The Degree Five L-function for GSp(4) and Bessel Coefficients:

- Integral representation
- (Tchlos-Bighash / Prasad)
- Application of previous to construct nearly equivalent representations

Converse Thm:

π rep. of $G(\mathbb{A})$, G reductive group. If $L(s,\pi)$ has nice analytic properties, should be able to reconstruct the representation

$$\pi(g) f(x) = f(xg).$$

$F = \#$ field

$$J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$$

$$G = GSp_4 = \{ g \in GL_4 : {}^t g J g = \lambda(g) J \}$$

$$P = \left\{ \left(\begin{array}{c|c} g & * \\ \hline & {}^t g^{-1} \cdot \lambda \end{array} \right) : g \in GL_2, \lambda \in GL_1 \right\}$$

$$U = \left\{ \underbrace{\left(\begin{array}{c|c} 1 & x \\ \hline & 1 \end{array} \right)}_{= n(x)} : {}^t x = x \right\}$$

$T \in M_2(F)$, ${}^tT = T$

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$\psi : F\backslash \mathbb{A} \longrightarrow \mathbb{C}$ additive character

$\psi_T : N(F)\backslash N(\mathbb{A}) \longrightarrow \mathbb{C}$

$$\psi_T(n(x)) = \psi(Tr(Tx)).$$

M normalizes N

$H_T =$ Connected component of the norm. of ψ_T

$$\psi_T(mnm^{-1}) = \psi_T(n)$$

$R = H_T N$

Remark:

$$H_T \cong \begin{cases} F \oplus F & T \text{ isotropic} \\ K \quad [K:F] = 2 & T \text{ anisotropic} \end{cases}$$

Let π be irreducible cuspidal automorphic representation, space V_π, and $\varphi \in V_\pi$.

$$Z_{\mathbb{A}} \subseteq G(\mathbb{A})$$

$$\pi(z)\varphi(g) = \omega_\pi(z)\varphi(g) \qquad \text{central character}$$

$$\nu : H_T(F)\backslash H_T(\mathbb{A}) \longrightarrow \mathbb{C}^\times$$

$$\nu|_Z = \omega_\pi$$

$$\nu \otimes \psi_T(hn) = \nu(h)\psi_T(n)$$

The Bessel coefficient of φ is defined as

$$\int_{Z(\mathbb{A})R(F)\backslash R(\mathbb{A})} \varphi(rg)(\nu\otimes\psi_T)^{-1}(r)\,dr = \varphi^{T,\nu}(g).$$

Weil representation:

$Sp_4 \times SO_T$

$\omega(g,h)\phi(x)$

$\phi \in C_c^\infty(M_2(\mathbb{A}))$

$$\Theta_\phi(\nu^{-1})(g) = \int_{SO_T(F)\backslash SO_T(\mathbb{A})} \nu^{-1}(h)\,\omega(g,h_g h_1)\phi(x)\,dh_1$$

$$f(s,-) \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\delta_P^s)$$

$$E(s,g) = \sum_{\gamma\in P(F)\backslash G(F)} f(s,\gamma g).$$

$$I(s) = \int_{Z_{\mathbb{A}}G^+(F)\backslash G^+(\mathbb{A})} E(s,g)\,\varphi(g)\,\Theta_\phi(\nu^{-1})(g)\,dg$$

$$G^+ = \{ g \in G \mid \exists h \in H_T \quad \lambda_G(g) = \lambda_{H_T}(h) \}$$

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Prop: $$I(s) = \int_{N(\mathbb{A})\backslash Sp_4(\mathbb{A})} f(s,g)\, \varphi^{T,\nu}(g)\, \omega(g,1)\, \phi(1)\, dg.$$

$$= \prod_v \int_{N(F_v)\backslash Sp_4(F_v)} f_v(s,g_v)\, \varphi_v^{T,\nu}(g)\, \omega_v(g,1)\, \phi_v(1)\, dg_v$$

$$\varphi = \otimes \varphi_v$$

$$f = \otimes f_v$$

$$\phi = \otimes \phi_v.$$

$$C = \mathbb{A}^{\times,2} F^{\times,2} \backslash \mathbb{A}^{\times,+} \cong Z_G(\mathbb{A})\, G_1(\mathbb{A})\, G^+(F) \backslash G^+(\mathbb{A})$$

$$C \times SO_T(\mathbb{A}) \cong Z_{\mathbb{A}} H_T(F) \backslash H_T(\mathbb{A})$$

used in unfoldings.

For almost all finite places,

$$\pi_v \cong \mathrm{Ind}_{B_v}^{G_v}(\gamma_v)$$

Prop v's from notation

$$\gamma_1 = \gamma\begin{pmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \gamma_2 = \gamma\begin{pmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{pmatrix}$$

$T \rightsquigarrow \chi$ quadratic character associated to quadratic space

$$\gamma_3 = \gamma\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{pmatrix} \qquad \gamma_4 = \gamma\begin{pmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{pmatrix}$$

$$\chi = \otimes \chi_{T,v}$$

Use a formula of Sugano to compute:

After normalization

$$I_v(s) = (1-\chi(\varpi)q^{-s})^{-1}\cancel{(1-q^{s-1})}\,(1-\chi(\varpi)\omega_\pi^{-1}(\varpi)\gamma_2 q^{-s})$$
$$(1-\chi(\varpi)\omega_\pi^{-1}(\varpi)\gamma_1\gamma_4 q^{-s})^{-1}(1-\chi(\varpi)\omega_\pi^{-1}(\varpi)\gamma_2\gamma_3 q^{-s})^{-1}$$
$$(1-\chi(\varpi)\omega_\pi^{-1}(\varpi)\gamma_3\gamma_4 q^{-s})^{-1}$$

$$= L(s, \pi_v \otimes \chi_{T,v}).$$

$$GO(2,2) \cong \overset{\pi_1 \boxtimes \pi_2}{GL_2 \times GL_2}/\Delta Z \qquad \Delta Z = (z, z^{-1})$$

$$H_T \subseteq GL_2$$

$\varphi_i \in V_{\pi_i}$ Find such forms so the following integral doesn't vanish

$$\int_{Z_{\mathbb{A}} H_T(F)\backslash H_T(\mathbb{A})} \varphi_i(hg)\,\nu^{-1}(h)\,dh \neq 0.$$

$$\begin{tikzcd} GSO(2,2) \arrow[dr, "\Theta"] & \\ & GSp_4 \\ GSO(4) \arrow[uu, "JL"] \arrow[ur, "\Theta"'] & \end{tikzcd}$$

D/F $\quad D_v = D \otimes F_v \simeq M_2(F_v)$

all but finitely many places

$D_v^\times \cong GL_2(F_v)$

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F totally real number field

S = set of places, $2 \mid \# S$.

D = quaternion alg

$T \subseteq D$

Prasad - Schulze-Pillot (2008): Can construct a rep on $D^\times(\mathbb{A})$ that has the following properties

$$\int_{Z(\mathbb{A})H_T(F)\backslash H_T(\mathbb{A})} \nu^{-1}(h)\varphi(hg) \neq 0 \qquad \nu\text{ - distinguished.}$$

π_1, π_2 reps. on $D^\times(\mathbb{A})$.

$\pi_i \longrightarrow \pi_i^{JL}$ rep. of $GL_2(\mathbb{A})$

$v \notin S$, $\qquad \pi_{i,v} \cong \pi_{i,v}^{JL}$

(Tunnell) $\pi_{i,v}$ is locally ν-distinguished, then $\pi_{i,v}^{JL}$ is not.

If $\pi_{i,v}$ is not, then $\pi_{i,v}^{JL}$ is ν-distinguished.

So π_i^{JL} are not ν-distinguished globally.

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Theorem: Given a finite set of places S of even cardinality of a totally real # field F, there exist two irred cuspidal auto. reps. σ_1, σ_2 s.t.

- $\sigma_{1,v} \simeq \sigma_{2,v} \quad \forall v \notin S$
- $\sigma_{1,v} \not\simeq \sigma_{2,v} \quad \forall v \in S$.
- σ_1 has nonzero TSV Bessel coefficient, σ_2 does not.