Average number of integral points in dynamical orbits:

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- $K/\mathbb{Q}$ or $K/\mathbb{F}_q(t)$, $\mathcal{O}_{K,S}$ = ring of S-integers
- $h: \mathbb{P}^1(K) \to \mathbb{R}_{\geq 0}$ log Weil ht.
- $\phi \in K(x)$, $d = \deg(\phi) \geq 2$.
- $\phi^n = \underbrace{\phi \circ \cdots \circ \phi}_{n\text{-times}}$ called n^{th} iterate
- $b \in \mathbb{P}^1(K)$, $\mathcal{O}_\phi(b) = \{b, \phi(b), \phi^2(b), \ldots\}$ = orbit.

Goal: Understand $\mathcal{O}_\phi(b) \cap \mathcal{O}_{K,S}$.

Example: $K = \mathbb{Q}$, $\phi(x) = \dfrac{899x^2 - 2002x + 275}{33x^2 - 589x + 275}$

$$0 \xrightarrow{\phi} 1 \xrightarrow{\phi} 3 \xrightarrow{\phi} 2 \xrightarrow{\phi} 5 \xrightarrow{\phi} -7 \xrightarrow{\phi} \cdots$$

$$\mathcal{O}_\phi(0) = \{0, 1, 3, 2, 5, -7\}.$$

Question: When is $\mathcal{O}_\phi(b) \cap \mathcal{O}_{K,S}$ infinite?

(1) ϕ = polynomial

(2) $\phi(x) = \frac{1}{x^2}$, $b = 2$ $\quad \mathcal{O}_\phi(b) \cap \mathbb{Z} = \{2, 2^4, 2^{16}, \ldots\}$

Guess: $\mathcal{O}_\phi(b) \cap \mathcal{O}_{K,S}$ = infinite $\Rightarrow$ $\phi^n \in K[x]$ for some $n \geq 1$.

Thm: (char$(K) = 0$ or ϕ = sep). If $\phi^n \in K[x]$, then $\phi^2 \in K[x]$.

Pf (Sketch):

Fact: $E \subseteq \mathbb{P}^1(\mathbb{C})$ is a finite subset and $\phi^{-1}(E) = E$ then

$\# E \leq 2$ (Riemann-Hurwitz formula)

$E = \{\infty, \phi(\infty), \dots, \phi^n(\infty)\}$

Thm (Silverman): If $\phi^2 \notin K[x]$, then

(1) $\mathcal{O}_\phi(\alpha) \cap \mathcal{O}_{K,S}$ is finite for all $\alpha \in \mathbb{P}^1(K)$

(2) $K = \mathbb{Q}$, write $\phi^n(\alpha) = \dfrac{a_n}{b_n}$, $\gcd(a_n, b_n) = 1$. Then

$$\lim_{n\to\infty} \frac{\log(a_n)}{\log|b_n|} = 1.$$

Analogous to:

Thm (Siegel): $E/\mathbb{Q}$ an elliptic curve. $E: y^2 = x^3 + Ax + B$.

(1) $E(\mathbb{Z})$ is finite

(2) $[m]: E \to E$, $P \in E(\mathbb{Q}) \setminus E(\mathbb{Q})_{tors}$

$X([m]^n P) = \dfrac{a_n}{b_n}$, $\gcd(a_n, b_n) = 1$. Then

$$\lim_{n\to\infty} \frac{\log|a_n|}{\log|b_n|} = 1$$

Natural Question: How large is $\#(\mathcal{O}_\phi(b) \cap \mathcal{O}_{K,S})$?

"For most" finiteness theorems in arithmetic dynamics we expect bound that only depends on $d = \deg(\phi)$ (K, S fixed)

Prop: For all $N > 0$ and $d \geq 2$, there exist $\phi \in \mathbb{Q}[x]$ s.t.

(1) $\phi^2(x) \notin \mathbb{Q}[x]$

(2) $\mathcal{O}_\phi(0)$ = infinite set

(3) $0, \phi(0), \phi^2(0), \ldots, \phi^N(0) \in \mathbb{Z}$

Weaker question: How large do we expect $\#(\mathcal{O}_\phi(b) \cap \mathcal{O}_{K,S})$ to be if we select b "at random"?

$$\overline{\mathrm{Avg}}(\phi, S) = \limsup_{B \to \infty} \frac{\sum_{\substack{P \in \mathbb{P}^1(K) \\ h(P) \leq B}} \#(\mathcal{O}_\phi(b) \cap \mathcal{O}_{K,S})}{\#\{P \in \mathbb{P}^1(K) : h(P) \leq B\}} \overset{?}{=} 0 \text{ expected.}$$

Basic: not varying ϕ

Thm: (WH) $\overline{\mathrm{Avg}}(\phi, S) = 0$ for all S

$\left(= O(\tfrac{1}{b}). \right)$

Remark: It suffices to control $b \in \mathbb{P}^1(K)$ s.t $\mathcal{O}_\phi(b)$ = infinite.

$\mathrm{PrePer}(\phi) = \{ b \in \mathbb{P}^1(\bar{K}) : \mathcal{O}_\phi(b) \text{ is finite} \}$

$\mathrm{PrePer}(\phi, K)$ = K-rational points (torsion pts on elliptic curves)

Northcott: $\mathrm{PrePer}(\phi)$ is a set of bounded ht.

Lemma (1): $\sup \{ n : \phi^n(b) \in \mathcal{O}_{K,S}$ for some $\hat{h}_\phi(b) > 0 \}$ is finite.

$\quad\quad\quad\quad\quad\quad\; \| $

$\quad\quad\quad\quad\quad N(\phi, S)$.

Clearly Lemma (1) & Northcott $\Rightarrow$ $\overline{\mathrm{Avg}}(\phi, S) \leq N(\phi, S)$.

Facts:

(1) $F \in K(x)$ nonconstant $\underbrace{\{ b \in \mathbb{P}^1(K) : F(b) \in \mathcal{O}_{K,S} \}}$. Then if F has at least 3 distinct poles, then $\to$ is finite.

(2) $\inf \{ \hat{h}_\phi(b) \mid b \in \mathbb{P}^1(K), \hat{h}_\phi(b) > 0 \}$ is strictly positive

$\to$ $E(\mathbb{Q})$ there is a minimum canonical ht of its nontorsion pts.

Say $\phi^n(b) \in \mathcal{O}_{K,S}$ and $\hat{h}_\phi(b) > 0$.

Without loss assume $n \geq 4$.

Can show $\phi^4 \in K(x)$ with at least 3-distinct poles ($\phi^2 \notin K[x]$)

$T_4(\phi, S) = \{ \alpha \in \mathbb{P}^1(K) : \phi^4(\alpha) \in \mathcal{O}_{K,S} \}$.

Then $T_4(\phi, S)$ is finite

$\phi^4(\phi^{n-4}(b)) = \phi^n(b) \in \mathcal{O}_{K,S}$.

$\to$ so $\phi^{n-4}(b) \in T_4(\phi, S)$

so $\quad h(\phi^{n-4}(b)) \leq C(\phi)$

$d^{n-4}\, \hat{h}_\phi(b) = \hat{h}_\phi(\phi^{n-4}(b)) \leq C_4(b)$.

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$$\to \quad d^{n-4}\, \hat{h}^{min}_{\phi,K} \le C_2(\phi)$$

$$\to \quad n \le \log_d \left(\frac{C_2(\phi)}{\hat{h}^{min}_{\phi,K}} \right) + 4$$

$T_n(\phi, S) = \{ b \in \mathbb{P}^1(K) : \phi^n(b) \in \mathcal{O}_{K,S} \}$

$T_n(\phi, S) = \emptyset$ empty set for all $n > N(\phi, S)$.

$$\overline{\mathrm{Avg}}(\phi, S) \le N(\phi, S) \cdot \sum_{n=0}^{N(\phi,S)} \bar{\delta}_K (T_n(\phi, S))$$

usual density.

Weak form Siegel:

<u>Prop:</u> $f \in K(x)$ nonconstant, $\{ b \in \mathbb{P}^1(K) : f(b) \in \mathcal{O}_{K,S} \}$

has density zero. (Pf of Siegel's thm)

Vary the function ϕ.

$\mathrm{Rat}_d = \{$ rational maps $\varphi \in K(x)$ of degree $d \}$. = affine variety

$\varphi = [F, G] = [a_d X^d + \dots + a_0 Y^d, b_d X^d + \dots + b_0 Y^d]$ resultant $R(F, G) \ne 0$.

$\updownarrow$

$[a_d, \dots, a_0, b_d, \dots, b_0] \in \mathbb{P}^{2d+1}$

$\mathrm{Rat}_d \subsetneq \mathbb{P}^{2d+1}$

- X/K variety

- $h_X : X(\bar{k}) \longrightarrow \mathbb{R}_{\geq 0}$ ample ht function

- $\phi : X \longrightarrow \mathrm{Rat}_d$ rational map

- $\beta : X \longrightarrow \mathbb{P}^1$ rational map

- $P \in X(K)$ then $\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}$ $\qquad \phi_P = \phi(P)$, $\beta_P = \beta(P)$

- $X(K, B) =$ K-rational pts of ht at most B

Thm (W.H): $X =$ curve. If $\deg(\beta) > \frac{2d-1}{d-1} \cdot \deg(\phi)$. Then

(1) $$\overline{\mathrm{Avg}}(\phi, \beta, S) = \limsup_{B \to \infty} \frac{\sum_{P \in X(K, B)} \# \mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}}{\# X(K, B)} = \text{bounded.}$$

(2) $\overline{\mathrm{Avg}}(\phi, \beta, S) = 0$ assuming ABC, Vojta conjecture over number fields and unconditionally for $K/\mathbb{F}_q(t)$.

Remarks: $X = \mathbb{P}^1$, $\phi =$ constant map, $\beta = id$, Thm Z.

Fact:
(1) $X^{PrePer}_{\phi, \beta}(K) = \{ P \in X(K) : \hat{h}_{\phi_P}(\beta_P) = 0 \}$

is finite.

(2) $\hat{h}^{min}_{X, \phi, \beta} = \inf \{ \hat{h}_{\phi_P}(\beta_P) : P \in X(K), \hat{h}_{\phi_P}(\beta_P) > 0 \}$

~~Both 2 facts follow from~~

T: $f(x,y) = r$ ~~[illegible]~~ Thue equation

f. field case

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$$h(x) \le c_1 h(T) + c_2$$

$\nearrow h(Y)$ for some absolute c_1, c_2.