \frac{-1728(4a^3)}{\Delta}.$$

Then $E_1 \cong_{/\bar{K}} E_2 \Leftrightarrow j_{E_1} = j_{E_2}$. This defines a function

$$\mathcal{E}ll_{\bar{K}} \longrightarrow \bar{K}$$

$$[E] \longmapsto j_E.$$

If $K = \mathbb{C}$, $E_1 \cong E_2 \Leftrightarrow \Lambda_1 = \alpha \Lambda_2$ for $\alpha \neq 0$.

$$\begin{array}{cc} E_1 & E_2 \\ \updownarrow & \updownarrow \\ \mathbb{C}/\Lambda_1 & \mathbb{C}/\Lambda_2 \end{array}$$

Therefore,

$$\mathcal{E}ll_{\mathbb{C}} \overset{\text{bij.}}{\longleftrightarrow} \{\Lambda \subseteq \mathbb{C} \text{ rank 2 lattice}\}/\sim$$

$$\Lambda_1 \sim \Lambda_2 \Leftrightarrow \Lambda_1 = \alpha \Lambda_2.$$

If $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\mathrm{Im}(\omega_1/\omega_2) > 0$, then

$$\Lambda = \omega_2\left(\mathbb{Z}\tfrac{\omega_1}{\omega_2} + \mathbb{Z}\right).$$

Letting $z = \omega_1/\omega_2$, then

$$\Lambda \sim \Lambda_z = \mathbb{Z}z + \mathbb{Z}, \quad z \in \mathfrak{H} = \text{upper half-plane}.$$

$$\mathfrak{H} \longrightarrow \{\Lambda \subseteq \mathbb{C}\}/\sim \longrightarrow \mathcal{E}ll_{\mathbb{C}} \longrightarrow \mathbb{C}$$

$$z \longmapsto [\Lambda_z]$$

$$[\Lambda] \longmapsto [\mathbb{C}/\Lambda]$$

$$[E] \longmapsto j_E$$

$$z \longmapsto j(E_{\Lambda_z}).$$

Papanikolas
4-16-09
p. 4

Define: $\Lambda \subseteq \mathbb{C}$ any lattice

$$g_2(\Lambda) = 60 \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0}} \omega^{-4} = 60 G_4(\Lambda)$$

$$g_3(\Lambda) = 140 \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0}} \omega^{-6} = 140 G_6(\Lambda)$$

$$\Delta(\Lambda) = g_2^3 - 27 g_3^2 .$$

Then

$$E_\Lambda : y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda) .$$

Using Λ_z, we have

$$G_4(z) = G_4(\Lambda_z) = \frac{\pi^4}{45}\left[1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n\right] , \quad q = e^{2\pi i z}$$

$$G_6(z) = G_6(\Lambda_z) = \cdots$$

$$\Delta(z) = \Delta(\Lambda_z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

$$j(z) = j(\Lambda_z) = \frac{1}{q} + 744 + 196884 q + \cdots$$

<u>Modular forms for $SL_2(\mathbb{Z})$:</u>

- $f : \mathfrak{H}^* \to \mathbb{C}$ holomorphic, $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$

$$\exists\, k \in \mathbb{Z} : \quad f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$\Lambda_z = \mathbb{Z}z + \mathbb{Z} = \mathbb{Z}(az+b) + \mathbb{Z}(cz+d) = (cz+d)\, \Lambda_{\frac{az+b}{cz+d}} .$$

Drinfeld Modular Forms:

$$\Omega = \mathbb{C}_\infty \setminus K_\infty$$

Restrict to the case $A = \mathbb{F}_q[\theta]$. We can do it in general, just becomes more complicated.

$$\begin{array}{c} \mathbb{C}_\infty \\ | \\ k_\infty = \mathbb{F}_q((\frac{1}{\theta})) \\ | \\ k = \mathbb{F}_q(\theta) \\ | \\ A = \mathbb{F}_q[\theta] \end{array}$$

$f : \Omega \to \mathbb{C}_\infty$ has weight k and type m if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k (ad-bc)^m f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)$.

(require f to be rigid analytic)

Example: Eisenstein series

$$E^{(k)}(z) = \sum_{(a,b)\in A^2}{}' \frac{1}{(az+b)^k} \quad \leftarrow \text{wt } k, \text{ type } 0.$$

don't divide by 0

(if $(q-1) \nmid k$, then $E^{(k)}(z) \equiv 0$)

Set $g(z) = (\theta^q - \theta)\, E^{(q-1)}(z)$ wt $q-1$, type 0

$$\Delta(z) = (\theta^{q^2} - \theta)\,(E^{(q-1)}(z))^{q+1} + (\theta^{q^2} - \theta)\, E^{(q^2-1)}(z)$$

wt q^2-1, type 0

Define a rank 2 Drinfeld module

$$\rho^{(z)} : \mathbb{F}_q[\theta] \longrightarrow \mathbb{C}_\infty[\tau]$$
$$\theta \longmapsto \rho^{(z)}_\theta = \theta + g(z)\tau + \Delta(z)\tau^2$$

Calculation: $\Lambda \subseteq \mathbb{C}_\infty$ rk 2

$$\exp_\Lambda(x) = x \prod_{\substack{\omega \in \Lambda \\ \omega \neq 0}} (1 - x/\omega) = x + c_q x^q + \cdots$$

$$x \frac{(\exp_\Lambda(x))'}{\exp_\Lambda(x)} = x \sum_{\omega \in \Lambda}{}' \frac{1}{(x-\omega)} = 1 - \sum_{k \geq 1} E^{(k)}(\Lambda)\, x^k$$

Use this with $\Lambda = \Lambda_z = Az + A$.

One can show: the lattice associated to $\rho^{(z)}$ is Λ_z.

Moreover, $\rho^{(z_1)} \simeq \rho^{(z_2)} \iff \Lambda_{z_1} \sim \Lambda_{z_2}$

$$\iff j(z_1) = j(z_2)$$

Where

$$j(z) = \frac{g(z)^{q+1}}{\Delta(z)}$$ wt 0, type 0 modular function.

Fourier Expansions:

$$t(z) = \frac{1}{\exp_C(\tilde{\pi} z)}$$ (replaces $q(z) = e^{2\pi i z}$ from classical theory)

$$\Delta(z) = -\tilde{\pi}^{q^2-1}\, t(z)^{q-1} \prod_{\substack{a \in A \\ \text{monic}}} f_a(t)^{(q^2-1)(q-1)}$$ due to (Gekeler)

where $f_a(x) = x^{q^{\deg a}} C_a(\frac{1}{x})$.

If you define
$h(z) = \Delta(z)^{\frac{1}{q-1}}$ has wt $q+1$, type 1.

<u>Higher levels:</u>

$\Gamma_0(N) \subseteq GL_2(A)$ (N can be poly here!)

$\Gamma_0(N) \backslash \Omega = Y_0(N)$ affine algebraic curve over k.

$\Gamma_0(N) \backslash \Omega^* = X_0(N)$ projective algebraic curve (Drinfeld modular curve)

$X_0(1) \cong GL_2(A) \backslash \Omega \longleftrightarrow$ isom. classes of rank 2 Drinfeld modules

$X_0(N) \longleftrightarrow$ isom classes (ρ, T)

cyclic submodule $\simeq A/N$.

rk 2 Drinfeld module

Deligne

$E/_k$ = elliptic curve over k w/ split mult. reduction at ∞

ie, E/k is a qt of $X_0(N)$.