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PSI

Level Stripping for Genus 2 Siegel Modular Forms:

§1 Galois Representations (intro)

Notation: Fix $\ell > 3$ a prime. Let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

Properties of $G_{\mathbb{Q}}$:

1) $G_{\mathbb{Q}} = \varprojlim_{\substack{K/\mathbb{Q} \text{ fin} \\ \text{Galois ext.}}} \mathrm{Gal}(K/\mathbb{Q})$

2) Equipping $\mathrm{Gal}(K/\mathbb{Q})$ with the discrete topology, we have that $G_{\mathbb{Q}}$ has the profinite topology.

3) For every prime p, we have an "element" $\mathrm{Frob}_p \subseteq G_{\mathbb{Q}}$

Thm (Cebotarev Density): Let $N \in \mathbb{Z}$. Then $\{\mathrm{Frob}_p : p \nmid N\}$ is dense in $G_{\mathbb{Q}}$.

Def: A d-dimensional Galois representation is a continuous homomorphism

$$\rho : G_{\mathbb{Q}} \to GL_d(R)$$

where R is some topological ring.

Examples: 1) Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\text{hom}} GL_1(\mathbb{C})$ (Dirichlet char mod N)

Set

$$\rho_\chi : G_{\mathbb{Q}} \xrightarrow{\text{res}} \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} GL_1(\mathbb{C})$$

This is an example of an Artin representation.

$$\rho_\chi(\mathrm{Frob}_p) = \chi(p) \quad \text{for } p \nmid N.$$

2) As $Gal(\mathbb{Q}(\zeta_{\ell^n})/\mathbb{Q}) \cong (\mathbb{Z}/\ell^n\mathbb{Z})^\times$,

we get

$$\rho_\ell : G_\mathbb{Q} \xrightarrow{res} \varprojlim_n Gal(\mathbb{Q}(\zeta_{\ell^n})/\mathbb{Q}) \cong GL_1(\mathbb{Z}_\ell).$$

This is an ℓ-adic representation.

$$\rho_\ell(\mathrm{Frob}_p) = p \qquad p \neq \ell.$$

3) Reducing ρ_ℓ mod ℓ we have

$$\overline{\rho}_\ell : G_\mathbb{Q} \to GL_1(\mathbb{Z}_\ell/\ell\mathbb{Z}_\ell) \cong GL_1(\mathbb{F}_\ell).$$

$$\overline{\rho}_\ell(\mathrm{Frob}_p) = p \pmod{\ell}, \quad p \neq \ell.$$

This is a mod ℓ representation.

§2 2-dim Galois reps:

Let $f \in S_k(N, \chi)$ $\quad$ k = weight, level N, character χ.

$f : \mathfrak{H} \to \mathbb{C}$ $\quad$ (holo, transforms nicely, vanishes at cusps)

Assume f is an eigenvector of the Hecke operators. (commuting linear operators).

We have a Galois rep. (Deligne, Shimura)

$$\rho_{f,\lambda} : G_\mathbb{Q} \longrightarrow GL_2(\mathcal{O}_K) \qquad \mathcal{O}_K = \text{r.o.i. of } K \subseteq \overline{\mathbb{Q}}_\ell$$

$\lambda | \ell$.

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The characteristic polynomial of $\rho_{f,\lambda}(\mathrm{Frob}_p)$ can be written in terms of the Hecke eigenvalues, $p \nmid N\ell$.

Reducing $\rho_{f,\lambda}$ modulo λ we have

$$\bar{\rho}_{f,\lambda} : G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}_\ell).$$

We call this type of representation "modular".

Serre's (refined) conjecture (Khare-Wintenberger 2007): If
$\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}_\ell)$ is irreducible, odd, then
$\bar{\rho}$ is modular. (of char. $\chi(\bar{\rho})$, level $(N(\bar{\rho})$, wt $k(\bar{\rho}))$.

Note: $\ell \nmid N(\bar{\rho})$.

Thm (Ribet '94): If $\bar{\rho}$ is modular of level $N\ell^r$ ($\ell \nmid N$, $r>0$), then $\bar{\rho}$ is modular of level N.

Pf: (idea): $\bar{\rho} = \bar{\rho}_{f,\lambda}$, f level $N\ell^r$. Find a level N eigenform g s.t. $f \equiv g \pmod{\lambda}$.

$$\Rightarrow \bar{\rho}_{f,\lambda} \cong \bar{\rho}_{g,\lambda}.$$

§3 4-dimensional Galois representations:

Let $F \in S^2_{(k_1,k_2)}(N,\chi)$ $\quad$ (k_1,k_2) = wt, level = N, χ = char.
2 = genus.

$F : \mathfrak{H}^2 \to V$ $\quad$ $\mathfrak{H}^2 = \{ Z = X+iY \in \mathrm{Mat}_2(\mathbb{C}),\ {}^tZ = Z,\ Y>0 \}$

V = f.d. $\mathbb{C}$-v.s.

Transform nicely, "vanish at cusps"

Assume F is an eigenvector of the Hecke ops.

We have a 4-d Galois rep.

(Weissauer et al) $\rho_{F,\lambda} : G_{\mathbb{Q}} \to GL_4(\mathcal{O}_K)$

char poly $\rho_{F,\lambda}(\mathrm{Frob}_p)$ can be written in terms of the Hecke eigenvalues.

Once again reduce modulo λ to obtain

$$\bar{\rho}_{F,\lambda} : G_{\mathbb{Q}} \to GL_4(\bar{\mathbb{F}}_\ell).$$

We call these representations coming from such a construction modular.

<u>Conjecture:</u> (Herzig-Tilouine, 2013): If $\bar{\rho} : G_{\mathbb{Q}} \to GL_4(\bar{\mathbb{F}}_\ell)$ is irreducible, odd, ordinary at ℓ, of modular weight, then $\bar{\rho}$ is modular of level coprime to ℓ.

<u>Thm (K.):</u> Suppose $\bar{\rho}$ is modular of level $N\ell^r$, $\ell \nmid N$, $r>0$ and of char. defined modulo $N\ell$. Then $\bar{\rho}$ is

modular of level N.

Pf idea: Same idea as Ribet.

$$\bar{\rho} = \bar{\rho}_{F,\lambda}.$$

Find G s.t. $F \equiv G \pmod{\lambda}$.
Then charpoly are equal. We then apply
Brauer-Nesbitt then gives $\overline{\rho_{F,\lambda}} \simeq \overline{\rho_{G,\lambda}}$.