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$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad s \in \mathbb{C}$$

- meromorphic cont. w/ simple pole at $s=1$

Euler (1749) "evaluate $\zeta(-m)$" $m \geq 0$:

$$1 - 2^{-s} + 3^{-s} - 4^{-s} + \cdots + (-1)^{n-1} n^{-s} + \cdots$$

$$= \sum_{n \geq 1} n^{-s} - 2 \sum_{n \geq 1} (2n)^{-s}$$

$$= \zeta(s) - 2^{1-s} \zeta(s)$$

$$g_s(t) = \sum_{n=1}^{\infty} (-1)^{n-1} n^s t^n$$

$$\left(t \frac{d}{dt}\right)^s (t^n) = n^s t^n \qquad s \in \mathbb{Z}_{\geq 0}$$

So

$$g_s(t) = \left(t \frac{d}{dt}\right)^s \Big(\underbrace{t - t^2 + t^3 - t^4 + \cdots}_{\frac{t}{1+t}} \Big)$$

$$g_s(1) = \left(t \frac{d}{dt}\right)^s \left(\frac{t}{1+t}\right)\Big|_{t=1}$$

Thus,

$$(1 - 2^{1-s}) \zeta(s) = \underbrace{\left(t \frac{d}{dt}\right)^s \left(\frac{t}{1+t}\right)\Big|_{t=1}}_{\in \mathbb{Q}}$$

So if $s = m \in \mathbb{Z}_{\geq 0}$.

Of course, this doesn't work b/c the geo series does not converge at $t=1$..

In fact, $\zeta(-m) \in \mathbb{Q}$

$$\zeta(1-k) = -\frac{B_k}{k},$$

where

$$\frac{e^t}{1-e^t} = \sum_{k \geq 1} B_k \frac{t^k}{k!}.$$

More generally, $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ $\quad N \geq 1$

$$L(s,\chi) = \sum_{n \geq 1} \chi(n) n^{-s}.$$

$k \geq 1$,

$$L(1-k,\chi) = \frac{-B_{k,\chi}}{k}.$$

where $$\sum_{a=1}^{N} \frac{\chi(a)\, t e^{at}}{e^{Nt}-1} = \sum_{n \geq 1} B_{n,\chi} \frac{t^n}{n!}.$$

$$L(0,\chi) = -B_{1,\chi}$$

Suppose $N = p > 2$.

Dirichlet class # formula $\Rightarrow$ (applied to $\mathbb{Q}(\mu_p)$) $\qquad \mathbb{Q}(\mu_p)^+$ totally real subfield

$$\prod_{\chi(-1)=-1} \frac{L(0,\bar{\chi})}{2} = \frac{h_p^-}{2p} \qquad h_p^- := h_p / h_p^+.$$

$$\chi : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mu_{p-1} \subseteq \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$$

$$\omega : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mu_{p-1} \subseteq \mathbb{Z}_p^\times$$ Teichmuller character

$$a \longmapsto \omega(a) \equiv a \pmod{p}.$$

is a generator of the character group.

$$\widehat{(\mathbb{Z}/p\mathbb{Z})^\times} = \langle \omega \rangle = \{1, \omega, \omega^2, \ldots, \omega^{p-2}\}.$$

Thus, we have

$$\prod_{\substack{i=1 \\ i \text{ odd}}}^{p-2} \left(-\frac{B_{1,\omega^i}}{2} \right) = \frac{h_p^-}{p}.$$

<u>Check:</u> $p B_{1,\omega^{-1}} \in \mathbb{Z}_p^\times$.

So

$$\operatorname{ord}_p \Big(\prod_{\substack{i=1 \\ \text{odd}}}^{p-4} \frac{-B_{1,\omega^i}}{2} \Big) = \operatorname{ord}_p(h_p^-).$$

C_0 = p-part of the class group of $\mathbb{Q}(\mu_p)$.

$\circlearrowleft$

$$\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q}) = \Delta \simeq (\mathbb{Z}/p\mathbb{Z})^\times$$

$$(\zeta_p \mapsto \zeta_p^a) \longleftarrow a$$

So the ω^i can be viewed as characters of Δ. Decompose

C_0 $\circlearrowleft$ $\mathbb{Z}_p[\Delta]$:

$$\varepsilon_i = \sum_{\sigma \in \Delta} \omega^i(\sigma)\, \sigma^{-1}$$

$\varepsilon_i C_0$ is the ω^i-isotypical piece of the class group.

<u>Thm: (Herbrand-Ribet):</u> $p \mid B_{1,\omega^i} \iff p \mid \#(\varepsilon_{-i} C_0)$

$$\left(B_{1,\omega^i} \equiv \frac{B_{i+1}}{i+1} \pmod p \qquad i = 1, 3, \ldots, p-4 \right)$$

Can we make the link more precise?

Yes: Main Conj. of Iwasawa Theory $\qquad \chi : (\mathbb{Z}/p\mathbb{Z})^\times \to \mu_{p-1} \subseteq \mathbb{Z}_p^\times$

$\exists$ p-adic analytic $L_p(s,\chi)$ on $\mathbb{Z}_p \setminus \{1\}$ s.t.

(Kubota-Leopoldt)

$$L_p(1-k,\chi) = (1 - \underbrace{\chi\omega^{-k}}_{\text{make this primitive}}(p)\, p^{k-1})\, L(1-k, \chi\omega^{-k})$$

If $\chi \neq \mathbb{1}$, then L_p is analytic on all of $\mathbb{Z}_p$.

Iwasawa gave a different construction. He constructed an $L_p(s,\chi)$ as an element of $\mathbb{Z}_p[\![T]\!]$.

$\exists!$ $\quad f_\chi^i \in \mathbb{Z}_p[\![T]\!] \quad$ s.t. $i = 2, 4, \ldots, p-3$

s.t.

$$L_p(k,\chi) = f_\chi^i(\gamma^k - 1) \qquad \text{where } \gamma \text{ is top. gen of } \mathbb{Z}_p.$$

$$C_n = p\text{-part of } Cl(\mathbb{Q}(\mu_{p^{n+1}}))$$

$$C_\infty = \varprojlim_{\text{norm}} C_n \circlearrowleft \Gamma \times \Delta$$

$$C_\infty = \bigoplus_{i=1}^{p-1} \varepsilon_i C_\infty$$

<u>Fact:</u> $\varepsilon_i C_\infty$ is f.g torsion $\mathbb{Z}_p[[\Gamma]]$-module.

$$\mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]] = \Lambda$$

$$\mathrm{char}_\Lambda(\varepsilon_i C_\infty) = (g_i)$$

<u>Main Conj:</u> $(g_i) = (f^i)$.

This is known by Mazur-Wiles originally, then Thaine-Kolyvagin.

<u>Analog of main conjecture</u>

- algebraicity result of values of L-functions
  $\downarrow$
  p-adic L-function

- cohomology group (Selmer group) (f.g, torsion, etc) } work of Greenberg

$$f_i : D^\times_{/\mathbb{A}} \longrightarrow V_k \quad \text{automorphic form.} \quad , i = 1, 2$$

↑ def. quat. alg.

J-L correspondence relates to classical modular form φ_i.

Consider $L(s, \varphi_1 \otimes \varphi_2)$. There are algebraicity result by Shimura.

Hida constructed a "p-adic L-function."

How to prove main conjectures?

- construct congruences using automorphic forms (on larger group) (using p-divisible of L-fctn)

Yoshida lift : $\Upsilon(g) = \langle \Theta, f_1 \otimes f_2 \rangle$ Siegel modular form.