Rationality of Canonical Height:

joint with Dragos Ghioca

Thm (Tate, Manin, Lang, Néron 1960s): Let E be an elliptic curve defined over $k = \mathbb{C}(X)$, X = curve, then the Néron-Tate canonical height $\hat{h}_E(P) \in \mathbb{Q}$ $\forall P \in E(\bar{k})$, and the local heights $\hat{\lambda}_E(P) \in \mathbb{Q}$ $\forall P \in E(\mathbb{C}_v) \setminus \{0\}$ $\forall$ place v of k.

Goals: (1) Explain connection to dynamical systems and give an elementary dynamical proof.

(2) More generally, for canonical heights for

$$f: \mathbb{P}^1 \to \mathbb{P}^1 \ / k = \mathbb{C}(X)$$

the local rationality fails.

(3) Application: Dynamical Mordell-Lang Conjecture.

Example: $E : \{y^2 = x(x-1)(x-t)\}$ $\quad k = \mathbb{C}(t)$

Construction of $\hat{h}_E : E(\bar{k}) \to \mathbb{R}_{\geq 0}$.

Tate:

$$\begin{tikzcd} E \arrow[r, "\times 2"] \arrow[d, "\pi"', "P\sim -P"'] & E \arrow[d, "P\sim -P"] \\ \mathbb{P}^1 \arrow[r, "f"] & \mathbb{P}^1 \end{tikzcd}$$

$\pi(x,y) = x$

$$f(x) = \frac{(x^2-t)^2}{4x(x-1)(x-t)}.$$

$$\hat{h}_E(P) = \lim_{n\to\infty} \frac{h(\overbrace{f^n(\pi P)}^{\pi(2^n P)})}{4^n}$$

h = logarithmic height

$$h(P) = \sum_{v \in M_k} \max(\|x_0\|_v, \dots, \|x_n\|_v)$$

$P = [x_0 : \dots : x_n]$.

$$\hat{h}_E(P) = \sum_{v \in M_k} \hat{\lambda}_{E,v}(P) \qquad P \in E(k)$$

$\hat{\lambda}_{E,v} : E(\mathbb{C}_v) \setminus \{0\} \to \mathbb{R}$

continuous

Dynamical canonical height : Call - Silverman (1994)

$f : \mathbb{P}^1 \to \mathbb{P}^1 \qquad \deg f = d \geq 2.$

$\hat{h}_f : \mathbb{P}^1(\bar{k}) \to \mathbb{R}_{\geq 0}$

$$\hat{h}_f(a) = \lim_{n\to\infty} \frac{1}{d^n} h(f^n(a)) = \sum_{v \in M_k} \hat{\lambda}_{f,v}(a)$$

(1) $\hat{h}_f(f(a)) = d\, \hat{h}_f(a)$

(2) $\exists\, c = c(f)$ s.t.

$$|\hat{h}_f(a) - h(a)| \leq c \quad \forall a \in \mathbb{P}^1(\bar{k})$$

<u>To compute these height</u>

$k = \mathbb{C}(X) \qquad d = \deg f$

$f : \mathbb{P}^1 \to \mathbb{P}^1$

$v \in M_k \leftrightarrow v = \mathrm{ord}_x, \ x \in X(\mathbb{C})$

$v(k^*) = \mathbb{Z}.$

Write

$$f(z:w) = (P(z,w) : Q(z,w)).$$

$\hat{\lambda}_{f,v}(a) = ? \qquad a \in \mathbb{P}^1(\mathbb{C}_v)$

Write $a = (x:y)$, $\min\{v(x), v(y)\} = 0$, $\mathcal{O}_f(a) = \{a, f(a), f^2(a), \dots\} \subseteq \mathbb{P}^1(\mathbb{C}_v)$.

Define $\sigma(f,a) = \min\{v(P(a)), v(Q(a))\}$

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Prop: (Call-Silverman) $\hat{\lambda}_{f,v}(a) = \sum_{n=0}^{\infty} \frac{\sigma(f, f^n(a))}{d^n} - \min\{0, v(a)\}$.

$0 \leq \sigma(f, \cdot) \leq v(\mathrm{Res}(P,Q))$.

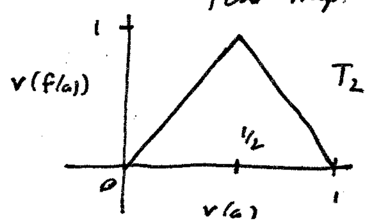
Upshot: $\hat{\lambda}_{f,v}(a)$ is in $\mathbb{Q} \cup \{\infty\}$ iff $\{\sigma(f, f^n(a))\}$ is eventually periodic.

Picture:

$$\begin{tikzcd} E(\mathbb{C}_v) \arrow[r, "\times 2"] \arrow[d, "\pi"'] & E(\mathbb{C}_v) \arrow[d, "\pi"] \\ \mathbb{P}^1(\mathbb{C}_v) \arrow[r, "f"] \arrow[d, "v"'] & \mathbb{P}^1(\mathbb{C}_v) \arrow[d, "v"] \\ {[0,1]} \arrow[r, "T_2"'] & {[0,1]} \end{tikzcd}$$

mult. reduction

Tent map.



Easy: Rational points in $[0,1]$
$\Updownarrow$
points with finite orbits

As $\{\sigma(f, f^n(a))\}$ is eventually periodic.

Known $f: \mathbb{P}^1 \to \mathbb{P}^1$ is a polynomial $\Rightarrow$ local (and global) rationality of heights.

Theorem 1: In every degree $d \geq 2$, $\exists\, f: \mathbb{P}^1 \to \mathbb{P}^1 / k$, $v \in M_k$
$a \in \mathbb{P}^1(\mathbb{C}_v)$ s.t. $\hat{\lambda}_{f,v}(a) \in \mathbb{R} \setminus \mathbb{Q}$.

Theorem 2: deg $d=2$

Given f, $v \in M_K$, exactly one of the following holds:

(1) f has potential good reduction

(2) f is strongly polynomial-like

(3) $\exists\, a \in \mathbb{P}^1(\mathbb{C}_v)$ with $\hat{\lambda}_{f,v}(a) \notin \mathbb{Q}$.