The BSD Conjecture and Generalized Kato Classes

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10-13-15
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§1 Background:

E elliptic curve $/\mathbb{Q}$.

$r(E) = \text{rank}(E(\mathbb{Q}))$, $r_{an}(E) = \text{ord}_{s=1} L(E,s)$.

BSD: $r(E) = r_{an}(E)$; $r_{an}(E) = r(E)$.

Theorem: (Gross-Zagier, Kolyvagin) If $r_{an}(E) \leq 1$, then $r(E) = r_{an}(E)$.

Moreover, $\# Ш(E) < \infty$.

Theorem: (Skinner-Urban, Venerucci, Zhang) If $r(E) \leq 1$ and $\# Ш(E) < \infty$,

then $r_{an}(E) = r(E)$.

There seem to be fundamental issues with using these args to get higher rank, so if we are stuck in rk 0, 1, then we want as fine of information as possible about the L-functions.

$$\rho: \text{Gal}(K/\mathbb{Q}) \to GL_n(\mathbb{C})$$

$$r(E,\rho) = \text{Hom}_{G_\mathbb{Q}}(\rho, E(K) \otimes \mathbb{C}).$$

$$r_{an}(E,\rho) = \text{ord}_{s=1} L(E,\rho,s).$$

BSD(E,ρ): $r(E,\rho) = r_{an}(E,\rho)$.

Let K be imag. quad., $\rho = \text{Ind}_{G_K}^{G_\mathbb{Q}} \psi$ where ψ is a ring class character : $\psi^c = \psi^{-1}$. $\rho = V_\psi$.

(GZK+) (1989)

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Theorem: (Gross-Zagier, Kolyvagin, Bertolini-D., Longo, ..., Nekovar): If

$r_{an}(E, V_\psi) \le 1$ then $r(E, V_\psi) = r_{an}(E, V_\psi)$.

Remark:

① $GKZ_+ \Rightarrow GZK$: Take ψ quadratic. Then $V_\psi = \chi_1 \oplus \chi_2$.

② Given E, we expect

$$r(E, V_\psi) = \begin{cases} 0 & 50\% \text{ of the time} \\ 1 & \text{"} \end{cases}$$

Theorem (Kato): If χ is a Dirichlet character, then if $r_{an}(E,\chi) = 0$

then $r(E,\chi) = 0$.

($\chi : Gal(\mathbb{Q}(\mu_n)/\mathbb{Q}) \to \mathbb{C}^\times$.
Get r by looking at Mordell-Weil group over cyclotomic field)

Remark:

① There is no rank 1 analogue.

② We expect $r(E,\chi) = 0$ for 100% of non-quadratic characters.

Ingredients:

① GKZ, GKZ_+: Heegner points

② S-U, V, Z.: GKZ + S-U MC.

③ Kato: Kato classes. Beilinson-Kato elements $K_2(X_0(N))$ + p-adic families.

2 Goals:

① Compare and synthesize these approaches.

② Prove similar results for more Artin representations.

Natural classes of ρ:

① Two dimensional irreducible ρ.

② V_ψ : ψ arbitrary char. of G_K, K quadratic
(not necessarily imag. quad.)

③ V_ψ : ψ is a ring class character.

Theorem (Bertolini-D.-Rotger): If ρ is an odd two-dimensional irred. Artin rep., $(\text{cond}(\rho), \text{cond}(E)) = 1$ Then $L(E,\rho,1) \neq 0$

$\Rightarrow E(H)^\rho = 0$

[e.g. $\rho = \mathrm{Ind}_K^{\mathbb{Q}} \psi$, - K quad imag.
- K real quad, ψ mixed sig.]

Theorem (D.-Rotger): If ψ is a ring class char. of K,

$$r_{an}(E, V_\psi) = 0 \Rightarrow r(E, V_\psi) = 0.$$

Remarks:

① There is no overlap between B-D-R and D-R.

② For ray class characters, we expect $r(E, V_\psi) = 0$ for 100% of ψ.

For ring class characters of K real quad., we expect

$$r(E, V_\psi) = \begin{cases} 0 & 50\% \text{ of the time} \\ 1 & 50\% \text{ of the time} \end{cases}$$

§2 Stark-Heegner Points:

K = real quadratic

E : $N = pM$, $p \nmid M$.

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<u>Generalized Heegner Hypothesis</u>: • p inert in K

• $\forall\, \ell \mid M$, ℓ split in K

$$\Rightarrow \quad L(E, V_{\psi}, 1) = 0 \quad \forall\, \psi, \quad (\mathrm{cond}(\psi), N) = 1.$$

$\mathcal{H}_p = \mathbb{P}_1(\mathbb{C}_p) \setminus \mathbb{P}_1(\mathbb{Q}_p)$ $\quad$ p-adic upper half-plane.

$$\Gamma = \Gamma_p(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}[\tfrac{1}{p}]) : M \mid c \right\} \circlearrowleft \mathcal{H}_p \times \mathcal{H}.$$

Given $z \in K \cap \mathcal{H}_p$. $\quad \tilde{\gamma}_z = \{z\} \times \mathrm{path}(z, \overset{\text{alg. conj.}}{z'}) \subseteq \mathcal{H}_p \times \mathcal{H}$

γ_z = image in $\Gamma \backslash \mathcal{H}_p \times \mathcal{H}$ = real 1-dim cycle on $\Gamma \backslash \mathcal{H}_p \times \mathcal{H}$.

<u>Differentials</u>: $E \rightsquigarrow f \in S_2(\Gamma_0(N)) \rightarrow f(z)\,dz \in \Omega^1(X_0(N))$

"Mock" Hilbert modular form on $\Gamma \backslash \mathcal{H}_p \times \mathcal{H}$ $\quad \omega_f \in$ "$\Omega^2(\Gamma \backslash \mathcal{H}_p \times \mathcal{H})$"

"<u>Definition</u>": $$P_z = \int_{\partial^{-1}(\gamma_z)} \omega_f \in \mathbb{C}_p^\times / q^{\mathbb{Z}} = E(\mathbb{C}_p).$$

<u>Conjecture</u>: The $[P_z]_{z \in \mathcal{H}_p \cap K}$ are defined over ring class fields of K and behaves just like Heegner points.

Let $\psi : G_K \to \mathbb{C}_p^\times$. $\quad \psi : \mathrm{Gal}(H_D/K) \to \mathbb{C}_p^\times$.

$$P_\psi = \sum_{\substack{z \\ \mathrm{disc}(z) = D}} \psi^{-1}(z) P_z \overset{?}{\in} (E(H_D) \otimes \mathbb{C}_p)^{\psi}.$$

Part II: Generalized Kato Classes:

p-adic Gross-Zagier for P_ψ:

Main ingredient $\mathcal{L}_p^{1/2}(\underline{f}, V_\psi)$ $\qquad U_p f = a_p f$, $a_p = \pm 1$ b/c p divides the level

$\underline{f}$ = Hida family specializing to f at wt 2, $\underline{f} \in \Lambda[[q]]$

$\mathcal{L}_p^{1/2}(\underline{f}, V_\psi)_k \sim L(\underline{f}_k, V_\psi, k/2) / * \langle \underline{f}_k, \underline{f}_k \rangle$ (interpolate only central critical values)

$k \in \mathbb{Z}_{\geq 2}$

Thm (Bertolini-D.): Let $P_\psi^+ = P_\psi + a_p(E) P_{\psi^{-1}}$. Then

$$\log_p(P_\psi^+) = \frac{d}{dk}\left(\mathcal{L}_p^{1/2}(\underline{f}_k, V_\psi)\right)_{k=2}.$$

(It is crucial that $p \| N$.)

(Could take this as definition of P_ψ if one isn't happy w/ earlier def.)

Generalized Kato Classes:

Basic Data: $f \in S_2(\Gamma_0(N))$

$g \in M_1(N, \chi)$, $h \in M_1(N, \bar{\chi})$

(Not assuming g and h are new)

level assumption: If N_f, N_g, N_h are the levels of the newforms, then $\gcd(N_f, N_g, N_h) = 1$.

$\underline{f}, \underline{g}, \underline{h}$ = Hida families specializing to f, g, h in wts (2,1,1)

Set f_k, g_l, h_m = wt k, l, m specialization of $\underline{f}, \underline{g}, \underline{h}$.

$L(f_k \otimes g_\ell \otimes h_m, c)$ $\qquad c = \frac{k+\ell+m-2}{2}$

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(properties of this due to Garrett)

$$\mathrm{sign}(L(f_k \otimes g_\ell \otimes h_m, c)) = \begin{cases} 1 & \text{if } k \geq \ell+m \text{ or } \ell \geq k+m \text{ or } m \geq k+\ell. \\ -1 & (k,\ell,m) \text{ is balanced} \end{cases}$$

Gross-Kudla-Schoen cycle: Fix (k,ℓ,m) balanced

$$\Delta_{k,\ell,m} \in CH^c(\underbrace{\mathcal{E}^{k-2}(N) \times \mathcal{E}^{\ell-2}(N) \times \mathcal{E}^{m-2}(N)}_{W})$$

Δ is a copy of $\mathcal{E}^c(N)$.

$$AJ_{et} : CH^c(W)_0 \longrightarrow H^1(\mathbb{Q}, \underbrace{H^{2c-1}_{et}(\bar{W}, \mathbb{Q}_p)(c)}_{\substack{\shortparallel \\ \mathrm{Ext}^1_{G_\mathbb{Q}}(\mathbb{Q}_p, \;)}}) .$$

$$\longrightarrow H^1(\mathbb{Q}, V_{f_k, g_\ell, h_m})$$

Def: The generalized Kato class associated to (k,ℓ,m) is a $\kappa(f_k, g_\ell, h_m) \in H^1(\mathbb{Q}, V_{f_k, g_\ell, h_m})$ obtained by

1. Taking $AJ_{et}(\Delta_{k,\ell,m}) \in H^1(\mathbb{Q}, V_{f_k, g_\ell, h_m})$.
2. if (k,ℓ,m) is unbalanced, we extend by p-adic continuity. (Need this since $(2,1,1)$ is not balanced.)

p-adic Gross-Kudla / YZZ formula:

For all (k,ℓ,m) with $k \geq \ell+m$ the class $\kappa(f_k, g_\ell, h_m)$

is non-crystalline at p iff $L(f_k \otimes g_\ell \otimes h_m, c) \neq 0$

More precisely, $\exp^*: H^1(\mathbb{Q}, V_{f_k, g_\ell, h_m}) \to \dfrac{H^1(\mathbb{Q}_p, V_{f_k, g_\ell, h_m})}{H^1_{cris}(\mathbb{Q}_p, V_{f_k, g_\ell, h_m})}$.

$$\exp^*(\kappa(f_k, g_\ell, h_m)) \longleftrightarrow L(f_k \otimes g_\ell \otimes h_m, c).$$

<u>Theorem (Rotger, D.) in progress:</u> If the Stark-Heegner point P_ψ^+ is non-zero then there is a non-trivial global class $\kappa(E,\psi) \in H^1_{Sel}(\mathbb{Q}, V_p(E) \otimes V_\psi)$.

<u>Sketch of proof:</u> ① Given ψ, $\exists$ two ray class characters η_1, η_2 of mixed signature s.t.

$$V_{\eta_1} \otimes V_{\eta_2} = V_\psi \oplus V_{\psi'} \qquad (\psi' \text{ some other ring class char})$$

$$\updownarrow \qquad \updownarrow$$

$$g \qquad h \quad \Theta\text{-series of wt } 1.$$

②

$$(\log P_\psi^+) \times (\log P_{\psi'}^+) = \frac{d}{dk} \mathcal{L}_p^{1/2}(\underline{f}, V_\psi)_{k=2} \times \frac{d}{dk} \mathcal{L}_p^{1/2}(\underline{f}\, V_{\psi'})_{k=2}$$

$$= \frac{d^2}{dk^2}(\mathcal{L}_p(\underline{f}, g, h)^{?}_{k=2}) \qquad (\text{Artin formalism})$$

$$= \frac{d^2}{dk^2}(\exp^*(\kappa(f_k, g_\ell, h_m)))_{k=2} \qquad (p\text{-adic GK-YZZ})$$

$$= \log(\kappa(f, g, h)) \leftarrow \text{Perrin-Riou, Venerucci.}$$

Venerucci: Kato classes $\longleftrightarrow$ Heegner pts
generalized Kato classes $\longleftrightarrow$ Stark Heegner pts.