An Automorphic Version of the Deligne Conjecture:

§1 The Deligne Conjecture

Let M be a motive $/\mathbb{Q}$ with coefficient in a number field E, pure of weight w and rank r.

- M_{dR} : vector space $/E$, dim r and E-rational Hodge filtration $\supset F^i(M) \supset F^{i+1}(M)$
- M_B : v.s. $/E$, dim r, and Hodge decomposition.

$$M_B \otimes \mathbb{C} \simeq \bigoplus_{p+q=w} M^{p,q}.$$

The action of infinite Frobenius on M_B extends to $M_B \otimes \mathbb{C}$ exchanges $M^{p,q}$ and $M^{q,p}$.

- $(M_\lambda)_{\lambda \text{ fin. place of } E}$ forms a compatible system of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ reps.

$\rightsquigarrow$ $L(s,M)$ can be defined.

We assume $M^{w/2, w/2} = \{0\}$ (*)

$\Downarrow$

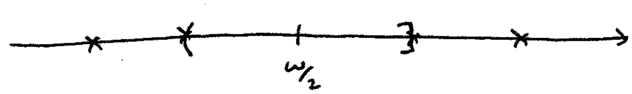
M has critical points

$\Downarrow$

The Deligne period can be defined.

<u>Def:</u> An integer m is <u>critical</u> for M if both $L_\infty(M,s)$ and $L_\infty(\check{M}, 1-s)$ are holomorphic at $s=m$.

<u>Remark:</u> $\{\text{Hodge numbers}\} = \{p : M^{p,w-p} \neq 0\}$.



$(*) \Rightarrow \{\text{critical points}\} = \{\text{integer in } (\ ,\ ]\}$.

Def: $M_B^+ = M_B^{F_\infty}$

$$F^+(M) = F^{w/2}(M)$$

$$I_\infty^+ : M_B^+ \otimes \mathbb{C} \hookrightarrow M_B \otimes \mathbb{C} \xrightarrow{\text{comp. isom}} M_{dR} \otimes \mathbb{C} \longrightarrow \left(M_{dR}/F^+(M)\right) \otimes \mathbb{C}.$$

Fix E-basis of M_B^+ and $M_{dR}/F^+(M)$ and extend them to $E \otimes \mathbb{C}$ basis of $M_B^+ \otimes \mathbb{C}$ and $(M_{dR}/F^+(M)) \otimes \mathbb{C}$.

Define $c^+(M) = \det(I_\infty^+)$ wrt the fixed basis.

$c^+(M) \in (E \otimes \mathbb{C})^\times$, well-defined up to $E^\times$.

For each $\sigma : E \hookrightarrow \mathbb{C}$, $L(s, M, \sigma)$.

$$L(s,M) = (L(s,M,\sigma))_\sigma \in E \otimes \mathbb{C} \qquad s \text{ critical}.$$

Conj. 1: (General Deligne Conjecture) Assume $(*)$. If m is critical for M, then

$$L(m,M) \underset{\substack{\uparrow \\ \text{equal up to } \overline{\mathbb{Q}}^\times}}{\sim} (2\pi i)^{\frac{mr}{2}} c^+(M).$$

Remark: $(*) \Rightarrow r = \operatorname{rank}(M)$ is even.

§2. Motives over Quadratic Imaginary Fields:

K imag. quad. field.

M motive $/K$ with coeff in E, pure of wt. w, rank n.

- M_{dR} : $E\otimes K$ module free rank n
- $\sigma: K \hookrightarrow \mathbb{C}$, M_σ E-v.s. dim n.

$I_{\infty,\sigma}: M_B\otimes \mathbb{C} \to M_{dR}\otimes_{K,\sigma}\mathbb{C}$ $\qquad M_\sigma\otimes\mathbb{C} \simeq \bigoplus M_\sigma^{p,q}$

↖ $E\otimes\mathbb{C}$-module.

We assume that M is regular $\forall \tau: E\hookrightarrow \mathbb{C}$, $\dim_{\mathbb{C}} M_\sigma^{p,q}(\tau) \le 1$.

Then for every $0\le s\le n$ we may define $Q^{(s)}(M)$ a motivic period.

When M is polarized, $Q^{(s)}(M)$ is the Petersson inner product of a rational vector in the bottom stage of $\Lambda^s \check{M}$.

Choose another motive M', wt w', rk n'. $\qquad 0\le t\le n'$ $\quad Q^{(t)}(M')$.

$\mathcal{M} = Res_{K/\mathbb{Q}}(M\otimes M')$.

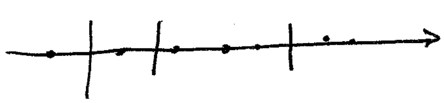
<u>Conj 2 (Variant of Deligne Conj.)</u>: If $m + \frac{n+n'-2}{2} \in \mathbb{Z}$ is critical for $Res_{K/\mathbb{Q}}(M\otimes M')$ then

$$L\left(m+\frac{n+n'-2}{2}, Res_{K/\mathbb{Q}}(M\otimes M')\right) \sim (2\pi i)^{mnn'}\prod_{s=0}^{n} Q^{(s)}(M)^{sp(s,M:M')}\prod_{s=0}^{n'} Q^{(t)}(M')^{sp(t,M':M)}.$$

<u>Remark:</u> (1) If $\mathcal{M} = Res_{K/\mathbb{Q}}(M\otimes M')$ does not satisfy (*) then there is no critical points. We may assume $\mathcal{M}$ satisfies (*).

(2) (*) $\Rightarrow$ $\forall$ $\begin{matrix}p\\p'\end{matrix}$ Hodge numbers for $\begin{matrix}M\\M'\end{matrix}$,

$$p+p' \neq \frac{w+w'}{2}$$



· HN for M

$\frac{w+w'}{2}$ ·

HN for M

$Sp(s, M; M')$ is the number of $\bullet$ between the barriers between the s-barriers and the $(s+1)$-barriers.

Def: ~~We~~ Let π be an auto. rep. of $GL_n(\mathbb{A}_K)$. We say a motive$/K$ is associated to π if π and M has the same local L-factors up to proper normalizations.

Tate conjecture $\Rightarrow$ If such M exists then it is unique up to isom. We denote it by $M(\pi)$.

§3. Automorphic Version of the Deligne Conjecture

Let π be a cuspidal cohomological conjugate self-dual and regular representation of $GL_n(\mathbb{A}_K)$.

$\forall\ 0 \le s \le n$ we can construct U_s a unitary group of signature $(s, n-s)/K$. We assume π descends to U_s by base change for $0 \le s \le n$. We can define a Shimura variety

$$Sh_s := Sh(\underset{\text{rational sim. grp.}}{GU_s}, X_s).$$

The descending of π contributes in a coh. space of Sh_s with coeff. in a local system W_s.

$$\mathrm{Im}(H_c^q(Sh_s, W_s) \to H^q(Sh_s, W_s)) =: \bar{H}^q(Sh_s, W_s).$$

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Moreover, $\bar{H}^q(Sh_s, W_s) \simeq \bigoplus_{w \in W' \leq W} \bar{H}^{q,w}(Sh_s, W_s)$

$$\bar{H}^{q,w}(Sh_s, W_s) \simeq \tilde{H}^{q-l(w)}(Sh_s, E_w)$$

auto vect. bundle.

obtained by the toroidal compactification

If $q = (n-s)s$, $\exists!$ w_0 longest in W' s.t. $l(w_0) = q$. Take rational $v \in \tilde{H}^0(Sh_s, E_{w_0})$. It lifts to an arithmetic holomorphic form β on $GU_s(\mathbb{A}_K)$. We define <u>automorphic period</u> $P^{(s)}(\pi)$, the Petersson inner product of β.

<u>Remark:</u> (1) If $M(\pi)$ exists uniquely, then we have constructed

a motive $\Lambda^s \check{M}(\pi)$.

Tate conj. $\Rightarrow$ $P^{(s)}(\pi) \sim Q^{(s)}(M(\pi))$.

(2) If $M(\pi)$ and $M(\pi')$ exist,

$$L(m, \tfrac{n+n'-2}{2}, Res_{K/\mathbb{Q}}(M(\pi) \otimes M(\pi'))) = L(m, \pi \times \pi').$$

<u>Def:</u> Assume $M(\pi)$ and $M(\pi')$ exist.

(1) We say m is critical for $\pi \times \pi'$ if $m + \frac{n+n'-2}{2}$ is critical for $M(\pi) \otimes M(\pi')$.

(2) $\forall\, 0 \leq s \leq n$ define $sp(s, \pi : \pi') = sp(s, M(\pi) : M(\pi'))$.

$\forall\, 0 \leq t \leq n'$ define $Sp(t, \pi' : \pi) = Sp(t, M(\pi') : M(\pi))$.

<u>Remark:</u> The Hodge type of $M(\pi)$ (resp $M(\pi')$) is determined by the inf. type of π (resp. π'). The above defs. still

make sense without knowing the existence of motives

<u>Conj 3:</u> (auto rep. version of Deligne Conj)

If $m \in \mathbb{Z} + \frac{n+n'}{2}$ is critical for $\pi \times \pi'$ then

$$L(m, \pi \times \pi') \sim (2\pi i)^{mnn'} \prod_{s=0}^{n} P^{(s)}(\pi)^{sp(s,\pi:\pi')} \prod_{t=0}^{n'} P^{(t)}(\pi')^{sp(t,\pi':\pi)}.$$

§4 Results and Ideas of Proof:

We assume $n \geq n'$.

<u>Thm 1:</u> Conj. 3 is true in the following cases:

(a) $n' = 1$.

(b) $n > n'$, (π, π') in good position

(c) $m = 1$

(all require regularity condition.)

<u>Remark:</u> (1) Write the infinity type of π (resp π') as $(z^{a_i} \bar{z}^{a_i})$

$a_1 > a_2 \dots > a_n$ (resp $(z^{b_j} \bar{z}^{b_j})$ $b_1 > b_2 > \dots > b_{n'}$).

(π, π') is in good position if $-b_j$, $1 \leq j \leq n'$ lie in different gaps between $a_1 > a_2 > \dots > a_n$

(2) (a) (Harris Crelle 97)

$n = n' - 1$ good position Grobner-Harris + L. (2015).

Ingredient: A Harris Case (a)

Ingredient B (Grobner-Harris)

$\pi^{\#}$ auto rep. generic tempered, enough regular, coh. of

$GL_{n-1}(\mathbb{A}_K)$, $(\pi, \pi^{\#})$ in good position

If m is critical for $\pi \times \pi^{\#}$,

$$L(m, \pi \times \pi^{\#}) \sim (AF)\; p(\pi)\, p(\pi^{\#})$$

arch. factor — Whittaker period

($\neq 0$ by Sun)

Ingredient C (Mahnkopf & Shahidi): If $\pi^{\#} = \pi_1 \boxplus \pi_2$ in (b) then

$$p(\pi^{\#}) \sim (AF)\cdot p(\pi_1)\, p(\pi_2) \times L(1, \pi_1 \times \pi_2^{\vee}).$$

Idea for Thm 1(b): Step 1 ✓ $\exists\ \pi^{\#} = \pi' \boxplus \chi_1 \boxplus \cdots \boxplus \chi_{n-n'-1}$ s.t. $(\pi, \pi^{\#})$ is in good position.

$$L(m, \pi \times \pi^{\#}) \overset{(B)}{\sim} (AF)\, p(\pi)\, p(\pi' \boxplus \chi_1 \boxplus \cdots \boxplus \chi_{n-n'-1})$$

$$\|$$

$$L(m, \pi \times \pi') \underbrace{\prod L(m, \pi \times \chi_i)}_{\text{Apply (A)}}.$$

$\Rightarrow$ $L(m, \pi \times \pi') \sim (AF)\, (AP)$ (Whittaker period.

auto period

Step 2: Write $p(\pi)$ in terms of AP.

$$\pi^{\#\#} = \chi_1 \boxplus \cdots \boxplus \chi_{n-1}$$

Repeat Step 1 $\Rightarrow$ $p(\pi) \sim (AF) \prod_{s=1}^{n-1} p^{(s)}(\pi)$.

Step 1 + Step 2. $\Rightarrow$ $L(m, \pi \times \pi') \sim (AF)$ (AP as expected)

Step 3: Calculate (AF) by replacing π and π' by $\underbrace{\text{simpler}}_{\text{induced from Hecke char.}}$ reps with same infinity type.

§5 Generalization to CM Fields:

F CM field, Φ CM type of F, $\Phi \sqcup \Phi^c = \{F \hookrightarrow \mathbb{C}\}$.

Motivic Part: $\forall\ \sigma \in \Phi$, $\forall\ 0 \leq s \leq n$ we can define $Q^{(s)}(M, \sigma)$.

Conj 2': $L(m + \frac{n+n'-2}{2}, Res_{K/\mathbb{Q}}(M \otimes M'))$

$$\sim (2\pi i)^{mnn'[F:\mathbb{Q}]/2} \prod_{\sigma \in \Phi} \left[\prod_{s=0}^{n} Q^{(s)}(M, \sigma)^{sp(s; M, M'_\sigma)} \right.$$

$$\prod_{t=0}^{n'} Q^{(t)}(M', \sigma)^{sp(t, M'; M, \sigma)}$$

Automorphic part: Ingredients B, C still true.

Ingredient A almost done by Lucco, Guerberoff.

$\forall\ I \in \Phi^{\{0,1,\dots,n\}}$, we can define $P^{(I)}(\pi)$.

Thm 2: (generalization of a conj. of Shin) $\exists$ complex numbers $P^{(s)}(\pi, \sigma)$, $0 \leq s \leq n$, $\sigma \in \Phi$

s.t. $P^{(I)}(\pi) \sim \prod_{\sigma \in \Phi} P^{(I(\sigma))}(\pi, \sigma)$.

Idea: We consider $P^{(*)}(\pi)$ as a function:

$$f: \underbrace{\Phi \times \cdots \times \Phi}_{n+1} \longrightarrow \mathbb{C}/\overline{\mathbb{Q}}^\times$$

$(\square) \Leftrightarrow \exists\ f_1, \dots, f_{n+1} : \Phi \to \mathbb{C}/\overline{\mathbb{Q}}^\times$ s.t.

$$f(x_1, \dots, x_{n+1}) = f_1(x_1) \cdots f_{n+1}(x_{n+1})$$

Lemma: $f: X \times Y \to Z$ $\quad X, Y$ sets, $(Z, \times)$..

$\exists\ g: X \to Z$, $h: Y \to Z$ s.t. $f(x,y) = g(x)h(y)$

$\Leftrightarrow \quad f(x,y)\, f(x',y') = f(x,y')\, f(x',y)$