Algebraic knot invariants, arithmetic knot theory, and asymptotics

Miller
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PS1

Knot theory:

study embeddings of manifolds into other manifolds

$= K \simeq S^1$ embedded in S^3

An n-knot is $\begin{array}{c} K \simeq S^n \\ \cap\!\shortmid \\ S^{n+2} \end{array}$

Knot theory studies these embeddings up to equivalence. (Won't give a precise definition of this.) Want to study invariants under this equivalence.

For our purposes we take K and S^{n+2} to be oriented.

'60s – Fox & Smythe found a way of constructing a knot invariant which took values in class groups of number fields.

$K_1 \downarrow$ trivial ideal class in $\mathbb{Z}[\frac{1}{53}][\sqrt{-211}]$

$K_2 \downarrow$ nontrivial ideal class in $\mathbb{Z}[\frac{1}{53}][\sqrt{-211}]$

$K_3 \downarrow$ other nontrivial ideal class in $\mathbb{Z}[\frac{1}{53}][\sqrt{-211}]$.

Alexander polynomials and modules

$K \simeq S^1$ - knot in S^3

One can naturally construct a knot invariant $M_{alex,K}$ of K

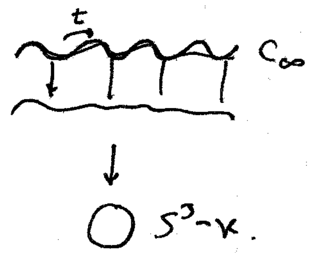
where $M_{alex,K}$ is a $\mathbb{Z}[t,t^{-1}]$-module satisfying

- $M_{alex,K}$ is annihilated by $\Delta_K[t]$ = Alexander poly. of K.

What is $M_{alex,K}$?

$H_1(S^3 \setminus K) = \mathbb{Z}$

$S^3 - K$ has an infinite cyclic cover C_∞



$M_{alex,K} = H_1(C_\infty, \mathbb{Z})$. $\mathbb{Z}[t,t^{-1}]$ acts via the action of shifting C_∞ over by 1.

$M_{alex,K}$ is naturally a module over $\mathbb{Z}[t,t^{-1}]/\Delta_K(t)$.

Note $\mathbb{Z}[t,t^{-1}]/\Delta_K(t) \subseteq \mathbb{Q}[t,t^{-1}]/\Delta_K(t)$. Note $\Delta_K(0) \neq 0$. If $\Delta_K(t)$ is square-free, then $\mathbb{Q}[t,t^{-1}]/\Delta_K(t)$ is a product of distinct number fields. If $\Delta_K(t)$ is irred, then $\mathbb{Q}[t,t^{-1}]/\Delta_K(t)$ is a number field. Usually $\mathbb{Z}[t,t^{-1}]/\Delta_K(t)$ is not just algebraic integers though.

Let $\mathcal{O}_\Delta = \mathbb{Z}[t,t^{-1}]/\Delta_K(t)$.

Which polynomials $\Delta(t)$ are $\Delta_K(t)$ for some K?

$\Delta_K(t)$ has $\Delta(0) \neq 0$. One needs:

- $\deg \Delta(t) = 2g$ for some g.
- $\Delta(t^{-1})\, t^{2g} = \Delta(t)$.
- $\Delta(1) = 1$.

If it satisfies these, then it is $\Delta_K(t)$ for some K.

<u>Example:</u> $g=1$, $\Delta_K(t) = mt^2 + (1-2m)t + m$, $m \in \mathbb{Z}$.

Fix Δ (sq. free). Let K be any knot with $\Delta_K = \Delta$. Then

(Kearton-Trotter) classified possible $M_{alex,K}$ as above

$\mathcal{O}_\Delta$-modules

$M \simeq I$ for some ideal I of $\mathcal{O}_\Delta$, $\exists$ a duality pairing

Blanchfield duality, Hermitian pairing

$$M \times M \to \mathcal{O}_\Delta$$
$$(a,b) = \overline{(b,a)}$$

The condition $\Delta(t^{-1})t^{2s} = \Delta(t)$ implies the involution of $\mathbb{Z}[t,t^{-1}]$ switching $t \leftrightarrow t^{-1}$ descends to an involution $x \mapsto \bar{x}$ of $\mathcal{O}_\Delta$.

Let $n = 2q \mp 1$, (q odd). Then a simple n-knot is a knot $K \subset S^{n+2}$ s.t. $\pi_i(S^{n+2} \setminus K) \cong \pi_i(S^1)$ for $i \leq q-1$. Can construct $M_{alex,K} \simeq H_q(C_\infty, \mathbb{Z})$. Then everything from above is still true

<u>Theorem (~~Keartin~~ Kearton, Trotter, Levine):</u> (70's) Simple n-knots are precisely classified by their alexander module plus the Blanchfield pairing.

How does this relate to arithmetic invariant theory?

Can we count these things asymptotically?

Cor: There are only finitely many simple n-knots with a given square-free Alexander polynomial.

We have $K \subseteq S^{n+2}$.
$\cong$
S^n

Def: A Seifert hypersurface for K is a hypersurface $V \subseteq S^{n+2}$ with boundary $\partial V = K$.

Theorem: Every 1-knot has a Seifert surface. Every simple n-knot has a $\underbrace{\text{simple}}_{\pi_i(V)=0 \text{ for } i \leq (q-1).}$ Seifert surface

where $n = 2q-1$.

If V is a simple Seifert hypersurface.
↳ $H_q(V;\mathbb{Z})$ is free and isomorphic to $\mathbb{Z}^{2g}$ for some g.

Moreover, there exists a $\mathbb{Z}$-valued "natural" bilinear pairing on $H_q(V;\mathbb{Z})$ called the Seifert pairing: $\langle \alpha, \beta \rangle$ (not symmetric or skew symmetric) If we consider the skew symmetric pairing $\langle \alpha, \beta \rangle - \langle \beta, \alpha \rangle$ is the intersection pairing, which is a unimodular skew-symmetric pairing, with matrix J after choosing an appropriate basis. Let P be the pairing matrix.

Knowing a Seifert hypersurface V and a Seifert pairing on V is enough to determine Alexander module of K.

$$\begin{tikzcd}[column sep=small]
 & & \text{ideal classes of rings } R = \mathbb{Z}[y]/f(y) \arrow[d, dash] \arrow[ddl] & \\
\begin{array}{c}\text{Simple Seifert surface} \\ V \text{ for } K \\ H_g(V;\mathbb{Z}) \cong \mathbb{Z}^{2g}\end{array} \arrow[r, "\overset{\text{bij}}{\simeq}"] \arrow[d] & \begin{array}{c}\text{data of} \\ H_g(V;\mathbb{Z}) = \mathbb{Z}^{2g} \\ \text{bil. pairings w/ matrix} \\ P - P^t = J \,/\, \text{change of basis}\end{array} \arrow[r] & f(y) \arrow[dr, mapsto] & \\
\begin{array}{c}\text{Simple } n\text{-knots} \\ K \\ \text{genus } 2g\end{array} \arrow[rr, "\simeq"', "\text{bij.}"'] & & \begin{array}{c}M_{alex,K} \\ \text{; Blanchfield pairings}\end{array} \arrow[r, "\text{finite to one}"] & \begin{array}{c}\{\text{Alex. polys}\} \\ f(t) = (t-1)^{2g}\Delta\left(\frac{1}{1-t}\right).\end{array}
\end{tikzcd}$$

Looking at $\{P \in Mat_{2g}(\mathbb{Z}) : P - P^t - J\} / \; P \sim GPG^t$ for any $G \in Sp_{2g}(\mathbb{Z})$

$\downarrow$

change variables to $Q = P + P^t$.

As we are looking at Sp_{2g} - equivalence classes of symmetric matrices.

Have an invariant polynomial $f(y) = \det(yP + J) \in \mathbb{Z}[y]$

and satisfies $f(y) = f(1-y)$.

$g = 1$ case: SL_2 acting on binary quadratic forms.

Alexander modules with Alexander polys $mt^2 + (1-4m)t + m$

correspond to narrow ideal classes of $\mathbb{Z}\left[\frac{1}{m}\right]\left[\frac{1+\sqrt{1-4m}}{2}\right]$

Seifert pairings (binary q.f.s of disc $= 1-4m$):

$\updownarrow$

narrow ideal classes of $\mathbb{Z}\left[\frac{1+\sqrt{1-4m}}{2}\right]$

for $0 < m < X$ # these $\sim X^{3/2}$

for $0 > m > X$ # these $\sim X/\log^2 X$ (conjecture)

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$\mathbb{Z}[\frac{1}{m}][\frac{1+\sqrt{1-4m}}{2}]$ has the same class number as

$\mathbb{Z}[\frac{1+\sqrt{1-4m}}{2}]$ for m prime, otherwise class group is 2-torsion (small)

Heuristically, ~~this~~ $\mathbb{Z}[\frac{1}{m}][\frac{1+\sqrt{1-4m}}{2}] \sim \frac{X^{3/2}}{\log X}$