INTRODUCTION TO THE HODGE CONJECTURE

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Abstract. These are notes from a talk introducing the Hodge conjecture to undergraduates attending the 2009 Clemson REU. The description given here of the Hodge conjecture was the author's attempt to bring the conjecture down to a reasonably understandable level to undergraduates. As such, liberties have been taken with the accuracy and generality of some of the statements. Please keep this in mind when reading the notes.

1. Introduction

Of the Clay Mathematics Institute's Millenium Problems, the Hodge conjecture is the most difficult to simplify into something understandable to non-experts. The description of the problem on the Clay website is given by Deligne. The first sentence of his description of the Hodge conjecture is:

"We recall that a pseudo-complex structure on a C^∞-manifold X of dim $2N$ is a $\mathbb{C}$-module structure on the tangent bundle $T(X)$."

To any non-expert in the field that sentence is surely somewhat daunting. The conjecture itself reads as follows:

Conjecture 1.1. (Hodge Conjecture) On a projective nonsingular algebraic variety over $\mathbb{C}$, any Hodge class is a rational linear combination of classes of algebraic cycles.

The goal of this lecture is to try and define (in some special cases) the objects that the Hodge conjecture is about. The Hodge conjecture proposes a deep connection between analysis, topology, and algebraic geometry. Very roughly it is saying that certain objects that are built via analysis (differential forms) actually can be built via algebraic methods (at least when one would actually hold out hope such a thing could be accomplished.) We begin with the analytic side of things.

2. The Analytic side of things

Though we are actually interested in statements over $\mathbb{C}$, we begin by working over $\mathbb{R}$ as this is more familiar to most undergraduates.

The author would like to thank Janine Janowski for taking notes during the talk and typing them into an earlier form of these notes.

Let V be a finite dimensional vector space over $\mathbb{R}$. We say a map $f: V \to V$ is *linear* if one has $f(\alpha v + w) = \alpha f(v) + f(w)$ for all $\alpha \in \mathbb{R}$ and all $v, w \in V$. We say that a map $f: V^k \to \mathbb{R}$ is k-*linear* if f is linear in each of the k-variables separately.

Definition 2.1. A k-linear map $f: V^k \to \mathbb{R}$ is said to be *alternating* if $f(x_1, \dots, x_k) = 0$ whenever $x_i = x_j$ for some $i \neq j$. The set of such maps is denoted $\mathrm{Alt}^k(V)$.

Exercise 1. Show that for any $k \geq 0$ one has $\mathrm{Alt}^k(V)$ is a $\mathbb{R}$-vector space.

Recall that S_k denotes the symmetric group on k-letters. If $\sigma \in S_k$, we can factor σ into a product of transpositions, say m of them. Define $\mathrm{sgn}(\sigma) = (-1)^m$. Let m and n be positive integers and define an (m,n)-shuffle to be a $\sigma \in S_{m+n}$ with $\sigma(1) < \sigma(2) < \dots \sigma(m)$ and $\sigma(m+1) < \dots < \sigma(m+n)$. The set of (m,n)-shuffles is denoted $S_{m,n}$. We can now define the exterior product. Given $f \in \mathrm{Alt}^P(V), g \in \mathrm{Alt}^q(V)$ we define a map

$$\wedge : \mathrm{Alt}^p(V) \times \mathrm{Alt}^q(V) \to \mathrm{Alt}^{p+q}(V)$$

given by

$$(f \wedge g)(x_1, \dots x_{p+q}) = \sum_{\sigma \in S_{p,q}} \mathrm{sgn}(\sigma) f(x_{\sigma(1)}, \dots x_{\sigma(p)}) g(x_{\sigma(p+1)}, \dots x_{\sigma(p+q)}).$$

Exercise 2. Show that $f \wedge g \in \mathrm{Alt}^{p+q}(V)$.

Exercise 3. In the special case that $p = q = 1$ show that

$$(f \wedge q)(x_1, x_2) = f(x_1)g(x_2) - f(x_2)g(x_1).$$

Exercise 4.
(1) $f \wedge g = (-1)^{pq} g \wedge f$
(2) $(f \wedge g) \wedge h = f \wedge (g \wedge h)$

We are now in a position to define differential forms. Let $U \subset \mathbb{R}^n$ be an open set. Though we continue to work over $\mathbb{R}$ for now, it is a good idea to keep in mind that we would like to be working over $\mathbb{C}$ so one should keep in the back of one's mind the case that $U \subset \mathbb{C}^n$.

Definition 2.2. A *differential* k-*form on* U is a smooth map

$$\omega : U \to \mathrm{Alt}^k(\mathbb{R}^n)$$

The set of differential forms on U is denoted $\Omega^k(U)$.

Exercise 5. Show that $\Omega^k(U)$ is a $\mathbb{R}$-vector space.

Exercise 6. Show $\mathrm{Alt}^0(\mathbb{R}^n) \cong \mathbb{R}$ and so $\Omega^0(U) = C^\infty(U)$ where $C^\infty(U)$ is the set of functions $f: U \to \mathbb{R}$ that have continuous partial derivatives to any order.

Recall we have $\omega : U \to \mathrm{Alt}^k(\mathbb{R}^n)$ and $\mathrm{Alt}^k(\mathbb{R}^n)$ is a finite dimensional $\mathbb{R}$-vector space. As such, we can take derivatives of ω as in multivariable calculus class. For $x \in U$, we set $D_x\omega$ to be the derivative of ω at x. Observe that we have $x \mapsto D_x\omega$ is a map $U \to (\mathbb{R}^n \to \mathrm{Alt}^k(\mathbb{R}^n))$. The *exterior derivative* is the map

$$d^k : \Omega^k(U) \to \Omega^{k+1}(U)$$

defined by

$$d_x^k\omega(x_1, \ldots, x_{k+1}) = \sum_{j=1}^{k+1}(-1)^{j-1}D_x\omega(x_j)(x_1, \ldots, \hat{x_j}, \ldots x_{k+1})$$

where $\omega(x_j)$ is really $\omega(0, \ldots, 0, x_j, 0, \ldots, 0)$ and $(x_1, \ldots \hat{x_j}, \ldots x_{k+1}) = (x_1, \ldots x_{j-1}, x_{j+1}, \ldots x_{k+1})$.

Example 2.3. Let $U \in \mathbb{R}^n$ and let $x_i : U \to \mathbb{R}$ be the projection onto the ith coordinate map, i.e.,

$$x_i(a_1, \ldots, a_n) = a_i.$$

Then one has $x_i \in \Omega^0(U)$ and so $d^0x_i \in \Omega^1(U)$. One can check that d^0x_i is the constant map that takes all the values x to ϵ_i where the ϵ_j are the dual basis to the vector space $\mathrm{Alt}^1(\mathbb{R}^n)$. Note that if $U \subseteq \mathbb{R}$ then we have the normal "dx" from calculus class.

Example 2.4. Let $U \subseteq \mathbb{C}^n$ be an open set. Take z_j be the projection map onto the j^{th} coordinate. We can also define a map $\overline{z}_j$ to be the composition of the projection to the j^{th} coordinate with the complex conjugation map, i.e.,

$$\overline{z}_j(a_1, \ldots, a_n) = \overline{a}_i.$$

Note that these maps are not holomorphic maps where the maps z_j were. Taking derivatives of these maps gives $dz_1, \ldots, dz_n, d\overline{z}_1, \ldots d\overline{z}_n \in \Omega^1(U)$.

Recall we defined $\wedge$ to be the wedge product on $\mathrm{Alt}^*(\mathbb{R}^n)$. We can also define $\wedge$ on $\Omega^*(U)$ via this previous definition.

Exercise 7. Show that $dz_j \wedge dz_j = 0$ and $d\overline{z}_j \wedge d\overline{z}_j = 0$ for all $1 \leq j \leq n$.

If we have $U \subseteq \mathbb{C}$ then we have dz and $d\overline{z}$. One can show that a C^∞ 1-form on U must be of the form

$$\omega = f(z, \overline{z})dz + g(z, \overline{z})d\overline{z}$$

for some $f, g \in C^\infty(U)$. Similarly, one can show that a C^∞ 2-form can be written

$$\omega = f(z, \overline{z})dz \wedge d\overline{z}$$

for some $f \in C^\infty(U)$. In this case one has that $\Omega^k(U) = 0$ for any $k \geq 3$ because one will necessary get either $dz \wedge dz$ or $d\overline{z} \wedge d\overline{z}$ in any expression for a C^∞ k-form when $k \geq 3$.

Suppose now that $U \in \mathbb{C}^n$. Let p, q be positive integers. Let $\Omega^{p,q}(U)$ be the space of forms generated by

$$dz_i \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}.$$

In this case one can write

$$\Omega^k(U) = \bigoplus_{p+q=k} \Omega^{p,q}(U). \tag{1}$$

In particular, this shows the space $\Omega^k(U)$ breaks down into component vector spaces that are somewhat easier to study.

Going back to the general case, one can show from the definition that the composition

$$\Omega^k(U) \xrightarrow{d^k} \Omega^{k+1}(U) \xrightarrow{d^{k+1}} \Omega^{k+2}(U) \tag{2}$$

is actually identically 0 for all k. If one has any experience with algebraic topology ones knows this is a very nice situation to be in!

Definition 2.5.
(1) If $\omega \in \Omega^k(U)$ is such that $d^k\omega = 0$, then ω is called a *closed form.*
(2) If $\omega \in \Omega^k(U)$ and $\omega \in d^{k-1}(\Omega^{k-1}(U))$ we say ω is *exact.*

Note that the fact that the composition given by equation (2) is identically 0 gives that the space of exact forms sits inside the space of closed forms. Using this we define the deRham cohomology groups by

$$H^k_{\mathrm{DR}}(U) = \text{closed forms/exact forms}.$$

Exercise 8. Show that $H^0_{\mathrm{DR}}(U)$ is the set of maps that are constant on each connected component of U. Thus, $\dim H^0_{\mathrm{DR}}(U)$ is the number of connected components of U.

An incredibly important and deep theorem in this subject is the Hodge decomposition theorem:

Theorem 2.6. *(Hodge Decomposition Theorem) One has that*

$$H^k_{\mathrm{DR}}(U) = \oplus_{p+q=k} H^{p,q}_{\mathrm{DR}}(U)$$

where $H^{p,q}_{\mathrm{DR}}(U)$ are the closed forms coming from $\Omega^{p,q}(U)$ modulo the exact forms coming from $\Omega^{p,q}(U)$.

This is truly an astounding theorem. We know that $\Omega^k(U)$ has such a decomposition, but it is not at all clear that given a closed form $f \in \Omega^k(U)$ and decompose it via the decomposition given in equation (1) that the resulting forms in $\Omega^{p,q}(U)$ would in fact also all be closed as well!

We now have an idea of the analytic side of things. However, one cannot hope that all of the cohomology classes constructed in this way are in any sense algebraic. As such, we need a way to pick out the ones that have a hope of being constructed via algebraic methods. This is the notion of a "Hodge class." In order to define a Hodge class, we need some more cohomology.

Our cohomology so far, namely deRham cohomology, lives over $\mathbb{R}$ or $\mathbb{C}$ as it was constructed via calculus. This is in no way an algebraic construction, so if we hope to single out classes that might come from algebra we need some "algebraic" cohomology. We do not have time to give definitions of these cohomology groups, so we merely state that they exist and arise via more algebraic methods. We denote the k^{th} singular (or Betti) cohomology group over $\mathbb{Z}$ by $H^k_{\text{B}}(U,\mathbb{Z})$. While one can do all of these using singular cohomology, the "better" way is to use sheaf cohomology. Given a ring R, we denote the k^{th} cohomology group over the sheaf $\underline{R}$ by $H^k(U,\underline{R})$ where $\underline{R}$ is the constant sheaf associated to R. One has

- $H^k_{\text{B}}(U,\mathbb{Z}) \cong H^k(U,\underline{\mathbb{Z}})$,
- $H^k(U,\underline{\mathbb{Z}}) \otimes \mathbb{C} \simeq H^k_{\text{DR}}(U)$,
- $H^k(U,\underline{\mathbb{Z}}) \otimes \mathbb{Q} \simeq H^k(U,\underline{\mathbb{Q}})$.

The rational Hodge classes are the classes in the deRham cohomology group that have a hope of being algebraic, i.e., they are the classes in

$$\text{Hdg}(U) := H^{2p}(U,\underline{\mathbb{Q}}) \cap H^{p,p}_{\text{DR}}(U).$$

Up to this point everything has been done for U an open subset of $\mathbb{R}^n$ or $\mathbb{C}^n$. This is the simplest case and is not really what the Hodge conjecture is all about. Quickly we give an idea of how one needs to generalize this.

The main idea is that we replace $\mathbb{R}^n$ or $\mathbb{C}^n$ be a real or complex manifold.

Definition 2.7. A *real (resp. complex) manifold of dimension* n is a topological space X along with a collection of open sets $\{U_i\}$ covering X and maps $\phi_i : U_i \to \mathbb{R}^n$ (resp. $\phi_i : U_i \to \mathbb{C}^n$) so that ϕ_i gives a homeomorphism of U_i with an open set $V_i \subset \mathbb{R}^n$ (resp. $V_i \subset \mathbb{C}^n$). One also requires that the maps be compatible, i.e.,

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is C^∞ (resp. holomorphic.)

In short, a real (resp. complex) manifold of dimension n is a topological space that locally looks like $\mathbb{R}^n$ (resp. $\mathbb{C}^n$.)

Example 2.8.
(1) The unit circle $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a real 1-manifold, i.e., a real manifold of dimension 1.
(2) The unit sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is a real $2-$manifold or a complex 1-manifold.

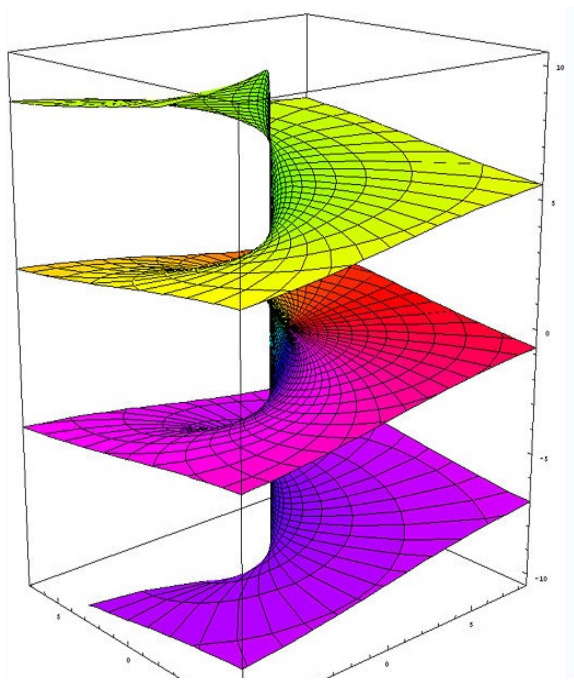
Manifolds arise very naturally when working in complex variables. For instance, suppose we want to define a continuous function $\log z$ on $\mathbb{C}$ that generalizes the familiar natural logarithm function on $\mathbb{R}$. Given any nonzero $z \in \mathbb{C}$, one can write $z = re^{i\theta}$ for some $r > 0$. Thus, it seems natural to define

$$\log z = \log r + i\theta$$

where $\log r$ means the usual natural logarithm of r as a function on $\mathbb{R}$. However, this is not a well-defined function. For instance, one has $r = re^{i\cdot 0} = re^{2\pi i}$ and so one would have

$$\log r = \log r + i \cdot 0 = \log r + 2\pi i.$$

This clearly does not make sense since $2\pi i \neq 0$. One can eliminate this problem if one defines log on a complex manifold instead so that instead of going around in a circle back to the same point, one goes upwards on the spiral as given in the following graphic:



Recall that in defining differential forms we had to consider $\mathrm{Alt}^k(V)$ for V a vector space. In our previous set-up of U being an open subset of $\mathbb{R}^n$ or $\mathbb{C}^n$, we had $V = \mathbb{R}^n$ or $V = \mathbb{C}^n$. However, in general a manifold is not a vector space. The correct way to generalize is to realize that given any point $x \in X$, the tangent space $T_x(X)$ is a vector space. For our previous cases, given any $x \in \mathbb{R}^n$ or $x \in \mathbb{C}^n$, one has $T_x(\mathbb{R}^n) \cong \mathbb{R}^n$ or $T_x(X) \cong \mathbb{C}^n$ so we did not actually need to consider the tangent space. Now to define differential forms on manifolds we consider maps

$$\omega : X \to \mathrm{Alt}^k(T_x(X)).$$

We use the fact that X locally looks like a Euclidean space to define differentiability, continuity, etc. For our purposes one can take it on faith it all works out. One then defines $\Omega^k(X)$ and $H^k_{\mathrm{DR}}(X)$ as before. There is one major difference though. In this level of generality one does not have a Hodge Decomposition Theorem. Thus, one must restrict to Kähler manifolds. One can think of these as manifolds where one does have a Hodge Decomposition Theorem. Though this is not the definition, all of the manifolds we will be interested, namely complex projective manifolds, are Kähler manifolds so the exact definition is not needed here.

3. Algebraic Side of things

To work on the algebraic side of things one must know a little algebraic geometry. In particular, one needs to know what complex projective space is. Consider the set of $(n+1)$-tuples $(z_1, \ldots, z_{n+1})$ in $\mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}$. One defines an equivalence relation on this set by declaring

$$(z_1, \ldots, z_{n+1}) \sim (w_1, \ldots, w_{n+1})$$

if there exists a non-zero $\lambda \in \mathbb{C}$ so that $z_i = \lambda w_i$ for all $1 \leq i \leq n+1$. The equivalence class containing $(z_1, \ldots, z_{n+1})$ is denoted by $[z_1 : \cdots : z_{n+1}]$. The set of these equivalence classes is denoted $\mathbb{CP}^n$. Recall that $\mathbb{CP}^2$ was defined when studying elliptic curves and defining the point at infinity.

The space $\mathbb{CP}^n$ is a complex n-manifold. In particular, for $z = [z_1 : \cdots : z_n] \in \mathbb{CP}^n$ one has that at least one $z_j \neq 0$. One can scale this equivalence class so one has

$$[z_1 : \ldots : z_{n+1}] = [\frac{x_1}{x_j} : \ldots : \frac{x_{j-1}}{x_j} : 1 : \frac{x_{j+1}}{x_j} : \ldots : \frac{x_{n+1}}{x_j}].$$

In other words, z lies in the set $U_j = \{[w_1 : \cdots : w_{j-1} : 1 : w_{j+1} : \cdots : w_{n+1}] : w_i \in \mathbb{C}\}$, which is homeomorphic to $\mathbb{C}^n$. Since z was an arbitrary point, we have that $\mathbb{CP}^n$ is a complex n-manifold.

Among the many brilliant ideas of Descartes was that one could relate geometry and algebra. Suppose one wishes to study a circle with radius a. One can study this using the tools of geometry as the ancient Greeks did, or one can study the solutions of $f(x, y) = 0$ where $f(x, y) = x^2 + y^2 - a^2$ algebraically. Once one gets used to the fact that one can study geometry in this way, namely, algebraically, there is no reason to limit oneself to equation $f(x, y) = 0$ where the $f(x, y)$ arises from a familiar geometric object. One can just as easily study $f(x, y) = 0$ for any polynomial $f(x, y)$. Moreover, there is no reason to restrict to polynomials of two variables. In fact, there is no reason to consider only the solution set of a single polynomial, one can study $f_1(x_1, \ldots, x_{n+1}) = f_2(x_1, \ldots, x_{n+1}) = \cdots = f_m(x_1, \ldots, x_{n+1}) = 0$ for $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_{n+1}]$. This is now moving into the world of algebraic geometry. Let $f_1, \ldots, f_m$ be homogeneous polynomials. Define

$$V(f_1, \ldots, f_m) = \{P \in \mathbb{CP}^n : f_1(p) = \ldots = f_m(p) = 0\}.$$

If this set is smooth and irreducible we call it an *complex projective algebraic variety*. The algebraic variety $V(f_1, \ldots, f_m)$ has a topology on it called the *Zariski topology* induced from the Zariski topology on $\mathbb{CP}^n$. We will not describe this topology here as it can be found in any basic algebraic geometry text. One can see the references given at the end of these notes for example.

Recall when studying elliptic curves one considers the equation $y^2 = x^3 + ax + b$. We said this equation (along with a distinguished point at infinity) gave an elliptic curves, call this curve E. Set $f(x, y) = y^2 - x^3 - ax - b$. Then studying the elliptic curve E amounts to studying $f(x, y) = 0$. In order to properly take into account the point at infinity, one should instead study the homogeneous equation $F(x, y, z) = zy^2 - x^3 - axz^2 - bz^3$. Recall that when studying elliptic curves one is particularly interested in $E(\mathbb{Q})$, i.e., the points $(X, Y, Z) \in \mathbb{Q}^3$ that satisfy $F(X, Y, Z) = 0$. Similarly, given any complex projective algebraic variety $V := V(f_1, \dots, f_m)$ and ring R, one can consider the points $P \in R^{n+1} \cap \mathbb{CP}^n$ so that $f_1(P) = \dots f_m(P) = 0$. We denote this set by $V(R)$.

Let X be a complex projective algebraic variety. We want to consider the space $X(\mathbb{C})$. This set is a complex manifold which we denote as X_{an}. While this is really just the set $X(\mathbb{C})$, we use this notation to indicate we are now considering it as a complex manifold and so it no longer has the Zariski topology.

Let $Y \subseteq X$ be a subvariety, i.e., it is a subset that is itself an algebraic variety. For example, if we consider the variety given by the sphere, any circle contained in the sphere is a subvariety. Let X have dimension r. Given a subvariety Y of codimension p, i.e., of dimension $r - p$, we obtain a complex submanifold Y_{an} of X_{an} of codimension p.

Note that the "proper" way to proceed here would be to define the Chow groups of X and then show we can consider these cycles as cycles in our original framework. However, since we have focused more on the analytic side we will continue to do so.

Let X be a complex projective algebraic variety of dimension r and Y a subvariety of codimension p. One can associate to Y_{an} a cohomology class $[Y] \in H^{2p}_{\text{DR}}(X)$ as follows. (Note that one can take this on faith as the reasoning is quite advanced.) Since Y_{an} has dimension $r - p$, there is a natural map

$$\Omega^{2(r-p)}(X_{\text{an}}) \to \Omega^{2(r-p)}(Y_{\text{an}})$$

which gives a map on cohomology

$$H^{2(r-p)}_{\text{DR}}(X_{\text{an}}) \to H^{2(r-p)}_{\text{DR}}(Y_{\text{an}}).$$

Since Y_{an} has dimension $r - p$, one has $\Omega^{2(r-p)}(Y_{\text{an}})$ has dimension 1 and so there is a natural map

$$H^{2(r-p)}_{\text{DR}}(Y_{\text{an}}) \to \mathbb{C}.$$

Thus, one has a map

$$H^{2(r-p)}_{\text{DR}}(X_{\text{an}}) \to H^{2(r-p)}_{\text{DR}}(Y_{\text{an}}) \to \mathbb{C},$$

i.e., an element in the dual of $H^{2(r-p)}_{\text{DR}}(X_{\text{an}})$. One now applies Serre duality to obtain an element $[Y] \in H^{2p}_{\text{DR}}(X_{\text{an}})$.

We have now covered enough to give an alternate statement of the Hodge conjecture in a form that uses what we have done.

Conjecture 3.1. (Hodge Conjecture again) Let X be a smooth complex projective algebraic variety and associate X_{an} as above. Recall the Hodge classes of X_{an} are given by

$$\mathrm{Hdg}(X_{\mathrm{an}}) = H^{2p}(X_{\mathrm{an}}, \underline{\mathbb{Q}}) \cap H^{p,p}_{\mathrm{DR}}(X_{\mathrm{an}}).$$

Given such a Hodge class, one can find algebraic subvarieties $\{Y_i\}$ so that the Hodge class is a rational sum of the algebraic cycles $[Y_i] \in H^{2p}_{\mathrm{DR}}(X_{\mathrm{an}})$.

Remark 3.2.
(1) Note that it is important that X be algebraic. The statement is NOT true for an arbitrary Kähler manifold.
(2) Originally when the conjecture was proposed by Hodge in the 50's it was over $\mathbb{Z}$ instead of $\mathbb{Q}$. However, Grothendieck constructed a counterexample over $\mathbb{Z}$ so he reformulated the conjecture over $\mathbb{Q}$.
(3) The Hodge conjecture is known for elliptic curves and even products of elliptic curves. However, those are about all of the cases it is known. It is not even known in general for abelian varieties!

Some references for those interested in reading more about the background material presented here are given in the bibliography.

References

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