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PSI

Oddness of Residually Reducible Galois Representations:

1. Introduction

$f \in S_k(N, \chi)$ Hecke eigenform, $k \geq 1$.

$\downarrow$

$\rho_f : G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\overline{\mathbb{Q}}_p)$

s.t. • $tr(\rho_f(Frob_\ell)) = a_\ell(f)$ (Hecke eigenvalue)

• $\det(\rho_f(c)) = (\chi \, \chi_{cycl}^{k-1})(c) = -1$. "odd" where c = complex conjugation

• irreducible

• geometric: unramified for all $\ell \nmid Np$ and $\rho_f|_{D_p}$ is potentially semi-stable.
(e.g. if $p \nmid N$ and $p > k$ then $\rho_f|_{D_p}$ is "short crystalline")

Fontaine-Mazur modularity Conjecture:

Let $\rho : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{Q}}_p)$ irred. and geometric and not twist of even representation with finite image. Then $\rho \simeq \rho_f$ (up to twist) for some f.

Thm (Kisin, Emerton, Khare-Wintenberger): FoMa is true if ρ is odd and $\bar{\rho} :=$ "ρ mod p" absolutely irreducible and extra technical assumptions on $\rho|_{D_p}$.

Thm (Calegari): Assume $\bar{\rho}$ abs. irred. and not dihedral and extra assumptions on $\rho|_{D_p}$. Then ρ is odd.

Goal: Give higher dim. generalization

Thm (Ribet): $p > 2$ $\rho : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{Q}}_p)$ abs. irred, short crystalline, and

unramified away from p, and $\bar\rho|_{D_p}$ distinguished s.t.
$\bar\rho$ is reducible. Then Vandiver's conjecture [$p \nmid$ class number of $\mathbb{Q}(\mu_p)^+$] implies ρ is odd.

<u>Proof (sketch):</u> Up to twist $\bar\rho^{ss} = \bar\chi_{cycl}^m \oplus \mathbb{1}$, $m \neq 0$.

Ribet (1976): $\exists$ lattice in ρ s.t. mod p reduction $\begin{pmatrix} \bar\chi_{cycl}^m & * \\ 0 & \mathbb{1} \end{pmatrix}$

non-split.

This gives $p \mid \# Cl(\mathbb{Q}(\mu_p)(\bar\chi_{cycl}^m))$.

Vandiver gives m must be odd. This implies $\det \rho$ odd. ∎

<u>Remark:</u> Ribet proved: $p \mid \frac{\zeta(k)}{\pi^k}$ (k even, "critical")

then $p \mid \# Cl(\mathbb{Q}(\mu_p)(\chi_{cycl}^{1-k}))$.

2. Polarizations and signs:

Let K/K^+ CM quad. ext., $\hat c \in G_{K^+}$ any lift of $1 \neq c \in Gal(K/K^+)$.
(think $K^+ = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{-D})$)

$R: G_K \longrightarrow GL_n(\bar{\mathbb{Q}}_p)$ abs. irred.

$\psi: G_{K^+} \longrightarrow \bar{\mathbb{Q}}_p^\times$.

<u>Def:</u> (R, ψ) is <u>polarized</u> if $R^\vee \simeq R^c \otimes \psi|_{G_K}$

$R^c(g) = R(\hat c g \hat c^{-1})$.

$R^\vee$ contragredient.

By Schur's lemma there exists $A \in GL_n(\overline{\mathbb{Q}}_p)$ s.t.

$R^\vee = A R^c A^{-1} \Psi$, $A^T = \pm A$. This $\pm$ is the sign associated to the pair (R, Ψ).

<u>Def:</u> (R, Ψ) is <u>odd</u> if sign $= +1$.

<u>Example:</u> $\tilde{R} : G_{K^+} \to GL_n(\overline{\mathbb{Q}}_p)$ s.t. $\tilde{R}^\vee \simeq \tilde{R} \otimes \Psi$

$\Rightarrow$ $\tilde{R}$ either $\underset{\text{(generalized)}}{\wedge}$ symplectic or orthogonal.

(e.g. $\overset{n=2}{K^+} = \mathbb{Q}$, $\rho_f^\vee \simeq \rho_f \otimes \det \rho_f^{-1}$)

pres. sense of oddness

Then $(\tilde{R}|_{G_K}, \Psi)$ odd $\Leftrightarrow$ $\begin{cases} \tilde{R} \text{ symplectic and } \Psi(\tilde{c}) = -1 \\ \tilde{R} \text{ orthogonal and } \Psi(\tilde{c}) = +1 \end{cases}$

<u>Thm (Bellaiche - Chenevier):</u> π regular algebraic cuspidal polarized rep. of GL_n/K. Then $(\rho_\pi, \chi_{cycl}^{n-1})$ is odd.

<u>Thm (B.):</u> Let $p > 2$. Assume there exists $R : G_K \to GL_{2n}(\overline{\mathbb{Q}}_p)$ abs. irred., short crystalline, (R, Ψ) polarized s.t.

$\overline{R}^{ss} \simeq \overline{\rho} \oplus \overline{\rho}^{c\vee} \otimes \Psi^{-1}|_{G_K}$ for $\rho : G_K \to GL_n(\overline{\mathbb{Q}}_p)$

s.t. $\overline{\rho}$ irred. and distinct from $\overline{\rho}^{c\vee} \otimes \Psi^{-1}|_{G_K}$. Then

$p \mid \# H^1_f(G_{K^+}, A_s^{-\Psi(\tilde{c})\,\text{sgn}}(\rho) \otimes \mathbb{Q}_p/\mathbb{Z}_p(\Psi))$ ← Bloch-Kato Selmer group.

Remarks:

- $\mathrm{Ext}^1_{G_K}(\bar{\rho}^{c\vee}, \bar{\rho}) \supset H^1_f(G_K, \bar{\rho} \otimes \bar{\rho}^c)$

$$= H^1_f(G_{K^+}, As^+(\bar{\rho})) \oplus H^1_f(G_{K^+}, As^-(\rho)).$$

As^+: Asai or tensor induction rep.

$As^- = As^+ \otimes \chi_{K/K^+}$

- Coeff. $As^{-\psi(c)\,sgn} \otimes \psi$ is critical (Deligne) iff $sgn(R, \psi) = +1$.

- Theorem provides evidence for Bloch-Kato conjecture for $As^{\pm}(\rho)$ (in critical case existence of R should be governed by $\frac{L(As^{\pm}\pi, s)}{\Omega}$)

- for $n=1$, $\rho = \mathbb{1}_{G_K}$, $\psi = \chi_{cycl}^{m-1}$ this recovers Ribet's situation as $As^+(\mathbb{1}_{G_K}) = \mathbb{1}_{G_{\mathbb{Q}}}$

B
- proof compares $\tilde{c}$-involution on $H^1(G_K, \rho \otimes \rho^c \otimes \psi)$ with involution coming from polarization and use result of Bellaiche - Chenevier on ext $\begin{pmatrix} \bar{\rho} & * \\ 0 & \bar{\rho}^{c\vee} \otimes \psi^{-1} \end{pmatrix}$ constructed like Ribet.

Fo Ma style conj:

Let $R: G_K \to GL_{2n}(\overline{\mathbb{Q}}_p)$ irred. and short crystalline.

$\psi: G_{K^+} \to \overline{\mathbb{Q}}_p^\times$

(R, ψ) polarized

Then $R \simeq \rho_\pi$ for auto. rep. π.

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Results: • [BLGGT] for $\bar{R}$ irred and odd prove potential automorphy.

• Thorne for $\bar{R}$ reducible of Schur type and odd

(not as in thm)

proves automorphy.

Cor: Let R be as in the theorem. Then Vandiver conj. generalization ($p \nmid \#$ non-critical $H^1_f(\mathrm{As}^{\pm}(\rho))$) $\Rightarrow$ (R, ψ) odd.