On Vanishing of torsion in the cohomology of Shimura varieties

Caraiani
Joint NYC
10-22-15
pg1

joint work w/ P. Scholze

§1 Motivation

$G/\mathbb{Q}$ connected reductive group.

$X = G(\mathbb{R})/K_\infty$ K_∞ = maximal compact X = symmetric domain

Eg: 1) $G = SL_{2/\mathbb{Q}}$ $X = \mathfrak{h}$.

2) $\mathrm{Res}_{E/\mathbb{Q}} SL_2$ $X = \mathfrak{h}^3$ hyperbolic 3-space, no complex structure

E = imag. quad. field.

$\Gamma \subset G(\mathbb{Q})$ congruence subgroup.

$X_\Gamma = \Gamma \backslash X$ locally symmetric space attached to G.

We will assume X_Γ is compact.

$H^*(X_\Gamma, \mathbb{C})$ can be computed in terms ~~of~~ of automorphic representations of G. (Betti cohomology here)

More precisely, in terms of $H^*(\mathfrak{g}, K, \pi_\infty)$ Lie algebra cohomology where $\pi = \pi_\infty \otimes \pi_f$ automorphic representation for G. This is known as Matsushima's formula.

Borel-Wallach: When π_∞ is tempered, π contributes to $H^i(X_\Gamma, \mathbb{C})$ only if $i \in [q_0, q_0 + l_0]$, $l_0 = \mathrm{rk}\, G - \mathrm{rk}\, K_\infty$, $q_0 = \frac{1}{2}(\dim X - l_0)$. When X_Γ is a Shim. Var, $l_0 = 0$ as tempered reps only contribute to the middle degree.

Remarks: self dual cusp form for GL_n contribute to the cohomology of Shimura varieties in middle degree is related to Ramanujan - Pet. Conj. (at finite places)

§2 Main Theorem

X_Γ compact unitary Shimura variety: (defined over a number field E)

$G/_{F^+}$, F^+ totally real field, $F = F^+K$ where K is imag. quad.

G unitary group.

Theorem $\mathbb{T}$ = spherical Hecke algebra. $\mathbb{T} \curvearrowright H^*(X_\Gamma, \mathbb{F}_\ell)$.

Let $m \subseteq \mathbb{T}$ be a maximal ideal. Assume $H^*(X_\Gamma, \mathbb{F}_\ell)_m \neq 0$. Then

1) $\exists\ \rho_m : \mathrm{Gal}(\bar{F}/F) \to GL_n(\bar{\mathbb{F}}_\ell)$ which matches m at unramified places (eigenvalues of Frob_p match Satake parameters of m)

2) If $\exists\ p$ split completely in F and such that ρ_m at every prime above p is decomposed generic then $H^i(X_\Gamma, \mathbb{F}_\ell)_m \neq 0$ only if $i = \dim X/2$.

Remarks: 1) Part 1 is not new, but we give a different proof. (already follows from Scholze) This different proof uses a p-adic rather than an ℓ-adic tower.

2) $\rho_m|_{G_v}$ is decomposed generic if it is unramified $\{\lambda_1, \dots, \lambda_n\}$ Frobenius eigenvalues, $\lambda_i/\lambda_j \notin \{1, q_v\}$ where q_v = residue char. of v.

This follows from a big image assumption. This does give some reducible cases as well. (e.g. some generic sum of chars)

3) Previous results:

- Lan-Suh (restrictions on level at ℓ and on weight)

- Harris-Taylor type $\mathcal{U}(n-1,1) \times \mathcal{U}(n,0)^{d-1}$
   - Emerton-Gee $\mathcal{U}(2,1)$
   - Shin w/ condition "supercuspidal at p"
   - Boyer very similar to our result

Caraiani

10-22-15

pg3

§3 Main Tool: Geometry of Hodge-Tate period morphism

$$\begin{tikzcd} \mathfrak{h} \arrow[r, hook, "\text{Borel embedding}"] \arrow[d] & \mathbb{P}^1(\mathbb{C}) \\ Y_\Gamma = \Gamma \backslash \mathfrak{h} & \end{tikzcd}$$

Hodge decomp. $\rightarrow$ Hodge de Rham filtration

$$\mathbb{C}^2 \simeq H^1(E, \mathbb{C}) \simeq H^1(E, \mathcal{O}_E) \oplus H^0(E, \Omega^1_E)$$

$$\begin{array}{c} H^1_{et}(E, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq \mathbb{C}_p^2 \\ \wr\wr \\ H^1(E, \mathcal{O}_E) \oplus H^0(E, \Omega^1_E)(-1) \end{array}$$

Hodge-Tate filtration

$$\underline{\mathcal{Y}_\Gamma = (Y_\Gamma \otimes_{\mathbb{Q}} \mathbb{C}_p)^{ad}} \quad \text{an "adic" space.}$$

$$\begin{array}{ll} \mathcal{Y}_\Gamma \\ \downarrow \\ \mathcal{Y}_{\Gamma(p)} & (E, E[p] = (\mathbb{Z}/p\mathbb{Z})^2) \\ | \\ \mathcal{Y}_\Gamma & (E, P \text{ order } M \end{array}$$

$\mathcal{Y}_{\Gamma(p^\infty)}$ $\qquad E, P, T_p E \simeq \mathbb{Z}_p^2$

dual to H^1_{et}

perfectoid modular curve

$$\begin{tikzcd}
\mathcal{Y}_{\Gamma(p^\infty)} \arrow[r, "\pi_{HT}"] \arrow[d] & \mathbb{P}^1(\mathbb{C}_p) \\
\mathcal{Y}_{\Gamma} &
\end{tikzcd}$$

Compute $R\pi_{HT*}\mathbb{F}_\ell$ on $\mathbb{P}^1(\mathbb{C}_p)$.

(Use Leray spectral sequence)

$$\mathbb{P}^1(\mathbb{C}_p) = \mathbb{P}^1(\mathbb{Q}_p) \sqcup \Omega^2$$

Drinfeld upper half-plane