

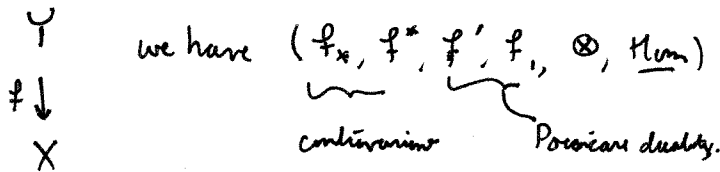
- I. The conjectural framework
- II. The no-pole case
- III. The pole case
- IV. Perspective.

I. The Conjectural Framework:

Grothendieck: X/\mathbb{Q} a variety

$MM(X)$ abelian category of mixed motivic sheaves over X .

$M \in MM(X)$ has a weight filtration. Pure if it only has one weight.



$m \in \mathbb{Z}$. $\mathbb{Q}(m) \in MM(\text{Spec } \mathbb{Q})$ pure of weight $-2m$.

Motivic cohomology. $H_{\mathbb{A}^1, \mu}^m(X, \mathbb{Q}(m)) = \text{Ext}_{MM(X)}^m(\mathbb{Q}_X(0), \mathbb{Q}_X(m))$

$\Delta^*(\mathbb{Q}(m)) =: \mathbb{Q}_X(m)$.

$MM(X) \xrightarrow{\Gamma_B} MM_{\mathbb{R}}(X)$ mixed \mathbb{R} -Hodge modules.

$\Gamma_B: H_{\mathbb{A}^1, \mu}^m(X, \mathbb{Q}(m)) \rightarrow H_{\mathbb{R}}^m(X, \mathbb{R}(m))$

absolute Hodge cohomology
(Deligne cohomology)

Example: $X = \text{Spec } K$ $[K:\mathbb{Q}] < \infty$

$$\Gamma_B: H_{\mathbb{A}^1, \mu}^1(X, \mathbb{Q}(1)) = K^{\times} \otimes \mathbb{Q} \xrightarrow{\Gamma_B} H_{\mathbb{R}}^1(X, \mathbb{R}(1)) = \mathbb{R}^{r_1+r_2}$$

Dirichlet regulator

Analytic class number formula $r_B \leftrightarrow \zeta_K^*(0)$ special value
 $= L(s, H^0(\text{Spec } K)(1))$
 motivic L-function

Lemma
 3-25-16
 p52

$H^0(\text{Spec } K)(1)$ has weight -2.

Today: Study another motive of weight -2, namely $H^4(S \times Y)(3)$

where S Shimura variety of $GSp(4)$, Y Shimura variety of $GL(2)$
 (let's pretend they are projective)

$$M = H^4(S \times Y)(3).$$

Conjecture (Tate): $\text{ord}_{s=0} L(s, M) = \dim_{\mathbb{Q}} \text{Hom}_{MM(\mathbb{Q})}(\mathbb{Q}(0), M)$
 $= \dim_{\mathbb{Q}} CH^2(S \times Y) / CH^2(S \times Y)_{\mathbb{h}}$

Conjecture (Beilinson): (i) The image of

$$r_B: H_{\mathbb{A}}^5(S \times Y, \mathbb{Q}(3)) \oplus CH^2(S \times Y) / CH^2(S \times Y)_{\mathbb{h}}$$

$$\rightarrow H_{\mathbb{K}}^5(S \times Y, \mathbb{R}(3)) \text{ is a } \mathbb{Q}\text{-structure}$$

(ii) $\det(\text{Im } r_B) = L^*(0, M) D(M)$ where $D(M)$ is an "elementary"
 \mathbb{Q} -structure on $\det H_{\mathbb{K}}^5(S \times Y, \mathbb{R}(3))$

Let $\pi \times \sigma$ an irreducible cuspidal automorphic representation of

$$GSp(4) \times_{\mathbb{G}_m} GL(2) = G.$$

Assumptions:

- generic $\Rightarrow \pi$ not Hecke character form
- cohomological of trivial weight $\Rightarrow \pi_f$ and σ_f are defined over the rationality field E .

Langlands: $L_S(s, \pi \times \sigma) = L_S(s-1, M(\pi \times \sigma))$

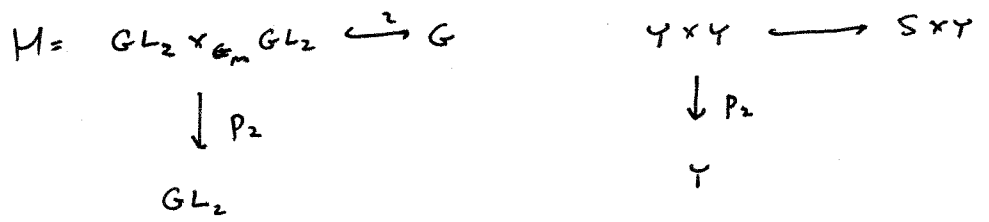
Let $\omega = \omega_\pi \omega_\sigma$ be the central character, ω° its finite order part

Dirichlet character of conductor $N \geq 1$.

II. The no-pole case:

π is stable

Assume $\omega^\circ \neq 1$. $(\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2 \setminus (0,0)$, $g_{\alpha, \beta} \in \mathcal{O}(Y)^* \otimes \mathbb{Q}$ Kato-Beilinson unit



$$\begin{array}{c}
 \mathcal{O}(Y \times Y)^* = H^1_{\mathbb{A}}(Y \times Y, \omega_{1,1}) \rightarrow H^5_{\mathbb{A}}(S \times Y, \mathbb{Q}(3)) \\
 \uparrow \\
 \mathcal{O}(Y)^*
 \end{array}$$

Def: $Z_\omega = \sum_{\alpha \in (\mathbb{Q}/\mathbb{Z})^*} \sum_{\beta \in (\mathbb{Q}/\mathbb{Z})^*} \left(\prod_{\alpha, \beta} g_{\alpha, \beta}^{\omega(\alpha, \beta)} \right) \in H^5_{\mathbb{A}}(S \times Y, \mathbb{E}(3))$

$\varphi \in \pi, \psi \in \sigma$ factorizable cusp forms.

ω_X, ω_Y associated harmonic diff. forms.

Lemma

3-25-16

PS4

Well-defined pairing $H_{2k}^5(S \times Y, \mathbb{R}(3)) \otimes_{\mathbb{R}} E \rightarrow \mathbb{R} \otimes E$

$$T \xrightarrow{\quad} T(\omega_X \otimes \omega_Y)$$

(Burger-Knaum-Kubota).

Thm: $r_B(Z_\omega) \in \omega_X \otimes \omega_Y = \int_{Z(\mathbb{A})/H(\mathbb{A})} \Psi(h_1, h_2) \varphi(h_1) E_{\mathbb{I}}(h_2, 1, \omega) d(h_1, h_2)$

$\xrightarrow{H(\mathbb{A})}$
 $Z(\mathbb{A})/H(\mathbb{A})$

Novodvasky integral

for some measure $d(h_1, h_2)$

$E_{\mathbb{I}}$ Jacquet Eisenstein series

$$E_{\infty}(x, y) = e^{-\pi(x^2 + y^2)}$$

$$E_p(x, y) = \mathbb{1}_{(N\mathbb{Z}, 1 + N\mathbb{Z})} \quad p \mid N$$

$$E_p(x, y) = \mathbb{1}_{\mathbb{Z}_p^*} \quad p \nmid N.$$

Proof: Commutative diagram

$$\begin{array}{ccc}
 \mathcal{O}(Y)^{\times} \otimes \mathbb{Q} & \longrightarrow & H_{2k}^5(S \times Y, \mathbb{Q}(3)) \otimes E \\
 \uparrow \alpha & & \downarrow r_B \\
 H_{2k}^1(Y, \mathbb{Q}(1)) & & \\
 \downarrow r_B & & \\
 H_{2k}^1(Y, \mathbb{R}(1)) & \xrightarrow{\quad ? \quad} & H_{2k}^5(S \times Y, \mathbb{R}(3)) \otimes E
 \end{array}$$

Scholl back

$\frac{1}{2}$ adelic translation of Kromer's second limit formula. \square

Note: $\omega^0 \neq 1 \Rightarrow E_{\pm}$ has no pole at $s=2$.

Lemma
3-25-16
P5

Cor: (Soudry-Murjama): $\Gamma_B(z)(\omega_\varphi \otimes \omega_\psi) = (*) \underbrace{L_S(0, M(\pi \times \sigma))}_{\neq 0}$.

\swarrow \searrow
 Compute Compute
 ramified integrals arch. integrals.

Question: if $\omega^0 = 1$ and $L_S(s, M(\pi \times \sigma))$ is holomorphic at 0 what can you say?

III. The pole case:

π endoscopy

More precisely, $L_S(s, \pi \times \sigma) = L_S(s, \pi, \sigma) L_S(s, \pi_2 \times \sigma)$

$\pi_2 \neq \pi_1$ rep of $GL(2)$ and $\pi_1 \cong \sigma$.

Note: $\omega = 1$ E_{\pm} has a pole at $s=2$

$[Y \times Y] \in CH^2(S \times Y) / CH^2(S \times Y)_A$

Thm: $\Gamma_B([Y \times Y])(\omega_\varphi \otimes \omega_\psi) = (*) \int_{H(\mathbb{A})} \varphi(h_1, h_2) \psi(h_1) d(h_1, h_2)$
 $\underbrace{H(\mathbb{A})}_{H(\mathbb{A})Z(\mathbb{A})} / H(\mathbb{A})$

Cor: $\Gamma_B([Y \times Y])(\omega_\varphi \otimes \omega_\psi) = (*) \operatorname{res}_{s=0} L_S(s, M(\pi \times \sigma))$

$\neq 0$.

IV. Perspective trilogy:

Lemma

3-25-16

196

①

$$GL(2) \times_{\mathbb{G}_m} GL(2) \longleftrightarrow GSp(4)$$

$$\begin{array}{ccc} & & \\ & \swarrow P_1 & \searrow P_2 \\ & GL(2) & GL(2) \end{array}$$

$$Y \times Y \xrightarrow{\tau} S \quad 2 \times (p_1^*(g_1) \cup p_2^*(g_2))$$

$$\begin{array}{ccc} & & \\ & \swarrow P_1 & \searrow P_2 \\ & Y & Y \end{array} \in H_M^4(S, \mathbb{Q}(3))$$

g_1, g_2 Siegel units.

$$\longleftrightarrow L_5(0, \text{spin}, M(\pi))$$

②

$$GL(2) \times_{\mathbb{G}_m} GL(2) \longleftrightarrow GSp(4) \times_{\mathbb{G}_m} GL(2)$$

$$\downarrow$$

$$GL(2)$$

$$2 \times p_2^*(g) \in H_M^5(S \times Y, \mathbb{Q}(3)) \longleftrightarrow L_5(0, \text{spin}, M(\pi \times \sigma^*))$$

③

$$GL(2) \times_{\mathbb{G}_m} GL(2) \xrightarrow{\tau} GSp(4) \times_{\mathbb{G}_m} GL(2) \times_{\mathbb{G}_m} GL(2)$$

$$Y \times Y \xrightarrow{\tau} S \times Y \times Y$$

$$\dim 1+1=2 \quad \dim 3+1+1=5$$

$$[Y \times Y] \in CH^2(S \times Y \times Y) = H_M^6(S \times Y \times Y, \mathbb{Q}(3))$$

$$\longleftrightarrow \text{Gan-Gross-Prasad relates this to } L_5(0, \text{spin}, M(\pi \times \sigma^*))$$

Analogue of Gross-Kudla-Schoen cycles.