p-adic families of Eisenstein series for unitary groups:

- Goal for talk: Discuss recent work in which p-adically interpolate special values of certain Eisenstein series (on unitary groups of signature (n,n))

(Conjectural)

Applications	Inspiration
① # Thy: Construct p-adic L-fctns for unitary groups (in preparation E.-Harris-Li-Skinner)	* Katz's p-adic L-fctns for CM fields
② Homotopy theory: Generalize an invariant ("Witten genus") joint ongoing project of Behrens-Hopkins-Naumann	* Katz again & Ando-Hopkins-Rezk..

The idea for constructing a p-adic family:

① Choose a "nice" family of E. series.
② Compute Fourier coeffs (where holo)
③ Apply certain weight-raising C^∞- (and p-adic) differential operators to obtain C^∞- (and p-adic) auto. forms.
$(D_\infty, D_{p\text{-adic}})$

$$\left((*)\, D_\infty E\left(\begin{smallmatrix}\text{ord}\\ \text{CM}\\ \text{pt}\end{smallmatrix}\ A/\mathcal{O}_K \right) = (*)'\, D_{p\text{-adic}}\, E(A) \right.$$

int. ring of # field K.

Eischen
4-24-13
pg 2

④ Use q-expansion principal & p-adic interpolation of F. coeffs (q. exp coeffs) to construct a p-adic family. (enough to know coeffs interpolate p-adically).

Setup:

$$\begin{array}{ll} K & \text{CM field} \supseteq \mathcal{O}_K \text{ integer ring} \\ 2| & \\ E & \text{tot. real} \end{array}$$

Fix rational prime p s.t. every prime in E dividing p splits completely in K.

Fix a choice of CM type ~~(primes over the primes over p for~~

Fix embeddings

$$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$$
$$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$$

Fix $W = 2n$ dimensional K-v.s. w/ Hermitian pairing $\langle , \rangle$ of signature (n,n).

$G = U(W)$.

$P =$ Siegel parabolic in G

$f \in \mathrm{Ind}_{P(\mathbb{A}_E)}^{G(\mathbb{A}_E)} (\chi |\cdot|_K^{-s})$ where $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ Hecke character. For simplicity assume $\mathrm{cond}(\chi) \mid p^\infty$.

Set

$$E_f(g) = \sum_{\gamma \in P(E)\backslash G(E)} f(\gamma g).$$

Prop.: $(E', 13)$ Let R be an $\mathcal{O}_K$-alg, $k \in \mathbb{Z}_{\geq n}$, $\nu(\sigma) \in \mathbb{Z}$. Let

$$F : (\mathcal{O}_K \otimes \mathbb{Z}_p) \times M_n(\mathcal{O}_E \otimes \mathbb{Z}_p) \longrightarrow R$$

be locally constant fctn supported on

$$(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times GL_n(\mathcal{O}_E \otimes \mathbb{Z}_p).$$

satisfying

$$F(ex, \mathbb{N}_{K/E}(e)^{-1} y) = \mathbb{N}_{k,\nu}(e) F(x,y)$$

where $\left(\mathbb{N}_{k,\nu} = \prod_{\sigma \in \Sigma} \sigma^{k+2\nu(\sigma)} (\sigma\bar{\sigma})^{-(k/2 + \nu(\sigma))} \right)$

Σ choice of CM type $\leftrightarrow$ choice of embedding at each arch. place.

$\forall\ e \in \mathcal{O}_K^\times,\ x \in (\mathcal{O}_K \otimes \mathbb{Z}_p),\ y \in M_n(\mathcal{O}_E \otimes \mathbb{Z}_p)$.

Then $\exists$ auto. form $G_{k,\nu,F}$ on $U(n,n)$, $\nu = (\nu(\sigma))_{\sigma \in \Sigma}$

of wt (k,ν) defined over R whose q-expansion at a cusp $m \in M_{K(\text{Levi} \subseteq G)}$ is of the form

$$\sum_{0 < \beta \in L_m} c(\beta) q^\beta$$

with $c(\beta)$ a (finite) $\mathbb{Z}$-linear combination of terms of the form

$$F(a, \mathbb{N}_{K/E}(a)^{-1}\beta)\, \mathbb{N}_{k,\nu}(a^{-1}\det\beta)\, \mathbb{N}_{E/\mathbb{Q}}(\det\beta)^{-n}$$

[When $R = \mathbb{C}$, this is the f.c. @ $s = k/2$ of a C^∞-auto form $G_{k,\nu,F}(z,s)$ of wt (k,ν) hol. at $s = k/2$.]

Thm (E, '13): There's a p-adic measure μ on

$$\mathcal{G} = \left((\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times GL_n(\mathcal{O}_E \otimes \mathbb{Z}_p) \right) / \overline{\mathcal{O}_K^\times}$$

($\mathcal{O}_K^\times$ embedded as itself in first term and as norm in second term)

with values in the space of p-adic auto. forms on $U(n,n)$ defined by

$$\int_{\mathcal{G}} H \, d\mu = G_{n,0,F}$$

for all $\hat{p}$-adically continuous functions H (on $\mathcal{G}$) with

$$F(x,y) = \frac{1}{\prod_{\sigma \in \Sigma} \sigma\left(x^{-1} N_{K/E}(x)^n \det y\right)^n} H(x, y^{-1}).$$

(As p-adic auto forms

$$G_{k,\nu,F} = G_{n,0}\, N_{k-n,\nu}\left(x^{-1} N_{K/E}(x)^n \det y\right) F(x,y).$$
)

<u>Remark:</u> For locally constant H,

CM point over Ch-alg $\quad R \hookrightarrow \mathbb{C}$, $R \hookrightarrow \mathbb{C}_p$

$$(*)_{k,\nu,d} \quad \int_{\mathcal{G}} H_k(x,y) \det\left(N_{K/E}(x)^{-1} y\right)^{-d} d\mu(\underline{A})$$

$$= (*')_{k,\nu} \quad G_{k+2d,\,-2d,\,F(x,y)}(z, k/2)$$

where z corresponds to CM pt. $\underline{A}$.