Congruence Primes for automorphic forms on unitary groups

The main focus of this talk is to provide a sufficient condition for a prime to be a congruence prime for an automorphic form on $U(n,n)(\mathbb{A}_F)$ with $F/\mathbb{Q}$ totally real. If there is time we will apply this result to Ikeda lifts.

We begin by surveying briefly some known results. We are loose with the statements leaving out technical conditions.

Roughly, given an automorphic form f that is a Hecke eigenform we say a prime ℓ is a congruence prime for f if there is an automorphic form orthogonal to f that has eigenvalues congruent to f's modulo ℓ. (We will give a precise definition in our set-up later.)

GL(2):

Theorem (Hida '81): ~~Let~~ $f \in S_k(\Gamma_0(N))$ be a newform. Then ℓ is a congruence prime for ~~k~~ f iff $\ell \mid L^{alg}(k, \mathrm{Sym}^2 f)$.

$GSp(4)$:

Given $f \in S_{2k-2}(\Gamma_0(N))$ a newform, one can associate an eigenform $SK(f) \in S_k(\Gamma_0^{(2)}(N))$ satisfying

$$L(s, SK(f); spin) = \zeta(s-k+1)\zeta(s-k+2) L(s,f).$$

This is referred to as the Saito-Kurokawa lift of f.

<u>Theorem (B'05, B'11, Agarwal-B'15):</u> Let $f \in S_{2k-2}(\Gamma_0(M))$ a newform with $k \geq 6$, M odd and sq. free. Let ℓ be a prime, $\ell \nmid M$, $\bar{\rho}_f$ irred. and $\ell \mid L^{alg}(k,f)$. If there is a fundamental disc $D < 0$, an integer $N > 1$, and a character χ of conductor N so that

$$\ell \nmid L(3-k,\chi)\, L^{alg}(k-1,f,\chi_D)\, L^{alg}(1,f,\chi)\, L^{alg}(2,f,\chi),$$

then ℓ is a congruence prime for $SK(f)$ with respect to cuspidal Siegel ~~modular~~ eigenforms with irreducible Galois representations.

(Note in practice one can always find such a D and χ.)

Let f_1 and f_2 be eigenforms of weight $k_1 > k_2 \geq 2$ of level $\Gamma_0(M)$ for M square-free. One can associate a cuspidal Siegel eigenform $Y(f_1 \otimes f_2)$, the Yoshida lift of f_1 and f_2, of level $\Gamma_0^{(2)}(M)$ and weights $Sym^{k_2-2} \otimes \det^{2+\frac{k_1-k_2}{2}}$.

Theorem (Böcherer-Dummigan-Schulze-Pillot '12, Agarwal-Klosin '13): With the set-up as above and assume $k_1 - k_2 \geq 6$. Let ℓ be a prime with $\ell \nmid M$, $\ell \mid L^{alg}\left(\frac{k_1+k_2}{2}, f_1 \otimes f_2\right)$ and $\ell \nmid L^{alg}\left(\frac{k_1+k_2}{2}+1, f_1 \otimes f_2\right)$. Then ℓ is a congruence prime for $Y(f_1 \otimes f_2)$ with respect to cuspidal Siegel eigenforms that are not endoscopic lifts.

$GSp(2n)$:

Theorem (Katsurada '08): Let $f \in S_k(Sp_{2n}(\mathbb{Z}))$ be an eigenform. Let $m \in \mathbb{Z}_{>0}$ with $m < k-n-2$, $m+n$ even, and $m \geq 3$ if $n \equiv 1 \pmod 4$ and $m \geq 1$ otherwise. Then ℓ is a congruence prime for f if ℓ divides the denominator of $L^{alg}(m, f; st)$.

Given $f \in S_{2k-n}(SL_2(\mathbb{Z}))$ a newform, one has an Ikeda lift associated to f, $IK(f) \in S_k(Sp_{2n}(\mathbb{Z}))$ satisfying

$$L(s, IK(f); st) = \zeta(s) \prod_{j=1}^{n} L(s+k-j, f).$$

Theorem (Katsurada '11, B.-Keaton '15): Let $k, n \in 2\mathbb{Z}$ with $k > n+1$. Let $\ell > 2k-n$ be a prime with $\bar{\rho}_f$ irred. and

$$\ell \mid L^{alg}(k, f) \prod_{j=1}^{\frac{n}{2}-1} L^{alg}(2j+1, ad^0 f).$$

If there exists a fundamental disc D, $\ell \nmid D$ with $(-1)^{n/2} D > 0$

and $\chi_D(-1) = -1$ with $\ell \nmid L^{alg}(k - n/2, f, \chi_D)$ and either

1) there exists $N > 1$ and a character χ of conductor N s.t.

$$\ell \nmid L^N(n-k+1, \chi) \prod_{j=1}^{n} L^{alg}(n+1-j, f, \chi)$$

or

2) for some m w/ $n/2 < m < k/2 - n/2$ one has

$$\ell \nmid \zeta^{alg}(2m) \prod_{j=1}^{n} L^{alg}(2m+k-j, f)$$

then ℓ is a congruence prime for f w.r.t. forms that are not Ikeda lifts.

$U(2,2)$:

There is previous work by Klosin here, but we omit the precise statements here since our results will contain these as a special case.

Notation and Set-up:

We will now give some more set-up and then the statement of the main result for $U(n,n)$, and a very brief outline of the proof.

Let $F/\mathbb{Q}$ be a totally real number field of degree d, K/F an imaginary quadratic extension of disc. D_K and character χ_K.

For $A \in \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F} \mathrm{Mat}_{n/\mathcal{O}_K}$, set $A^* = {}^t\bar{A}$. Here we have $\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}$ the Weil restriction defined by

$$\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F} X_{/\mathcal{O}_K}(A) = X(A \otimes_{\mathcal{O}_F} \mathcal{O}_K)$$

for A an $\mathcal{O}_F$-algebra.

Let $GU(n,n)$ be the unitary similitude group scheme over $\mathcal{O}_F$ associated to $J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$:

$$GU(n,n) = \left\{ A \in \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F} GL_{2n/\mathcal{O}_K} : A J_n A^* = \mu_n(A) J_n \right\}$$

where μ_n is a morphism $\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F} GL_{2n/\mathcal{O}_K} \to \mathbb{G}_{m/\mathcal{O}_F}$.

Set $G_n = \ker \mu_n$.

Let $\mathfrak{n} \subseteq \mathcal{O}_F$ be an ideal. Let $K_{0,n}(\mathfrak{n}) \subseteq G_n(\mathbb{A}_F)$ be the usual congruence subgroup.

$$K_{0,n,v}(\mathfrak{n}) = \left\{ g \in G_n(F_v) : a_g, b_g, d_g \in \mathrm{Mat}_n(\mathcal{O}_{K,v}),\ c_g \in \mathrm{Mat}_n(\mathfrak{n}\mathcal{O}_{K,v}) \right\}$$

where $\mathcal{O}_{K,v} = \mathcal{O}_{F,v} \otimes_{\mathcal{O}_F} \mathcal{O}_K$

$$K^{+}_{0,n,2} = \left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in G_n(\mathbb{R}^2) : A, B \in GL_n(\mathbb{C}^2),\ AA^* + BB^* = 1_n,\ AB^* = BA^* \right\}$$

$$K_{0,n}(\mathfrak{n}) = K^{+}_{0,n,2}\, K_{0,n,f}(\mathfrak{n}).$$

Let K be an open compact subgroup of $G_n(\mathbb{A}_{F,f})$. For $\kappa, \nu \in \mathbb{Z}^2$, we let $M_{n,\kappa,\nu}(K)$ denote the $\mathbb{C}$-space of automorphic forms: ($S_{n,\kappa,\nu}(K)$ the cusp forms.) functions $f : G_n(\mathbb{A}_F) \to \mathbb{C}$ satisfying

(i) $f(\gamma g) = f(g) \quad \forall\ \gamma \in G_n(F),\ g \in G_n(\mathbb{A}_F)$

(ii) $f(gk) = f(g) \quad \forall\ k \in K,\ g \in G_n(\mathbb{A}_F)$

(iii) $f(gu) = (\det u)^{-\nu}\, j(u, i1_n)^{-\kappa} f(g)$ for all $g \in G_n(\mathbb{A}_F)$, $u \in K_{0,n,2}$.

(iv) $f_c(Z) = (\det g_2)^{\nu}\, j(g_2, i1_n)^{\kappa} f(g_2 c)$ is a holo. function of $Z = g_2\, i1_n \in \mathbb{H}_n^2$ for every $c \in G_n(\mathbb{A}_{F,f})$ where $g_2 \in G_n(\mathbb{R}^2)$ and $\mathbb{H}_n = \{ Z \in Mat_n(\mathbb{C}) : -i1_n(Z - Z^*) > 0 \}$.

ψ Hecke char of K of cond $|\mathfrak{n}$:

$M_{n,\kappa,\nu}(\mathfrak{n}, \psi) = \{ f \in M_{n,\kappa,\nu}(K_1(\mathfrak{n})) : f(gk) = \psi_{\mathfrak{n}}(\det(a_k))^{-1} f(g) \quad g \in G_n(\mathbb{A}_F),\ k \in K_0(\mathfrak{n}) \}$

For each $q \in GL_n(\mathbb{A}_{K,f})$ and $h \in S_n(F)$ there are complex numbers $C_f(h,q)$ so that

$$f\left(\begin{bmatrix} 1_n & \sigma \\ 0 & 1_n \end{bmatrix} \begin{bmatrix} q & 0 \\ 0 & \hat{q} \end{bmatrix} \right) = \sum_{h \in S_n(F)} C_f(h,q)\, e_{\mathbb{A}_F}(\mathrm{tr}\, h\sigma)$$

for every $\sigma \in S_n(\mathbb{A}_F)$ where

$$S_n = \{ h \in \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F} \mathrm{Mat}_{/\mathcal{O}_K} : h^* = h \}$$

and $\hat{q} = (q^*)^{-1}$.

Here we define $e_{/A_F}$ as follows: Let $\alpha = (\alpha_v) \in /A_F$. Set $e_v(\alpha_v) = e(\alpha_v) = e^{2\pi i \alpha_v}$ for $v \in \mathfrak{a}$ and write $e_{\mathfrak{a}}(\alpha) = e\left(\sum_{v \in \mathfrak{a}} \alpha_v\right)$, if $v \nmid \infty$, set $e_v(\alpha_v) = e(-y)$ where $y \in \mathbb{Q}$ is chosen so that $\mathrm{Tr}_{F_v/\mathbb{Q}_p}(\alpha_v) - y \in \mathbb{Z}_p$ if $p \mid v$. Set $e_{/A_F}(\alpha) = \prod_v e_v(\alpha_v)$.

Let $\mathcal{B} \subseteq GL_n(/A_{K,f})$ be a subset of cardinality h_K with the property that the canonical projection (from CFT) $c_K : /A_K^\times \to Cl_K$ restricted to $\det \mathcal{B}$ is a bijection. Recall $/A_K^\times / K^\times K_\infty^+ \hat{\mathcal{O}}_K^\times \cong Cl_K$. We call such a $\mathcal{B}$ a base and note

$$GL_n(/A_K) = \coprod_{b \in \mathcal{B}} GL_n(K) GL_n^+(K_{\infty}) \, b \, GL_n(\hat{\mathcal{O}}_K).$$

Set $p_b = \begin{bmatrix} b & 0 \\ 0 & \hat{b} \end{bmatrix}$.

Def: Let ℓ be a prime and $\mathcal{O}$ the ring of integers in some algebraic extension $E/\mathbb{Q}_\ell$ with maximal ideal λ.

1) Let $f \in \underbrace{M_{n,k,v}(\mathfrak{n},\psi)}_{\text{really only need for } M_{n,k,v}(K)}$. We say f has $\mathcal{O}$-integral f.c wrt $\mathcal{B}$ if there exists a base $\mathcal{B}$ s.t for all $b \in \mathcal{B}$ and all $h \in S_n(F)$ we have $e_{\mathfrak{a}}(-i \operatorname{tr} h_{\mathfrak{a}}) \, c_f(h, p_b) \in \mathcal{O}$.

2) Let $f, g \in M_{n,k,v}(\mathfrak{n},\psi)$ and suppose both have $\mathcal{O}$-integral f.c. wrt $\mathcal{B}$. Let $E'/\mathbb{Q}_\ell$ be a finite extension with $E' \subseteq E$, $\mathcal{O}'$ the r.o.i and ϖ' a uniformizer. We say f is congruent to g modulo ϖ^n and write $f \equiv g \pmod{\varpi^n}$ if for all $b \in \mathcal{B}$ and $h \in S_n(F)$ we have

$$e_2(-i\,\mathrm{tr}\, h_2)\, C_f(h, p_b) - e_2(-i\,\mathrm{tr}\, h_2)\, C_g(h, p_b) \in \varpi^n \mathcal{O}.$$

Def. Given f and ℓ, if there exists such a g with $\langle f, g\rangle = 0$ we say ℓ is a <u>congruence prime</u> <u>for f</u>.

<u>Main Result:</u>

The main result in this case is the following. One can think of this as a $U(n,n)$ version of Katsurada's $GSp(2n)$ result, though we work in more generality because we allow $F \neq \mathbb{Q}$. The $GSp(2n)$ case with $F \neq \mathbb{Q}$ is current work of my student Huixi Li as part of his PhD thesis.

<u>Theorem (B'-Klosin '16):</u> Let $F/\mathbb{Q}$ be totally real of degree d, K/F imag. quad. and assume $Cl_K^- = Cl_K$ (Cl_K^- is part of class group where $\mathrm{Gal}(K/F)$ acts via -1) Let $n \in \mathbb{Z}_{>0}$ with $\gcd(2n, h_K) = 1$. Let $k = (k, \dots, k) \in \mathbb{Z}^d$ so that $k > 0$. Let ℓ be a natural prime with $\ell > k$ and $\ell \nmid D_K h_K$. Let $f \in S_{n,k}(K_{0,n}(\mathfrak{n}))$ be a Hecke eigenform with $\mathcal{O}$-integral f.c. Let ξ be a Hecke character of K s.t. $\xi_\infty(z) = \left(\frac{z}{|z|}\right)^{-t}$ for $t = (t, \dots, t) \in \mathbb{Z}^d$ with $-k \leq t \leq \min\{-6, -4n\}$. Let

$$-b = \mathrm{val}_\varpi \left(\frac{\pi^{dn^2}}{\mathrm{vol}(\mathcal{F}_{K_{0,n}(\mathfrak{n})})} \cdot \overline{L_c^{\mathrm{alg}}(2n + t/2, f, \xi; \mathrm{st})} \right) < 0.$$

Then there exists $f' \in S_{n,k}(K_{0,n}(\mathfrak{n}))$, orthogonal to f, with $\mathcal{O}$-integral f.c. so that $f \equiv f' \pmod{\varpi^b}$.

Here $L(s, f, \xi; \mathrm{st})$ is the twisted standard L-function associated to f.

A very brief outline of the proof: We use ξ to define a theta series

Θ_ξ with integral f.c. Similarly we define an Eisenstein series with

integral f.c. (really, $\pi^{-n(n+1)d/2} E$ has integral f.c.).

Write $\Xi = E\Theta_\xi$. Then we have

$$\Xi = C_f f + g$$

for some $g \in S_{n,k}(K_{0,n}(\mathfrak{n}))$ where $\langle f, g\rangle = 0$ and $C_f = \dfrac{\langle \Xi, f\rangle}{\langle f, f\rangle}$.

We then calculate that

$$C_f = (*) \frac{\pi^{dn^2}}{\mathrm{vol}\left(\mathcal{F}_{K_{0,n}(\mathfrak{n})}\right)} \overline{L^{alg}(2n+t/2, f, \xi; st)}.$$

Thus, if $\mathrm{val}_\varpi(C_f) = -b < 0$, then we can write $C_f = A/\varpi^b$, $\varpi \nmid A$,

and so $\varpi^b \Xi = Af + \varpi^b g$. Reducing modulo ϖ^b

gives a congruence $Af \equiv \varpi^b g \pmod{\varpi}$.

Application to Ikeda lifts:

For this we let $F = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{-D_K})$ w/ disc $-D_K$. Let $n = 2m$ (resp $n = 2m+1$).

Let $\phi \in S_{2k+1}(D_K, \chi_K)$ (resp. $S_{2k}(SL_2(\mathbb{Z}))$) be a newform.

Ikeda has shown there is a Hecke eigenform $I_\phi \in S_{n,2k+2m,-k-m}(G_n(\hat{\mathbb{Z}}))$

so that for any Hecke character ψ of K

one has

$$L^{D_K}(s, I_\phi, \psi; s+1) = \prod_{j=1}^{n} L^{D_K}(s+k+m-n-i+1, BC(\phi)\otimes\psi).$$

(definition of base change L-function is on page 17 of the paper.)

The main work in applying the general result to this case is one does not immediately know I_ϕ has nice arithmetic properties.

We show that there is a base $\mathcal{B}$ so that the fourier coefficients of I_ϕ are algebraic integers for this base.

The main work goes into investigating the non-vanishing of I_ϕ modulo ℓ.

<u>Theorem (B'-Klosin '16)</u>: Let $\ell \nmid 2D_K$ be prime. If $n \not\equiv 2 \pmod 4$ there is a base $\mathcal{B}$ so that at least one f.c of I_ϕ is non-vanishing modulo ℓ. If $n \equiv 2 \pmod 4$ and $\bar{\rho}_\phi(G_K)$ ($\bar{\rho}_\phi : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_\ell)$ and $G_K = Gal(\overline{\mathbb{Q}}/K)$) is a non-abelian subgroup of $GL_2(\overline{\mathbb{F}}_p)$, then wrt $\mathcal{B}$ at least one f.c. of I_ϕ is non-vanishing modulo ℓ.

Note a result connecting forms ϕ arising from a Hecke char of an imag quad. field and the ℓ-adic Galois rep of ϕ having abelian image when restricted to G_K was given by Ribet.

This result requires some delicate Galois args. which we won't get into here.

One can now apply the main result in this case (and consider the factorization of the L-function for Ikeda lifts and the relation between $\langle I_\phi, I_\phi \rangle$ and $\langle \phi, \phi \rangle$) to conclude the following result.

<u>Theorem (B'-Klosin '16):</u> With the set-up as before, set

$$\mathcal{U} = \prod_{j=1}^{n} \pi^{-2n-2k-2m-t+2j-2} L^{D_K}(n+t/2+k+m-j+1, BC(\phi)\otimes \xi^{-1}\beta)$$

$$\times \prod_{j=2}^{n} \frac{L(j, \chi_K^j)}{\pi^j} L^{\Sigma}_{D_K}(2n+t/2, \Psi_\beta^{-1}(I_\phi), \xi^{-1}; st) \in \bar{\mathbb{Q}}$$

$\Psi_\beta : f \mapsto \beta \otimes f$ where β is everywhere unramified char of K of ∞ type $\left(\frac{z}{|z|}\right)^{-2v}$. Then $\Psi_\beta^{-1}(I_\phi) \in S_{n,2k+2m}(G_n(\hat{\mathbb{Z}}))$. Want $-2v = 2k+2m$.

Set

$$\mathcal{V} = \begin{cases} \displaystyle\prod_{j=2}^{n} \frac{L(j+2k-1, \mathrm{Sym}^2\phi \otimes \chi_K^{j+1})}{\pi^{2k+2j-1}\langle \phi, \phi \rangle} & n = 2m+1 \\ \displaystyle\prod_{j=2}^{n} \frac{L(j+2k, \mathrm{Sym}^2\phi \otimes \chi_K^{j})}{\pi^{2k+2j}\langle \phi, \phi \rangle} & n = 2m. \end{cases}$$

If $\mathrm{val}_\varpi(\mathcal{U}) = 0$ and $\mathrm{val}_\varpi(\mathcal{V}) = b > 0$, then there is a nonzero $f' \in S_{n,2k+2m}(G_n(\hat{\mathbb{Z}}))$, orthogonal to $\Psi_\beta^{-1}(I_\phi)$, s.t. $f' \equiv \Psi^{-1}(I_\phi) \pmod{\varpi^b}$.

The reason for Ψ_p is because this map gives an isomorphism between $\mathcal{M}_{n,k}(\mathbb{K})$ and $\mathcal{M}_{n,k,v}(\mathbb{K})$, and an earlier result where for $v=0$.