

On Vanishing of torsion in the cohomology of Shimura varieties

Caraiami

Joint NYC

10-22-15

191

Joint work w/ P. Scholze.

§ 1 Motivation

G/\mathbb{Q} connected reductive group.

$X = G(\mathbb{R})/K_\infty$ $K_\infty = \text{maximal compact}$ $X = \text{symmetric domain}$

Eg: 1) $G = SL_2/\mathbb{Q}$ $X = \mathfrak{h}$.

2) $\text{Res}_{E/\mathbb{Q}} SL_2$ $X = \mathfrak{h}^3$ hyperbolic 3-space, no complex structure

$E = \text{imag. quad. field.}$

$\Gamma \subset G(\mathbb{Q})$ congruence subgroup.

$X_\Gamma = \Gamma \backslash X$ locally symmetric space attached to G .

We will assume X_Γ is compact.

$H^*(X_\Gamma, \mathbb{C})$ can be computed in terms of automorphic representations of G . (Betti cohomology here)

More precisely, in terms of $H^*(\mathcal{O}_f, \kappa, \pi_\infty)$ Lie algebra cohomology

where $\pi = \pi_\infty \otimes \pi_f$ automorphic representation for G . This is known as

Matsushima's formula.

Borel-Wallach: When π_∞ is tempered, π contributes to $H^i(X_\Gamma, \mathbb{C})$

only if $i \in [g_0, g_0 + l_0]$, $l_0 = \text{rk } G - \text{rk } K_\infty$, $g_0 = \frac{1}{2}(\dim X - l_0)$. When X_Γ is a Shim. var, $l_0 = 0$ so tempered reps only contribute to the middle degree.

Remark: Self dual cusp form for G/\mathbb{Q} contribute to the cohomology of Shimura varieties in middle degree is related to Ramanujan - Pet. Conj. (at finite places)

§2 Main Theorem

Caraiami

10-22-15

Pg 2

X_F compact unitary Shimura variety: (defined over a number field E)

G/F^+ , F^+ totally real field, $F = F^+K$ where K is imag. quad.

G unitary group.

Theorem $\Pi =$ spherical Hecke algebra. $\Pi \subset H^*(X_F, \mathbb{F}_\ell)$.

Let $\mathfrak{m} \in \Pi$ be a maximal ideal. Assume $H^*(X_F, \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0$. Then

1) $\exists \rho_{\mathfrak{m}}: \text{Gal}(\bar{F}/F) \rightarrow \text{GL}_n(\bar{\mathbb{F}}_\ell)$ which matches \mathfrak{m} at unramified places. (eigenvalues of Frob_p match Satake parameters of \mathfrak{m})

2) iff $\exists p$ split completely in F and such that $\rho_{\mathfrak{m}}$ at every prime above p is decomposed generic then $H^i(X_F, \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0$ only if $i = \dim X/2$.

Remarks: 1) Part 1 is not new, but we give a different proof.

(already follows from Scholze) This different proof

uses a p -adic rather than an l -adic tower.

2) $\rho_{\mathfrak{m}}|_{G_v}$ is decomposed generic if it is unramified
 $\{\lambda_1, \dots, \lambda_n\}$ Frobenius eigenvalues, $\lambda_i/\lambda_j \notin \{1, q_v\}$

where $q_v =$ residue char. of v .

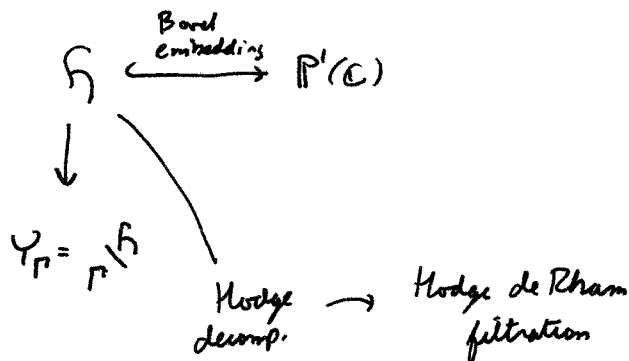
This follows from a big image assumption. This does give some reducible cases as well. (e.g. some generic sum of characters)

3) Previous results:

• Lan-Suh (restrictions on level at l and on weight)

- Harris-Taylor type $U(n-1, 1) \times U(n, 0)^{d-1}$
- Emerton-Gee $U(2, 1)$
- Shin w/ condition "supercuspidal at p"
- Boya very similar to our result

§ 3 Main Tool: Geometry of Hodge-Tate period morphism



$$\mathbb{C}^2 \simeq H^1(E, \mathbb{C}) \simeq H^1(E, \mathcal{O}_E) \oplus H^0(E, \Omega_E^1)$$

$$H^1_{\text{ét}}(E, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq \mathbb{C}_p^2$$

|S

$$H^1(E, \mathcal{O}_E) \oplus H^0(E, \Omega_E^1)(-1)$$

Hodge-Tate filtration

$$Y_p = (Y_p \otimes_{\mathbb{Q}} \mathbb{C}_p)^{\text{ad}} \quad \text{an "adiv" space.}$$

$$\downarrow$$

$$Y_{\Gamma(p)} \quad (E, E_{\Gamma(p)} = (\mathbb{Z}/p\mathbb{Z})^2)$$

$$|$$

$$Y_{\Gamma} \quad (E, P \text{ order } M)$$

