Higher Regulators and non-critical values of degree 8 L-functions for $GSp(4) \times GL(2)$

I. The conjectural framework

II. The no-pole case

III. The pole case

IV. Perspective.

I. The Conjectural Framework:

Grothendieck: $X_{/\mathbb{Q}}$ a variety

$MM(X)$ abelian category of mixed motivic sheaves over X.

$M \in MM(X)$ has a weight filtration. Pure if it only has one weight.

$$\begin{array}{c} Y \\ f\downarrow \\ X \end{array} \quad \text{we have } (\underbrace{f_*, f^*}_{\text{contravariant}}, \underbrace{f^!, f_!}_{\text{Poincaré duality.}}, \otimes, \underline{Hom})$$

$m \in \mathbb{Z}$. $\mathbb{Q}(m) \in MM(\operatorname{Spec}\mathbb{Q})$ pure of weight $-2m$

Motivic cohomology. $H^m_{\mathcal{M}}(X, \mathbb{Q}(n)) = \operatorname{Ext}^m_{MM(X)}(\mathbb{Q}_X(0), \mathbb{Q}_X(n))$

$\Delta^*\mathbb{Q}(n) =: \mathbb{Q}_X(n)$.

$MM(X) \xrightarrow{r_B} MM_{\mathbb{R}}(X)$ mixed $\mathbb{R}$-Hodge module.

$r_B : H^m_{\mathcal{M}}(X, \mathbb{Q}(n)) \to H^m_{\mathcal{H}}(X, \mathbb{R}(n))$

absolute Hodge cohomology
(Deligne cohomology)

Example: $X = \operatorname{Spec} K$ $[K:\mathbb{Q}] < \infty$

$r_B : H^1_{\mathcal{M}}(X, \mathbb{Q}(1)) = K^\times \otimes \mathbb{Q} \xrightarrow{r_B} H^1_{\mathcal{H}}(X, \mathbb{R}(1))$
$= \mathbb{R}^{r_1 + r_2}$

Dirichlet regulator

Analytic class number formula $r_B \leftrightarrow \zeta_K^*(0)$ special value

$= L(s, H^0(\operatorname{Spec} K)(1))$

motivic L-function

$H^0(\operatorname{Spec} K)(1)$ has weight -2.

Today: Study another motive of weight -2, namely $H^4(S\times Y)(3)$

where S Shimura variety of $GSp(4)$, Y Shimura variety of $GL(2)$

(let's pretend they are projective)

$M = H^4(S\times Y)(3)$.

Conjecture (Tate): $-\operatorname{ord}_{s=0} L(s,M) = \dim_{\mathbb{Q}} \operatorname{Hom}_{MM(\mathbb{Q})}(\mathbb{Q}(0), M)$

$= \dim_{\mathbb{Q}} CH^2(S\times Y)/CH^2(S\times Y)_h$

Conjecture (Beilinson): (i) The image of

$r_B : H^5_{\mathcal{M}}(S\times Y, \mathbb{Q}(3)) \oplus CH^2(S\times Y)/CH^2(S\times Y)_h$

$\longrightarrow H^5_{\mathcal{H}}(S\times Y, \mathbb{R}(3))$ is a $\mathbb{Q}$-structure

(ii) $\det(\operatorname{Im} r_B) = L^*(0,M)\, D(M)$ where $D(M)$ is an "elementary" $\mathbb{Q}$-structure on $\det H^5_{\mathcal{H}}(S\times Y, \mathbb{R}(3))$

Let $\pi \times \sigma$ an irreducible cuspidal automorphic representation of

$GSp(4) \times_{\mathbb{G}_m} GL(2) = G$.

Assumptions:

- generic $\Rightarrow$ π not Siegel modular form
- cohomological of trivial weight $\Rightarrow$ π_f and σ_f are defined over the rationality field E.

Langlands: $L_S(s, \pi\times\sigma) = L_S(s-1, M(\pi\times\sigma))$

Let $\omega = \omega_\pi \omega_\sigma$ be the central character. ω° its finite order part Dirichlet character of conductor $N \geq 1$.

II. The no-pole case:

π is stable

Assume $\omega^\circ \neq 1$. $(\alpha,\beta) \in (\mathbb{Q}/\mathbb{Z})^2 \setminus (0,0)$, $g_{\alpha,\beta} \in \mathcal{O}(Y)^\times \otimes \mathbb{Q}$ Kato-Siegel unit

$$\begin{array}{ccc} H = GL_2 \times_{\mathbb{G}_m} GL_2 \xhookrightarrow{\iota} G & \qquad & Y \times Y \longrightarrow S \times Y \\ \big\downarrow p_2 & & \big\downarrow p_2 \\ GL_2 & & Y \end{array}$$

$$\begin{array}{c} \mathcal{O}(Y\times Y)^\times = H^1_{\mathcal{M}}(Y\times Y, \mathbb{Q}(1)) \longrightarrow H^5_{\mathcal{M}}(S\times Y, \mathbb{Q}(3)) \\ \big\uparrow \\ \mathcal{O}(Y)^\times \end{array}$$

Def: $Z_\omega = \iota_* p_2^* \left(\prod_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} g_{a,\beta}^{\omega^\circ(a)} \right) \in H^5_{\mathcal{M}}(S\times Y, E(3))$

$\varphi \in \pi, \psi \in \sigma$ factorizable cusp forms.

Lemma

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PS4

ω_φ, ω_ψ associated harmonic diff. forms.

Well-defined pairing $H^5_{\mathcal{H}}(S\times Y, \mathbb{R}(3))\otimes_{\mathbb{R}} E \longrightarrow \mathbb{R}\otimes E$

$$T \longmapsto T(\omega_\varphi \otimes \omega_\psi)$$

(Burgos-Kramer-Kühn)

Thm: $$r_B(Z_\omega)(\omega_\varphi\otimes\omega_\psi) = (*)\int_{Z(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} \psi(h_1,h_2)\varphi(h_1)\,E_\Phi(h_2,1,\omega^0)\,d(h_1,h_2)$$

Novodvorsky integral

for some measure $d(h_1,h_2)$

E_Φ Jacquet Eisenstein series

$\Phi_\infty(x,y) = e^{-\pi(x^2+y^2)}$

$\Phi_p(x,y) = \mathbb{1}_{(N\mathbb{Z}_p,\,1+N\mathbb{Z}_p)}$ $\quad p\mid N$

$\Phi_p(x,y) = \mathbb{1}_{\mathbb{Z}_p^2}$ $\quad p\nmid N$.

Proof: Commutative diagram

$$\begin{tikzcd}
\mathcal{O}(Y)^\times\otimes\mathbb{Q} \arrow[r] \arrow[d, equal] \arrow[ddd, bend right, "u"'] & H^5_{\mathcal{M}}(S\times Y, \mathbb{Q}(3))\otimes E \arrow[ddd, "r_B"] \\
H^1_{\mathcal{M}}(Y,\mathbb{Q}(1)) \arrow[dd, "r_B"] & \\
& \\
H^1_{\mathcal{H}}(Y,\mathbb{R}(1)) \arrow[r, "\imath_*"] & H^5_{\mathcal{H}}(S\times Y, \mathbb{R}(3))\otimes E
\end{tikzcd}$$

$u \mapsto \log|u|$

Scholl Bach

& adelic translation of Kronecker second limit formula. ∎

<u>Note:</u> $\omega^\circ \neq 1 \Rightarrow E_\Phi$ has no pole at $s=2$.

<u>Cor:</u> (Soudry - Moriyama): $r_B(Z_\infty)(\omega_\varphi \otimes \omega_\psi) = \underset{\neq 0}{(*)}\, L_S(0, M(\pi\times\sigma))$.

computes ramified integrals — compute arch. integrals.

<u>Question:</u> if $\omega^\circ = 1$ and $L_S(s, M(\pi\times\sigma))$ is holomorphic at 0 what can you say?

II. The pole case:

π endoscopic

More precisely, $L_S(s,\pi\times\sigma) = L_S(s,\pi_1\times\sigma)\, L_S(s,\pi_2\times\sigma)$

$\pi_1 \not\cong \pi_2$ rep of $GL(2)$ and $\check{\pi}_1 \simeq \sigma$.

<u>Note:</u> $\omega = 1$ E_Φ has a pole at $s=1$

$[X\times Y] \in CH^2(S\times Y)/CH^2(S\times Y)_h$

<u>Thm:</u> $r_B([Y\times Y])(\omega_\varphi\otimes\omega_\psi) = (*) \displaystyle\int_{H(\mathbb{Q})Z(\mathbb{A})\backslash H(\mathbb{A})} \varphi(h_1,h_2)\,\psi(h_1)\, d(h_1,h_2)$

<u>Cor:</u> $r_B([Y\times Y])(\omega_\varphi \otimes \omega_\psi) = (*)\, \operatorname{res}_{s=0} L_S(s, M(\pi\times\sigma))$

$\neq 0$.

IV. Perspective trilogy:

①

$$\begin{tikzcd} GL(2) \times_{\mathbb{G}_m} GL(2) \arrow[r, hook] \arrow[d, "p_1"'] \arrow[dr, "p_2"] & GSp(4) \\ GL(2) & GL(2) \end{tikzcd}$$

$$\begin{tikzcd} Y \times Y \arrow[r, hook, "\iota"] \arrow[d, "p_1"'] \arrow[dr, "p_2"] & S \\ Y & Y \end{tikzcd}$$

$\iota_*(p_1^*(g_1) \cup p_2^*(g_2)) \in H^4_M(S, \mathbb{Q}(3))$

g_1, g_2 Siegel units.

$\longleftrightarrow L_S(0, \text{spin}, M(\pi))$

②

$$\begin{tikzcd} GL(2) \times_{\mathbb{G}_m} GL(2) \arrow[r, hook] \arrow[d] & GSp(4) \times_{\mathbb{G}_m} GL(2) \\ GL(2) & \end{tikzcd}$$

$\iota_* p_2^*(g) \in H^5_M(S \times Y, \mathbb{Q}(3)) \longleftrightarrow L_S(0, \text{spin}, M(\pi \times \sigma))$.

③ $GL(2) \times_{\mathbb{G}_m} GL(2) \xrightarrow{\iota} GSp(4) \times_{\mathbb{G}_m} GL(2) \times_{\mathbb{G}_m} GL(2)$

$Y \times Y \xrightarrow{\iota} S \times Y \times Y$

dim $1+1=2$ dim $3+1+1=5$

$[Y \times Y] \in CH^3(S \times Y \times Y) = H^6_M(S \times Y \times Y, \mathbb{Q}(3))$

$\longleftrightarrow$ Gan-Gross-Prasad relates this to $L_S(0, \text{spin}, M(\pi \times \sigma \times \sigma'))$

analogue of Gross-Kudla-Schoen cycles.