p-adic Beilinson and Bloch-Kato:

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in progress w/ Rob de Jeu

k = number field, $[k:\mathbb{Q}]=d$.

$$\zeta_k^*(0) = \frac{-h_k R_k}{\omega_k} \cdot \xrightarrow{\text{via f.e.}} \zeta_k^*(1)$$

$$\text{reg}: \mathcal{O}_k^\times \otimes \mathbb{R} \longrightarrow \mathbb{R}^{r_1+r_2-1}$$

We are interested in $\zeta_k(r)$ $(r \geq 2)$.

$K_1(\mathcal{O}_k) = \mathcal{O}_k^\times$, $r \geq 2$ $K_{2r-1}(\mathcal{O}_k) = K_{2r-1}(k)$ is finitely generated: it has

$$\text{rank} = \begin{cases} r_1 + r_2 & \text{if } r \text{ is odd} \\ r_2 & \text{if } r \text{ is even} \end{cases}$$

(these are K-groups, though we don't define them, just state the properties we need.)

$\sigma : k \hookrightarrow \mathbb{C}$

$$\sigma^* : K_{2r-1}(k) \longrightarrow K_{2r-1}(\mathbb{C}) \overset{\text{Borel}}{\simeq} (2\pi i)^{r-1}\mathbb{R}.$$

Assume r is odd, k is totally real.

Let $\alpha_1, \ldots, \alpha_d$ be a $\mathbb{Z}$-basis of $K_{2r-1}(k)/_{tors}$.

$R_\infty(r,k) = \det((\sigma^*(\alpha_i))_{\sigma,i})$.

Fix a $\mathbb{Q}$-basis of k, say $a_1, \ldots, a_d$.

$$D_k^{1/2} = \det((\sigma(a_i))_{\sigma,i}).$$

Thm (Borel):

$$\zeta_k(r) D_k^{1/2} = q(r,k) R_\infty(r,k).$$

for some $q(r,k) \in \mathbb{Q}$.

Let $\chi : \mathrm{Gal}(\bar{k}/k) \longrightarrow \overline{\mathbb{Q}}_p$.

Fact (Deligne - Ribet): $\exists !$ $\mathcal{L}_p(s,\chi,k) : \mathbb{C} \to \overline{\mathbb{Q}}_p$ s.t. it is meromorphic in a nbd of zero and it interpolates the usual Dirichlet L-fctn, i.e.,

$$\mathcal{L}_p(m,\chi,k) = L(m,\chi,k)\, L_p(m,\chi,k)$$

for $m<0$, $m \equiv 1 \pmod{p-1}$.

Let $v | p$. $\sigma : k \hookrightarrow k_v$

Syntomic regulator (defined by Besser, de Jeu).

$$\sigma^* : K_{2r-1}(k) \longrightarrow K_{2r-1}(k_v) \xrightarrow{\mathrm{reg}_v} k_v$$

$\sigma : k \hookrightarrow \overline{\mathbb{Q}}_p$

$$\det((\sigma^*(\alpha_i))_{\sigma,i}) =: R_p(r,k).$$

$$\det((\sigma^*(a_i))_{\sigma,i}) =: D_{k,p}^{1/2}$$

P-adic Beilinson conjecture: ① $\mathcal{L}_p(r,\omega_p^{1-r},k)\, D_{k,p}^{1/2} = q_p(r,k)\, R_p(r,k) \cdot L_p(r,\omega_p^{1-r},k)$.

for some $q_p(r,k) \in \mathbb{Q}$.

② $q_p(r,k) = q(r,k)$

③ $R_p(r,k) \neq 0$, $\mathcal{L}_p(r,\omega_p^{1-r},k) \neq 0$.

Here ω_p = Teichmuller character:

$$\omega_p : \mathrm{Gal}(\bar{k}/k) \to \mathrm{Gal}(k(\mu_p)/k) \hookrightarrow \mathbb{Z}_p^\times.$$

$k(\omega_p^{1-r})$ is totally real.

<u>Thm (Besser, Buckingham, de Jeu, Roblot):</u> ① & ② of the conj. is true for all abelian $K/\mathbb{Q}$.

Let $L = K_{2r-1}(k)/K_{2r-1}(k)_{tors}$, $L_\infty = \left(\dfrac{\bigoplus_\sigma \mathbb{C}}{\bigoplus_\sigma (2\pi i)^r \mathbb{Z}} \right)^+ \simeq \mathbb{R}(r-1)^d \oplus (\mathbb{Z}/2\mathbb{Z})^d$. ($k$ - tot real, r odd)

We have a map

$$L \to L \otimes \mathbb{R} \to L_\infty.$$

For $v | p$, $L_v = H^1(k_v, \mathbb{Z}_p(r)) \cdot \prod_{\ell \neq p} H^1(k_v, \mathbb{Z}_\ell(r))_{tors}$

$$\cap$$

$$H^1(k_v, \hat{\mathbb{Z}}(r)).$$

Write $L_p = \prod_{v|p} L_v$.

$$k \otimes \mathbb{R} \simeq (k \otimes \mathbb{C})^+ \longrightarrow L_\infty$$

$$\underbrace{\Theta_\infty : \det_{\mathbb{R}}(k \otimes \mathbb{R}) \simeq \mathbb{R}}$$

this gives a measure μ_∞ on $k \otimes \mathbb{R}$.

$\exists$ an exponential map

$$\begin{array}{ccc} \prod_{v|p} H^1(k_v, \mathbb{Q}_p(r)) & \xrightarrow{\simeq} & k \otimes \mathbb{Q}_p. \\ \uparrow & & \\ L_p & & \end{array}$$

So this induces a measure μ_p on L_p coming from $k \otimes \mathbb{Q}_p$.

<u>Conjecture (Bloch-Kato):</u>

$$\mu_\infty(L_\infty/L) \cdot \prod \mu_p(L_p) = \frac{\# H^0(k, \mathbb{Q}/\mathbb{Z}(1-r))}{\# Ш(k(r))}.$$

Thm (B-K, Perrin-Riou, Benois-Quang Do): $K/\mathbb{Q}$ abelian

$$\mu_p(L_p) = \prod_{v|p} (1-q_v^{-r}) \, |D_k|_p^{r-1} \, |(r-1)!|_p^d \cdot \prod_{v|p} H^0(k_v, \mathbb{Q}_p/\mathbb{Z}_p(1-r))$$

$$\mu_\infty({}^{L_\infty}/_L) = \frac{R_\infty(r,k)}{D_k^{1/2}}$$ (equality up to powers of 2).

So we write B-K as:

$$\boxed{\frac{R_\infty(r,k)}{D_k^{1/2}} \cdot \zeta_k(r)^{-1}} \, D_k^{1-r} \, (r-1)!^{-d} \cdot (\text{cohomology grps}) = \frac{\# H^0(k, \mathbb{Q}/\mathbb{Z}(1-r))}{\# Ш(k(r))}.$$

$$\| \quad q(r,k)^{-1}$$

$$\text{B-K} \iff q(r,k) = (\text{cohomology groups, disc, rational \#}).$$

$$q_p(r,k) = \frac{\mathcal{L}_p(-)\, D_{k,p}^{1/2}}{R_p(-)\, L_p(-)}$$

Thm (Bayer-Neukirch): If $\mathcal{L}_p(r, \omega_p^{1-r}, k) \neq 0$, then

$$|\mathcal{L}_p(r, \omega_p^{1-r}, k)|_p = \frac{\# H^0(k, \mathbb{Q}_p/\mathbb{Z}_p(1-r))}{\# H^1(k, \mathbb{Q}_p/\mathbb{Z}_p(1-r))}.$$

$R_p(r,k)$ is defined

$$K_{2r-1}(k) \to K_{2r-1}(k_v) \xrightarrow{\mathrm{reg}_v} k_v.$$

Thm (Nizoł): The following diagrams commute

$$\begin{tikzcd} K_{2r-1}(k_v)\otimes\mathbb{Q} \arrow[r] & k_v \arrow[r, "\overset{exp.}{\sim}"] & H^1(k_v, \mathbb{Q}_p(r)) \\ K_{2r-1}(k)\otimes\mathbb{Q} \arrow[u] \arrow[r] & K_{2r-1}(k)\otimes\mathbb{Q}_p \arrow[r, phantom, "\simeq"] & H^1(k, \mathbb{Q}_p(r)) \arrow[u] \end{tikzcd}$$

We can relate $R_p(r,k)$ to

$$\mu_p(L_p) = L_p(-)\cdot D\,(r-1)!\ \cdot \text{cohomology grps.}$$

... Calculations match.

This shows p-adic Beilinson implies B-K.

- We just need $|q_p(r,k)|_p = |q(r,k)|_p$.

- p-adic Beilinson is known for $p=2$, $K/\mathbb{Q}$ abelian.

If k is CM, abelian over $\mathbb{Q}$, r even, then

$$\mathcal{L}_p(r, \chi\otimes\omega_p^{1-r}, k^+) = \frac{\#H^1(k, \mathbb{Z}_p(1-r))}{\#H^2(k, \mathbb{Z}_p(1-r))}\cdot\frac{\#H^2(k^+, \mathbb{Z}_p(1-r))}{\#H^1(k^+, \mathbb{Z}_p(1-r))}$$

where $\chi: Gal(k/k^+) \to \{\pm 1\}$.