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Modularity of Residually Reducible Galois Representations

Introduction:

p prime, $N \in \mathbb{Z}_+$: One version of the Fontaine-Mazur conjecture reads.

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{Modular Hecke eigenforms} \\ \text{of level } N\end{array}\right\} \arrow[r, leftrightarrow, "1-1"'] \arrow[r, bend left, "\text{Eichler-Shimura-Deligne-Serre}"'] & \left\{\begin{array}{l}\rho: \overset{G_{\mathbb{Q}}}{\overset{\shortparallel}{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}} \xrightarrow{\text{cont.}} GL_2(\overline{\mathbb{Q}}_p) \\ \text{unram. away from } Np \\ \text{\& potentially semistable at } p\end{array}\right\} \arrow[l, bend left=60, "\text{almost.}"]
\end{tikzcd}$$

- If
$$\begin{tikzcd}
G_{\mathbb{Q}} \arrow[r] \arrow[rr, bend right, "\bar{\rho}"'] & GL_2(\overline{\mathbb{Z}}_p) \arrow[r] & GL_2(\overline{\mathbb{F}}_p)
\end{tikzcd}$$

is irred, (Taylor-Wiles, Breuil-Conrad-Diamond-Taylor, Diamond-Flach-Guo, Kisin).

- If $\bar{\rho}$ is reducible (Skinner-Wiles)

More generally, if G is a reductive algebraic group, $F = \#$ field.

$$\left\{\begin{array}{c}\text{"Nice" auto. reps.} \\ \text{of } G(\mathbb{A}_F)\end{array}\right\} \xleftrightarrow[??]{1-1} \left\{\begin{array}{c}\text{"Nice" reps.} \\ \rho: \cancel{G_{\mathbb{Q}}}\ G_F \to GL_n(\overline{\mathbb{Q}}_p)\end{array}\right\}.$$

What is known?

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1) If $G = GL_2$, F = tot. real

($\underbrace{\text{Fujiwara, Kisin, Taylor}}_{\bar{\rho} \text{ irred}}$, $\underbrace{\text{Skinner-Wiles}}_{\bar{\rho} \text{ red}}$)

2) If $G = GL_2$, F imag. quad. (Berger - K.)
$\bar{\rho}$ red

3) If G = unitary, ($\underbrace{\text{Clozel-Harris-Taylor \& students of Taylor}}_{\bar{\rho} = \text{irred.}}$)

<u>Some Theorems:</u>

1a) Imag. quad. case

p = prime ≥ 3, $[E : \mathbb{Q}_p] < \infty$, $\mathcal{O} \subset E$, ϖ = unif.,
$\mathbb{F} = \mathcal{O}/\varpi$.
F = imag. quad. field, $p \nmid \# Cl_F$, $p \nmid d_F \neq 3, 4$.
Fix $\mathfrak{p} \mid p$.
Σ = finite set of primes of F, $\mathfrak{p} \in \Sigma$, $G_\Sigma = \mathrm{Gal}(F_\Sigma / F)$.
Ψ = (unramified) Hecke char. of F s.t.

$$\Psi_\infty(z) = \frac{z}{\bar{z}}$$

$\Psi_\mathfrak{p} : G_\Sigma \to \mathcal{O}^\times$, $\chi_0 = \Psi_\mathfrak{p} \pmod{\varpi}$.

<u>Theorem 1 (Berger - K.):</u> Let $\rho : G_\Sigma \to GL_2(\bar{\mathbb{Q}}_p)$ be cont. and irred. Suppose

- $\det \rho = \Psi_\mathfrak{p}$.

- $\bar{\rho}^{ss} = 1 \oplus \chi_0$ with χ_0 satisfying

  (*) $\dim_{\mathbb{F}} \mathrm{Ext}^1_{\mathbb{F}[G_\Sigma]}(1, \chi_0) = 1$

- ρ is crystalline (or ordinary) at p
- ρ is minimally ramified.

Then ρ is modular; i.e., $\exists$ an automorphic representation π of $GL_2(/\mathbb{A}_F)$ s.t.

$$L(s, \pi) = L(s, \rho) \quad \text{(up to a twist of } \rho\text{)}.$$

<u>Remarks:</u>

1) This is similar to a result of Skinner-Wiles for $\mathbb{Q}$, but their method fails for F = imag. quad. field. A key step in [SW] is a construction of ordinary minimal lift of $\rho_0 = \begin{pmatrix} 1 & * \\ 0 & \chi_0 \end{pmatrix} \not\cong \begin{pmatrix} 1 & 0 \\ 0 & \chi_0 \end{pmatrix}$ to an upper triangular rep. $\rho: G_{\mathbb{Q}} \to GL_2(\mathcal{O})$. But;

<u>Thm: (Berger-K.):</u> For F = imag. quad, there does not exist an ordinary minimal lift of such ρ_0 to an upper triangular rep. into $GL_2(\mathcal{O})$.

2) The unramifiedness of ψ condition can be replaced by requiring that $H_c^2(S_{K_f}, \mathbb{Z}_p)^{tors} = 0$.

1b): GSp_4 case

N = sq. free integer, k = even, $p > k \geq 4$, $F = \mathbb{Q}$, $\forall\, \ell \mid N$ is s.t.

$\ell \not\equiv 1 \pmod{p}$. $f \in S_2(N)$, $g \in S_k(N)$,
$\Sigma = \{ \ell \mid N, p \}$. Assume $\bar{\rho}_f$, $\bar{\rho}_g$ abs. irred.

Theorem 2: (Berger - K.) Suppose

(*) $\dim_{\mathbb{F}} H^1_\Sigma(\mathbb{Q}, \mathrm{Hom}(\bar{\rho}_g, \bar{\rho}_f(k/2-1))) = 1$

(**) $R_{\bar{\rho}_f} = R_{\bar{\rho}_g}$ = discrete valuation rings

(BK) The Bloch-Kato conjecture holds for $\mathrm{Hom}(\rho_g, \rho_f(k/2-1))$.

Let $\rho : G_{\mathbb{Q},\Sigma} \to GL_4(\overline{\mathbb{Q}}_p)$ be cont., irred s.t.

- $\bar{\rho}^{ss} = \bar{\rho}_f(k/2-1) \oplus \bar{\rho}_g$
- ρ is crystalline at p
- ρ is ess. self-dual

Then ρ is modular, i.e., $\exists$ a Siegel modular form F of weight $k/2+1$, level N, trivial char. s.t.

$$\rho_F \cong \rho.$$

Method:

$\mathbb{F}$ = # field, $\rho_i : G_\Sigma \to GL_{n_i}(\mathbb{F})$ $i = 1, 2$ irred.

Consider $\rho_0 : G_\Sigma \to GL_{n_1+n_2}(\mathbb{F})$,

$$\rho_0 = \begin{pmatrix} \rho_1 & * \\ 0 & \rho_2 \end{pmatrix} \not\cong \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}.$$

We will study crystalline (at p) lifts (deformations) of ρ_0 to local, complete, Noetherian $\mathcal{O}$-algebras w/ residue field $\mathbb{F}$.

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We know via Mazur that there exists a (R_Σ, ρ_Σ)
where R_Σ = univ. def. ring and ρ_Σ = univ. def.

<u>Goal:</u>

$$\phi : R_\Sigma \xrightarrow{\sim} \mathbb{T}_\Sigma .$$

<u>Def:</u> I_u = smallest ideal I of R_Σ s.t. $\operatorname{tr} \rho_\Sigma = \Psi_1 + \Psi_2 \mod I$.

$\operatorname{tr}\rho' + \operatorname{tr}\rho''$

pseudo chars

where $\Psi_i = \operatorname{tr} \rho_i \mod m_{R_\Sigma}$.

R_Σ / I_u controls all the "reducible" lifts

I_u controls the "irred" ones.

The key step is to reduce the problem to modularity of reducible lifts. (i.e., to show "$R/I_u = \mathbb{T}/\phi(I_u)$")

<u>Thm (Berger-K.):</u> R, S comm. rings, $r \in R$, $\bigcap_n r^n R = 0$.
Let A = domain. Suppose S is a fin. gen. free A-module.
Consider

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ \downarrow & \circlearrowleft & \downarrow \\ R/rR & \xrightarrow[\sim]{\bar\phi} & S/\phi(r)S \end{array}$$

If $\operatorname{rk}_A S/\phi(r)S = 0$, then ϕ is an isom.

Cor: Set $S = \mathbb{T}_\Sigma$. Suppose $I_u = (r)$ and that

$$\# \mathbb{T}_\Sigma / \phi(I_u) < \infty.$$

If $\phi : R_\Sigma \to \mathbb{T}_\Sigma$ induces an isom.

$$\bar{\phi} : R_\Sigma / I_u \xrightarrow{\sim} \mathbb{T}_\Sigma / \phi(I_u),$$ then

ϕ is an isom.

Upshot: To prove $R = T$, it is enough to show

- I_u is principal
- every reducible deformation is modular.

Remark: The condition $\mathrm{rk}_A \, S/\phi(r)S = 0$ can be replaced by demanding that

$$rR/r^2R \simeq \phi(r)S/\phi(r)^2S.$$

Then this criterion gives an alternative to the criterion of Wiles and Lenstra.

$I_u = (r)$:

Thm (Berger-K.) Building on work of Bellaïche-Chenevier: Let A be Noetherian, local ring, $(n_1+n_2)! \in A^\times$, $S = A[G_\Sigma]$.
Let $\rho : S \to M_{n_1+n_2}(A)$ morphism of A-algebras s.t. $\rho \equiv \rho_0 \pmod{m_A}$.

1) If A is reduced, infinite but $\# A/I_u A < \infty$

then $I_{red,A}$ is principal.

2) If $\exists\ \tau : S \to S$ an anti-autom s.t. $\tau^2 = 1$ & $tr(\rho \circ \tau) = tr\rho$, $tr\rho_i \circ \tau = tr\rho_i$, then $I_{red,A}$ is principal.

Modularity of reducible lifts "$R_\Sigma / I_{red} \simeq \mathbb{T}_\Sigma / \phi(I_{red})$"

1) $\# R_\Sigma / I_{red} \leq \# \mathcal{O} / L\text{-value}$ (Bloch-Kato conj.)

2) $\# \mathbb{T}_\Sigma / \phi(I_{red}) \geq \# \mathcal{O} / L\text{-value}$ (congruences imag. quad: Berger; Sp_4 Agarwal, K., B-D.-S.P)