Local Langlands for simple supercuspidal representations of $SO(2l+1, F)$:

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I. Preliminaries

F = p-adic field, $\mathcal{O} \supseteq \wp \ni \varpi$

$\psi : F \to \mathbb{C}^\times$
$\psi|_\wp = 1$
$\psi|_\mathcal{O} \neq 1$.

LLC for $GL(n, F)$:

$\{ \mathrm{Gal}(\bar{F}/F) \to GL_n(\mathbb{C}) \}$ really should be W_F' Weil-Deligne group, but for this talk we will ignore this.

$\updownarrow \leftarrow$! bijection

$\{$ irred adm representations of $GL_n(F) \}$

Here $\rho \mapsto \pi_\rho$ satisfying:

- $n = 1$ gives Artin map from LCFT.
- $L(s, \rho) = L(s, \pi_\rho)$
- $\varepsilon(s, \pi_\rho, \psi) = \varepsilon(s, \rho, \psi)$.

This is a theorem of Harris-Taylor, Henniart.

Q: What is the bijection?

A: Bushnell-Henniart several papers except bad primes are missing (for GL_n).

LLC more generally (conjecture at this point): $\exists$! bijection

$\{ \mathrm{Gal}(\bar{F}/F) \to {}^L G \}$

$\updownarrow$

$\{$ finite unions of irred. adm reps of $G(F) \}$

G connected reductive p-adic group.
($SL(n,F)$, $Sp(2n,F)$, $SO(2l+1,F)$, $SO(2l,F)$, exceptional groups)

Ex:

$G(F)$	${}^L G$
$Sp(2n, F)$	$SO(2l+1, \mathbb{C})$
$SO(2l+1, F)$	$Sp(2n, \mathbb{C})$
$SO(2l, F)$	$SO(2l, \mathbb{C})$

[redacted]

II. Simple supercuspidal reps of $SO(2l+1, F)$: Moshe

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and $\gamma(s, \pi \times \tau, \psi) = \gamma(s, \Pi \times \tau, \psi)$ for all reps. τ of $GL(1,F)$, $GL(2,F), \ldots, GL(N-1,F)$.

__Soudry:__ Let π be a (generic) representation of $SO(2l+1, F)$, τ a (generic) representation of $GL(n,F)$.

π generic means $\pi \hookrightarrow \mathrm{Ind}_U^G \psi$ $= \{ f : G \to \mathbb{C} : f(ug) = \psi(u) f(g) \}$

$$\psi : \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \to \mathbb{C}^\times$$

U

$u \longmapsto \psi(u_{12} + \cdots + u_{l\,l+1})$

τ generic means $\tau \hookrightarrow \mathrm{Ind}_{U_n}^{GL(n,F)} \psi$

$W \in \mathrm{Im}(\pi \hookrightarrow \mathrm{Ind}_U^G \psi)$ Whittaker function

__Specialize__ $n = 1$ $\quad \tau : GL(1,F) \to \mathbb{C}^\times$

Define __zeta integral__

$$\Phi(W, \tau) = \int_{F^\times} \int_{\bar{X}} W(\bar{x} h) \tau(h) \, d\bar{x} \, dh$$

$u \backslash SO(2n)$

$\bar{X}$ = space of unipotent matrices.

$$\Phi^*(W, \tau) = \Phi(W, M(\tau, s)\tau)$$

$\exists !$ function $\gamma(s, \pi \times \tau, \psi)$ s.t.

__Thm (Soudry):__ $\Phi(W,\tau) \underbrace{\gamma(s, \pi \times \tau, \psi)}_{\text{indep. of } s \text{ for } n=1} = \Phi^*(W, \tau)$

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Can define $W(g) = \begin{cases} \psi(u)\,\chi(k) & \text{if } g = uk \in UK \\ 0 & \text{o/w.} \end{cases}$

This is in image $(\pi \hookrightarrow \mathrm{Ind}_U^G \psi)$.

$$\Phi(W,\tau) = \int_{F^\times} \int_{F^{\ell-1}} W\begin{pmatrix} a & & & & \\ ax_1 & 1 & & & \\ \vdots & & \ddots & & \\ \vdots & & & 1 & \\ \vdots & & \cdots & -ax_1 & a^{-1} \end{pmatrix} \tau(a)\, dx_1 \ldots dx_{\ell-1}\, da$$

__Thm:__ (-) Let τ be a tamely ramified character of $F^\times$ $(\tau|_{1+\mathfrak{p}} = 1)$.

Then $\gamma(s, \pi_\chi \times \tau, \psi) = \tau(-1)^\ell\, \tau(\varpi)\, \chi(g_\chi)\, q^{1/2 - s}$.

__Thm__ (-, Liu): $\exists!$ supercuspidal Π_χ of $GL(2\ell, F)$ s/t.

$\gamma(s, \Pi_\chi \times \tau, \psi) = \gamma(s, \tilde{\pi}_\chi \times \tau, \psi)$. for all tamely ramified chars. τ of $GL(1,F)$.

__Cor:__ Π_χ is the lift of $\tilde{\pi}_\chi$.