Introduction to Drinfeld Modules and Function Field Arithmetic

Matthew Papanikolas

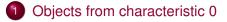
Texas A&M University

Clemson University

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Function fields and Drinfeld modules

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Arithmetic objects from characteristic 0

- The multiplicative group and exp(z)
- Elliptic curves and elliptic functions
- Abelian extensions of imaginary quadratic fields

The multiplicative group

We have the usual exact sequence of abelian groups

$$0
ightarrow 2\pi i \mathbb{Z}
ightarrow \mathbb{C} \stackrel{exp}{
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,

where

$$\exp(z) = \sum_{i=0}^{\infty} \frac{z^i}{i!} \in \mathbb{Q}[[z]].$$

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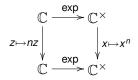
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where

$$\exp(z) = \sum_{i=0}^{\infty} rac{z^i}{i!} \in \mathbb{Q}[[z]].$$

For any $n \in \mathbb{Z}$,



which is simply a restatement of the functional equation

$$\exp(nz)=\exp(z)^n.$$

Torsion in the multiplicative group

The *n*-th roots of unity are defined by

$$\mu_n := \left\{ \zeta \in \mathbb{C}^{\times} \mid \zeta^n = 1 \right\} = \left\{ \exp\left(2\pi i a/n\right) \mid a \in \mathbb{Z} \right\}$$

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Explicit class field theory for \mathbb{Q} :

• Gal
$$(\mathbb{Q}(\mu_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^{\times}$$
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Explicit class field theory for \mathbb{Q} :

- $\operatorname{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^{\times}.$
- Kronecker-Weber Theorem: Every abelian extension of Q is contained in Q(μ_n) for some n.

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Explicit class field theory for \mathbb{Q} :

- $\operatorname{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^{\times}.$
- Kronecker-Weber Theorem: Every abelian extension of Q is contained in Q(μ_n) for some n.
- If ℓ is a prime $\ell \nmid n$, then the Artin automorphism $\sigma_{\ell} \in \text{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$ acts by

$$\sigma_{\ell}(\zeta) = \zeta^{\ell}.$$

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• Smooth projective algebraic curve of genus 1.

$$E: y^2 = 4x^3 + ax + b, \quad a, b \in \mathbb{C}$$

- *E*(ℂ) has the structure of an abelian group through the usual chord-tangent construction.
- The identity element of *E*(ℂ) is the point *O*, which lies on the line at infinity in ℙ².

Weierstrass uniformization

There exist $\omega_1, \omega_2 \in \mathbb{C}$, linearly independent over \mathbb{R} , so that if we consider the lattice

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,$$

then the Weierstrass p-function is defined by

$$\wp_{\Lambda}(z) = rac{1}{z^2} + \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0}} \left(rac{1}{(z-\omega)^2} - rac{1}{\omega^2}
ight).$$

The function $\wp(z)$ has double poles at each point in Λ and no other poles.

We obtain an exact sequence of abelian groups,

$$0 o \Lambda o \mathbb{C} \stackrel{\mathsf{exp}_E}{ o} E(\mathbb{C}) o 0,$$

where

$$\exp_E(z) = (\wp(z), \wp'(z)).$$

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We obtain an exact sequence of abelian groups,

$$0 o \Lambda o \mathbb{C} \stackrel{\mathsf{exp}_{\mathcal{E}}}{ o} \mathcal{E}(\mathbb{C}) o 0,$$

where

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Moreover, we have a commutative diagram

where [n]P is the *n*-th multiple of a point *P* on the elliptic curve *E*.

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Periods of E

How do we find ω_1 and ω_2 ?

An elliptic curve E,

$$E: y^2 = 4x^3 + ax + b, \quad a, b \in \mathbb{C},$$

has the geometric structure of a torus in $\mathbb{P}^2(\mathbb{C})$. Let

$$\gamma_1, \gamma_2 \in H_1(E, \mathbb{Z})$$

be generators of the homology of *E*.

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Then we can choose

$$\omega_1 = \int_{\gamma_1} \frac{dx}{\sqrt{4x^3 + ax + b}}, \qquad \omega_2 = \int_{\gamma_2} \frac{dx}{\sqrt{4x^3 + ax + b}}.$$

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Multiplication by n on E

• Note that if P = (x, y) is a point on E, then [n]P has the form

$$[n]P = \left(f_n(x,y),g_n(x,y)\right),$$

where f_n and g_n are rational functions in x, y, and the coefficients of the defining polynomial for E.

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• For example,

$$[2]P = \left(\frac{x^4 - 2ax^2 - 8bx + a^2}{4y^2}, \frac{x^6 + 5ax^4 + 20bx^3 - 5a^2x^2 - 4abx - a^3 - 8b^2}{8y^3}\right)$$

Multiplication by *n* on *E*

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 In particular, if x, y, a, b are all in a field K, then [n]P has coordinates in K.

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Torsion points on elliptic curves The group E[n]

Suppose *E* is defined over a field *K* ⊆ ℂ. For a field *L* with *K* ⊆ *L* ⊆ ℂ, we set

$$E(L) := \{(x, y) \in E \mid x, y \in L\}.$$

Then E(L) is a subgroup of E.

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Then E(L) is a subgroup of E.

• For each $n \in \mathbb{Z}$, we define the torsion subgroup

$$E[n] := \{P \in E(\mathbb{C}) \mid [n]P = O\} \subseteq E(\mathbb{C}).$$

Then as an abstract group,

$$E[n] \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}.$$

Division fields

• Since *E*[*n*] is a finite group, the field *K*(*E*[*n*]) satisfies

 $[K(E[n]):K]<\infty.$

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Division fields

Since *E*[*n*] is a finite group, the field *K*(*E*[*n*]) satisfies
 [*K*(*E*[*n*]) : *K*] < ∞.

• Moreover, K(E[n])/K is Galois: for $\sigma \in Gal(\overline{K}/K)$,

 $[n](\sigma P) = \sigma([n]P) \Rightarrow \sigma(E[n]) \subseteq E[n].$

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$$[n](\sigma P) = \sigma([n]P) \Rightarrow \sigma(E[n]) \subseteq E[n]$$

• Because $E[n] \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, we find that

 $\operatorname{Gal}(K(E[n])/K) \hookrightarrow \operatorname{GL}_2(\mathbb{Z}/n\mathbb{Z}).$

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Abelian extensions of imaginary quadratic fields

Elliptic curves with complex multiplication (CM)

• Consider the example

$$E: y^2 = x^3 - x.$$

Then for *i* = √−1, the morphism [*i*](*x*, *y*) := (−*x*, *iy*) induces and embedding

 $\mathbb{Z}[i] \subseteq \operatorname{End}(E).$

In this case we say *E* has CM by $\mathbb{Z}[i]$.

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• Let $K = \mathbb{Q}(i)$ and $n \ge 3$.

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In this case we say *E* has CM by $\mathbb{Z}[i]$.

- Let $K = \mathbb{Q}(i)$ and $n \ge 3$. Then
 - Gal(K(E[n])/K) is abelian (and explicitly given depending on the primes dividing n),
 - Every abelian extension of K is contained in K(E[n]) for some n.
 - For a prime $\ell \nmid 2n$ and $P \in E[n]$, we have

$$\sigma_{\ell}(\boldsymbol{P}) = [\ell]\boldsymbol{P}.$$

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General imaginary quadratic fields

Let $K = \mathbb{Q}(\sqrt{-d})$ for some $d \ge 1$. Let h_K be the class number of K.

- There are exactly *h_K* isomorphism classes (over ℂ) of elliptic curves with CM by *O_K*.
- For any such curve $E: y^2 = x^3 + ax + b$, set

$$j_E = \frac{6912a^3}{4a^3 + 27b^2}.$$

- The field H := K(j_E) is the Hilbert class field of K; that is, H is the maximal abelian unramified extension of K.
- Morever, as long as $j_E \neq 0$ or 1728,

$$K^{ab} = \bigcup_{n} K(j_E, x(E[n])) = \bigcup_{n} H(x(E[n])).$$

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Function fields and Drinfeld modules

- Function fields
- Drinfeld modules
 - The Carlitz module
 - Drinfeld modules of rank 1 and abelian extensions
 - Drineld modules of higher rank

Function fields

Let p be a fixed prime; q a fixed power of p.

${\sf A} \mathrel{\mathop:}= \mathbb{F}_q[heta]$	\longleftrightarrow	\mathbb{Z}
$k \mathrel{\mathop:}= \mathbb{F}_q(\theta)$	\longleftrightarrow	\mathbb{Q}
\overline{k}	\longleftrightarrow	$\overline{\mathbb{Q}}$
$k_\infty \mathrel{\mathop:}= \mathbb{F}_q((1/ heta))$	\longleftrightarrow	\mathbb{R}
$\mathbb{C}_\infty := \widehat{\overline{k_\infty}}$	\longleftrightarrow	\mathbb{C}
$ f _{\infty}=q^{\deg f}$	\longleftrightarrow	.

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Twisted polynomials

• Let $\tau : \mathbb{C}_{\infty} \to \mathbb{C}_{\infty}$ be the *q*-th power Frobenius map: $\tau(x) = x^q$.

• For a subfield $\mathbb{F}_q \subseteq K \subseteq \mathbb{C}_{\infty}$, the ring of *twisted polynomials* over K is

 $K[\tau] =$ polynomials in τ with coefficients in K,

subject to the conditions

$$\tau \mathbf{c} = \mathbf{c}^{\mathbf{q}} \tau, \quad \forall \mathbf{c} \in \mathbf{K}.$$

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In this way,

 $K[\tau] \cong \{\mathbb{F}_q \text{-linear endomorphisms of } K^+\}.$

For $x \in K$ and $\phi = a_0 + a_1 \tau + \cdots + a_r \tau^r \in K[\tau]$, we write

$$\phi(\mathbf{x}) := \mathbf{a}_0 \mathbf{x} + \mathbf{a}_1 \mathbf{x}^q + \cdots + \mathbf{a}_r \mathbf{x}^{q^r}.$$

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Functions on algebraic curves

- Let X be a smooth projective curve over \mathbb{F}_q , with function field $K = \mathbb{F}_q(X)$.
- Suppose we have fixed maps,

$$X \to \mathbb{P}^1 \quad \Leftrightarrow \quad \mathbb{F}_q(\theta) \hookrightarrow K.$$

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- Suppose we have fixed maps,

$$X \to \mathbb{P}^1 \quad \Leftrightarrow \quad \mathbb{F}_q(\theta) \hookrightarrow K.$$

- Fix a point ∞ on X that sits above the infinite point on \mathbb{P}^1 .
- Throughout the following we set

 $A := \{ f \in K \mid f \text{ is regular on } X \text{ away from } \infty \}.$

• So if
$$X = \mathbb{P}^1$$
, then $A = \mathbb{F}_q[\theta]$.

Drinfeld modules

Function field analogues of \mathbb{G}_m and elliptic curves Fix a curve X/\mathbb{F}_q and ring $A \subseteq K = \mathbb{F}_q(X)$ as above.

Definition

A Drinfeld A-module is an \mathbb{F}_q -algebra homomorphism,

 $\rho: \mathbf{A} \to \mathbb{C}_{\infty}[\tau],$

such that

$$\rho_f = f + a_1 \tau + \cdots a_s \tau^s, \quad \forall f \in A.$$

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• ρ makes \mathbb{C}_{∞} into an *A*-module in the following way:

$$f * \mathbf{x} := \rho_f(\mathbf{x}), \quad \forall f \in \mathbf{A}, \mathbf{x} \in \mathbb{C}_{\infty}.$$

• If $a_1, \ldots, a_r \in K \subseteq \mathbb{C}_{\infty}$ for all $f \in A$, we say ρ is defined over K.

• $s = r \deg(f)$, where r is called the *rank* of ρ .

The Carlitz module

The analogue of \mathbb{G}_m

• We define a Drinfeld $\mathbb{F}_q[\theta]$ -module $C : \mathbb{F}_q[t] \to \mathbb{C}_{\infty}[\tau]$ by

$$C_{\theta} := \theta + \tau.$$

Thus, for any $x \in \mathbb{C}_{\infty}$,

$$C_{\theta}(x) = \theta x + x^q.$$

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Thus, for any $x \in \mathbb{C}_{\infty}$,

$$C_{\theta}(x) = \theta x + x^q.$$

• And for example,

$$egin{aligned} \mathcal{C}_{ heta^2} &= \mathcal{C}_{ heta}\mathcal{C}_{ heta} = (heta + au)(heta + au) = heta^2 + (heta + heta^q) au + au^2, \ \mathcal{C}_{ heta^2}(x) &= heta^2 x + (heta + heta^q) x^q + x^{q^2}. \end{aligned}$$

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Carlitz exponential

We set

$$\exp_{\mathcal{C}}(z) = z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{(\theta^{q^i} - \theta)(\theta^{q^i} - \theta^q) \cdots (\theta^{q^i} - \theta^{q^{i-1}})}.$$

• $exp_{\mathcal{C}}: \mathbb{C}_{\infty} \to \mathbb{C}_{\infty}$ is entire, surjective, and \mathbb{F}_q -linear.

• Functional equation:

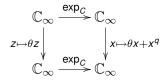
$$\exp_C(heta z) = heta \exp_C(z) + \exp_C(z)^q, \ \exp_C(f(heta) z) = C_f(\exp_C(z)), \quad orall f(t) \in \mathbb{F}_q[t].$$

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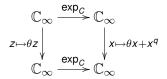
Carlitz uniformization and the Carlitz period

We have a commutative diagram of $\mathbb{F}_q[t]$ -modules,



Carlitz uniformization and the Carlitz period

We have a commutative diagram of $\mathbb{F}_q[t]$ -modules,



The kernel of $\exp_C(z)$ is

$$\ker(\exp_{\mathcal{C}}(z)) = \mathbb{F}_{q}[\theta]\widetilde{\pi},$$

where

$$\widetilde{\pi} = \theta \sqrt[q-1]{-\theta} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1}.$$

Torsion points on the Carlitz module

- Recall $k = \mathbb{F}_q(\theta), A = \mathbb{F}_q(\theta)$.
 - For $f \in \mathbb{F}_q[\theta]$, we set

$$C[f] = \{ x \in \mathbb{C}_{\infty} \mid C_f(x) = 0 \},\$$

= *f*-torsion submodule of *C*.

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Torsion points on the Carlitz module

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• For example,

$$egin{aligned} \mathcal{C}[heta] &= \{x \in \mathbb{C}_\infty \mid heta x + x^q = 0\} \ &= \{ \exp_{\mathcal{C}} igg(rac{a}{ heta} ig) \mid a \in \mathbb{F}_q \} \ &= \{ \zeta \mid q \sqrt[q]{- heta} \mid \zeta \in \mathbb{F}_q \}. \end{aligned}$$

- Preliminary observations:
 - $C[\theta] \cong A/\theta$ as an A-module;
 - $k(\mathbb{C}_{\infty}[\theta])/k$ is an abelian extension.

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Explicit class field theory for $\mathbb{F}_q(\theta)$

• For every $f \in A$,

$$\operatorname{Gal}(k(C[f])/k) \cong (A/f)^{\times}.$$

Indeed, given ℓ ∈ A irreducible with ℓ ∤ f, the Frobenius automorphism σ_ℓ ∈ Gal(k(C[f])/k) acts by

$$\sigma_{\ell}(\zeta) = C_{\ell}(\zeta), \quad \zeta \in C[f].$$

 Moreover, every abelian extension of k that is unramified away from ∞ is contained in k(C[f]) for some f ∈ A.

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Drinfeld A-modules for general A

Do they always exist? In general, defining a ring homomorphism
 A → S to some target ring S is non-trivial.

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Drinfeld A-modules for general A

- Do they always exist? In general, defining a ring homomorphism $A \rightarrow S$ to some target ring *S* is non-trivial.
- Yes, in fact for any *A*, there are Drinfeld *A*-modules of every possible rank.
- Example (Thakur): Let $A = \mathbb{F}_3[\theta, \eta]/(\eta^2 \theta^3 + \theta + 1)$. Then there is a rank 1 Drinfeld *A*-module,

$$\rho: \boldsymbol{A} \to \mathbb{C}_{\infty}[\tau],$$

with

$$\rho_{\theta} = \theta + \eta(\theta^{3} - \theta)\tau + \tau^{2},$$

$$\rho_{\eta} = \eta + \eta(\eta^{3} - \eta)\tau + (\eta^{9} + \eta^{3} + \eta)\tau^{2} + \tau^{3}.$$

In fact ρ is defined over the fraction field of *A*.

Rank 1 Drinfeld A-modules

Let A be given, K its fraction field. For simplicity, assume the point ∞ has degree 1.

- Let *h* be the class number of *A*. Let *H* be the Hilbert class field of *A* (maximal abelian unramified extension).
- Then there exist *h* isomorphism classes of rank 1 Drinfeld
 A-modules. Moreover, representatives ρ¹,..., ρ^h for these classes can be chosen (uniquely) so that each is defined over *H*:

$$o^i: \mathbf{A} \to \mathbf{H}[\tau].$$

(Uniqueness arises from normalizing the leading coefficients to be specific constants.)

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Explicit class field theory for K

Fix such a rank 1 Drinfeld A-modules, $\rho : A \rightarrow H[\tau]$.

• For any ideal $\mathfrak{f} \subseteq A$, the extension $H(\rho[\mathfrak{f}])/H$ is abelian and

 $\operatorname{Gal}(H(\rho[\mathfrak{f}])/H) \cong (A/\mathfrak{f})^{\times}.$

- Moreover, H(ρ[f])/K is abelian. (Recall that Gal(H/K) is isomorphic to the class group of A, so we can pin down the total Galois group precisely.)
- As in previous cases, the Artin automorphisms act via the ρ-action on the torsion points:

$$\sigma_{\ell}(\zeta) = \rho_{\ell}(\zeta), \quad \zeta \in \rho[\mathfrak{f}], \ell \nmid \mathfrak{f}.$$

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Drinfeld modules of arbitrary rank

- Suppose $\rho : \mathbf{A} \to \mathbb{C}_{\infty}[\tau]$ is a rank *r* Drinfeld A-module.
- Then there is an unique, entire, \mathbb{F}_q -linear function

$$\exp_{\rho}: \mathbb{C}_{\infty} \to \mathbb{C}_{\infty},$$

so that

$$\exp_{\rho}(fz) = \rho_f(\exp_{\rho}(z)), \quad \forall f \in A.$$

Clemson University (Lecture 1)

Intro. to Drinfeld Modules

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Periods of Drinfeld modules

• Furthermore, there are $\omega_1, \ldots, \omega_r \in \mathbb{C}_{\infty}$ and ideals $I_1, \ldots, I_r \subseteq A$, so that

$$\ker(\exp_{\rho}(z)) = I_1\omega_1 + \cdots + I_r\omega_r =: \Lambda,$$

where Λ is a discrete *A*-submodule of \mathbb{C}_{∞} of projective rank *r*.

Periods of Drinfeld modules

Furthermore, there are ω₁,..., ω_r ∈ C_∞ and ideals I₁,..., I_r ⊆ A, so that

$$\ker(\exp_{\rho}(z)) = I_{1}\omega_{1} + \cdots + I_{r}\omega_{r} =: \Lambda,$$

where Λ is a discrete A-submodule of \mathbb{C}_{∞} of projective rank r.

• Chicken vs. Egg:

$$\exp_{\rho}(z) = z \prod_{0 \neq \omega \in \Lambda} \left(1 - \frac{z}{\omega}\right).$$

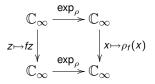
Again we have a uniformizing exact sequence of F_q[t]-modules

$$0 o \Lambda o \mathbb{C}_\infty \stackrel{\mathsf{exp}_
ho}{ o} \mathbb{C}_\infty o 0.$$

• How do we find the periods?

Torsion points on higher rank modules

In reality, exp_ρ is the unique power series that makes the following diagram commute for *f* ∈ *A*:



 Furthermore, the *f*-torsion submodule is isomorphic to *r* copies of *A*/*f*, which leads to a Galois representation

$$\operatorname{Gal}(L^{sep}/L) \to \operatorname{GL}_r(A/f),$$

where *L* is a field of definition for ρ .

 One can develop a theory of "*l*-adic" Galois representations (see Pink, Taguchi, Tamagawa, et al.)