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Level stripping of genus 2 Siegel modular forms

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Abstract

In this dissertation we consider stripping primes from the level of genus 2 cuspidal Siegel eigenforms. Specifically, given an eigenform of level $N\ell^r$ which satisfies certain mild conditions, where $\ell \nmid N$ is a prime, we construct an eigenform of level N which is congruent to our original form. To obtain our results, we use explicit constructions of Eisenstein series and theta functions to adapt ideas from a level stripping result on elliptic modular forms. Furthermore, we give applications of this result to Galois representations and provide evidence for an analog of Serre's conjecture in the genus 2 case.

For Anna and Evelyn

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Chapter 1

Introduction

In modern number theory, a primary object of interest is the absolute Galois group of the rationals, i.e., $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. As this group tends to be quite unapproachable by any direct means, it is necessary to consider more sophisticated techniques. For instance, throughout this dissertation we will be broadly interested in using representation theory to extract information about $G_{\mathbb{Q}}$. In particular, we will consider representations of $G_{\mathbb{Q}}$, called Galois representations, which arise from certain automorphic forms. This method for studying $G_{\mathbb{Q}}$ has become commonplace over the past fifty to sixty years. In fact, all techniques of this type used to better understand $G_{\mathbb{Q}}$ can be fit into a much more general framework known as the Langland's program, which is one of the primary engines driving modern number theory.

For motivation, we consider the simplest case, i.e., the one dimensional complex Galois representations. To be more precise, these are continuous homomorphisms of the form

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_1(\mathbb{C}).$$

A natural question to ask about these representations, and a question which we will

return to many times is, how many of these representations arise from automorphic forms? We will not be overly concerned with making this question and the results surrounding more precise in this setting, as it is well documented in the literature, but we will consider its higher dimensional analogues in considerable depth in Chapter 5.

We begin by simply considering the basic properties of ρ. As a result of the continuity condition, one can show that the image of ρ is finite (see Chapter 5). By the Kronecker-Weber theorem, we have that the map ρ factors through the finite Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, where N is some positive integer and ζ_N is a primitive N^{th} root of unity. As $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times$, we see that ρ can be viewed as an element of the dual group of $(\mathbb{Z}/N\mathbb{Z})^\times$, i.e., ρ can be viewed a Dirichlet character modulo N. As this process is reversible, we have a bijection between one dimensional complex representations of $G_\mathbb{Q}$ and Dirichlet characters.

In order to connect this classification of one dimensional complex representations of $G_\mathbb{Q}$ with the theory of automorphic forms, we have the groundbreaking work done in Tate's thesis ([70]). Without saying too much, Tate's thesis gives, among other things, that there is a bijection between the set of all Hecke characters of finite order and the set of all Dirichlet characters. In the one dimensional case, it is the Hecke characters which play the role of the automorphic forms. While a detailed explanation of what this means and the implications thereof would take us too far afield, the interested reader is referred to Section 3.1 in [15] for an exposition of Tate's thesis and Section 2.1 in [29] for a particularly nice interpretation of Hecke characters as automorphic forms. In summary, to answer the rough question given above, we have that all one dimensional complex Galois representation arise from automorphic forms.

As was mentioned previously, these one dimensional complex representations always have finite image. As $G_\mathbb{Q}$ is far from a finite group, it stands to reason that

we have lost considerable information by considering only these representations. As will be discussed later, due to the continuity condition, there is a richer theory of one dimensional Galois representations whose image is contained in the ℓ-adic numbers or a finite extension thereof, where ℓ is a rational prime. However, these one dimensional ℓ-adic representations can still only capture the Abelian structure of $G_{\mathbb{Q}}$, and hence considerable information is still lost. In order to capture any non-Abelian structure of $G_{\mathbb{Q}}$, we must consider higher dimensional Galois representations. The natural next step is to consider two dimensional Galois representations.

There are several known constructions of these two dimensional Galois representations. In particular, from the work of Deligne [20], we can construct a two dimensional Galois representation from an elliptic Hecke eigenform whose image lies in some ℓ-adic number field. We will refer to such a Galois representation as modular. In this case, the determinant and trace of the representation can be expressed in terms of the Hecke eigenvalues. Unlike in the one dimensional setting, this construction of Deligne falls far short of providing us with all two dimensional Galois representations. For example, if the determinant of the image of complex conjugation under the representation is 1, then such a representation cannot arise from such a construction. While we know that we will not obtain all such representations, one can still ask, which Galois representations do arise from elliptic Hecke eigenforms? Note, here elliptic Hecke eigenforms are playing the role of the appropriate automorphic forms just as the Hecke characters did in the previous discussion.

By moving to the two dimensional this question has already become much harder to answer. In fact, at the writing of this dissertation, it is still an open problem and an active area of research. However, considerable progress has been made on this question, with far reaching consequences. The most notable example, perhaps, is the proof of the Taniyama-Shimura conjecture, see [13], [72], and [83]. The content of

their proof being that they show that a certain large class of two dimensional Galois representations, namely those arising from elliptic curves, are in fact modular. As a direct application to classical number theory, the proof of this conjecture implies Fermat's last theorem.

As we have mentioned, general two dimensional Galois representations are quite difficult to work with. In order to make things a bit easier to, we consider two dimensional residual representations, i.e., two dimensional representations whose image lies in the algebraic closure of some finite field. To this end, let K be a number field with ν a prime lying over ℓ and let

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_{K_\nu})$$

be a Galois representation, where K_ν is the completion of K at ν. Let $\overline{\rho}$ denote the residual representation of ρ obtained by composition with the map $\mathrm{GL}_2(\mathcal{O}_{K_\nu}) \to \mathrm{GL}_2(\mathbb{F}_\nu)$, where $\mathbb{F}_\nu$ is the residue field of $\mathcal{O}_K$ at ν. Then, it makes sense to ask the analogous question, which two dimensional Galois representations of the form

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\nu)$$

arise as the residual representation of a modular Galois representation?

By passing to the residual representation, this becomes an easier question. In fact, by Serre's conjecture ($3.2.4_?$,[64]), which is now known to be a theorem ([43]), we know precisely the conditions necessary for the semisimplification of $\overline{\rho}$ to be modular. Furthermore, Serre's refined conjecture ($3.2.4_?$,[64]) tells us the precise character, level, and weight of such an eigenform. Note, the equivalence of Serre's conjecture and Serre's refined conjecture is known by the work of Coleman-Voloch [18], Gross

[30], Ribet [59], and others (see [22]). In the process of proving this equivalence, Ribet presented the following result which we will be interested in extending.

Theorem 1. *[60, Theorem 2.1]*
Let $\ell \geq 3$ be a prime. Suppose that f is an eigenform of level $N\ell^r$ with $r > 0$ and $(N, \ell) = 1$. Then, there exists an eigenform of level N whose eigenvalues away from the level of f are congruent to the eigenvalues f modulo ℓ.

In Chapter 3, we give a detailed proof of this result, as the techniques will be quite similar to the proof of the main result of this dissertation.

In summary, we see that while we are not able to provide any answer to the question regarding general two dimensional Galois representations, we do have quite a satisfying theory of two dimensional residual representations. In this dissertation, we will be interested in adapting some of this theory of two dimensional residual representations to the four dimensional setting.

In order to transfer to the setting we will primarily be interested in, we let

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}_4(\mathcal{O}_{K_\nu})$$

be a Galois representation, where we are keeping the same notation as above. In this setting there is a conjecture of Herzig and Tilouine which serves as an analogue to Serre's conjecture. In particular, this conjecture gives conditions on when the residual representation of ρ should arise from a genus 2 Siegel eigenform, see Section 5.3 for details. Given a conjecture of this form it is natural to want some type of refined conjecture to make precise the character, level, and weight of such an eigenform. The desired weight is discussed in detail in [33]. Concerning the level, a natural starting place is Theorem 55, which is our analogue to Theorem 1 in the genus 2 setting.

Similar results have been obtained using an extension of Hida theory to Siegel modular forms, see Theorem 3.2 in [71]. However, the proof of Theorem 36 given by Ribet uses strictly classical methods. It is this approach which we adapt to the genus 2 setting. Note, we also do not require an ordinarity assumption for our argument, which is necessary for the proof of Theorem 3.2 in [71].

While we are interested in the applications of such a result to Galois representations, the proof of our main result is contained wholly within the realm of the theory of modular forms, just as the proof of Theorem 1 is. For this reason, we spend considerable time in the realm of modular forms. Once we have the tools necessary for a our main result, the application to Galois representations comes almost as a corollary, as we shall see.

Chapter 2

Modular forms and Hecke operators

2.1 Elliptic modular forms

In this section we give an introduction to the theory of elliptic modular forms. This section will provide us with a framework, as well as motivation, for studying the more general theory of Siegel modular forms in Section 2.3. For more details concerning the theory of elliptic modular forms, the reader is referred to [23], [46], and [55].

We begin by introducing some notation that will be used throughout. For a ring R we will use $\mathrm{M}_n(R)$ to denote the set of $n \times n$ matrices with entries coming from R. We set $\mathrm{GL}_n(R)$ to be the subset of $\mathrm{M}_n(R)$ whose elements have unit determinant. Note, we will also use $\mathrm{GL}(V)$ to denote the automorphisms of a vector space V, though we use this notation to signify that we have not chosen a basis for this vector space. Furthermore, we use $\mathrm{SL}_n(R)$ to denote the subset of $\mathrm{GL}_n(R)$ having determinant 1_R.

The Poincare upper half plane is defined by

$$\mathfrak{h}_1 = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}.$$

It is well known that the group $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathfrak{h}_1$ via fractional linear transformation, i.e.,

$$\gamma \cdot \tau = \frac{a_\gamma \tau + b_\gamma}{c_\gamma \tau + d_\gamma},$$

where $\tau \in \mathfrak{h}_1$, $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$, and we are using the $+$ to denote elements of $\mathrm{GL}_2(\mathbb{R})$ having positive determinant. Note, throughout we will denote our matrices by

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix},$$

and drop the subscript when the matrix is clear from context.

For our purposes we are primarily interested in the action of certain subgroups of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{h}_1$. These subgroups of interest are given by

$$\Gamma_0^1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1^1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^1(N) : a \equiv d \equiv 1 \pmod{N} \right\},$$

$$\Gamma^1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1^1(N) : b \equiv 0 \pmod{N} \right\},$$

where N is a positive integer. We refer to any subgroup of $\mathrm{SL}_2(\mathbb{Z})$ which contains

$\Gamma^1(N)$ as a level N congruence subgroup. Note, $\Gamma_0^1(N), \Gamma_1^1(N)$ are both congruence subgroups of level N. Furthermore, as $\Gamma^1(1) = \mathrm{SL}_2(\mathbb{Z})$, we refer to $\mathrm{SL}_2(\mathbb{Z})$ as the level 1 congruence subgroup. The need for the superscript 1 will become clear later when we consider subgroups of $\mathrm{Sp}_{2n}(\mathbb{Z})$.

When considering congruence subgroups, we are able to extend the action on $\mathfrak{h}_1$ to an action on $\mathfrak{h}_1^* = \mathfrak{h}_1 \cup \mathbb{P}^1(\mathbb{Q})$ in a natural way. It is clear that the set $\mathbb{P}^1(\mathbb{Q})$ is stable under the action of any congruence subgroup. Furthermore, if we fix a congruence subgroup Γ, then we refer to an equivalence class of elements in $\mathbb{P}^1(\mathbb{Q})$ as a cusp. We denote the cusps by $\{a/b\}_\Gamma$ or $\{\infty\}_\Gamma$. We have that $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$, hence $\mathfrak{h}_1^*$ has a unique cusp under the action of $\mathrm{SL}_2(\mathbb{Z})$.

Let k be a positive integer and let $f : \mathfrak{h}_1^* \to \mathbb{C}$ be a function. We have an action of the group $\mathrm{GL}_2^+(\mathbb{Q})$ on f via the weight k slash operator which is defined by

$$(f|_k\gamma)(\tau) = (\det\gamma)^{\frac{k}{2}}(c\tau + d)^{-k} f(\gamma \cdot \tau).$$

We will often drop the k from the subscript when it is clear.

Suppose that f is invariant under the weight k action of a congruence subgroup Γ, i.e., $(f|_k\gamma)(\tau) = f(\tau)$ for all $\gamma \in \Gamma$. There is a minimal positive integer h such that

$$\gamma_h = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma.$$

For example, the congruence subgroups $\Gamma_0^1(N)$ and $\Gamma_1^1(N)$ each contain the element γ_1. Combining the existence of such a matrix with the invariance of f we obtain

$$f(\tau) = (f|\gamma_h)(\tau) = f(\gamma_h \cdot \tau) = f(\tau + h).$$

From this we have that f has a Fourier expansion of the form

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_f(n) \exp\left(\frac{n\tau}{h}\right),$$

where $\exp(\cdot) = e^{2\pi i(\cdot)}$. We call this the Fourier expansion at $\{\infty\}$.

For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ it is clear that $(f|\gamma)$ is invariant under the action of $\gamma^{-1}\Gamma\gamma$ and that $\Gamma^1(N) \subseteq \gamma^{-1}\Gamma\gamma$ if $\Gamma^1(N) \subseteq \Gamma$. From this it follows that $(f|\gamma)$ has a Fourier expansion as well. Note, the matrix γ sends the cusp $\{\infty\}$ to the cusp $\{a_\gamma/b_\gamma\}$. For this reason we refer to the Fourier expansion of $(f|\gamma)$ as the Fourier expansion of f at the cusp $\{a/b\}$. If the Fourier expansion of f at the cusp $\{a/b\}$ satisfies $a_{f|\gamma}(n) = 0$ when $n < 0$ we say that f is holomorphic at the cusp $\{a/b\}$ and if $a_{f|\gamma}(n) = 0$ when $n \leq 0$, we say that f vanishes at the cusp $\{a/b\}$. We are now prepared to define elliptic modular forms.

Definition 2. Let k be a positive integer and let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup of level N. Let

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$$

be a Dirichlet character and consider χ as a character of Γ by $\chi(\gamma) = \chi(d)$. Let $f : \mathfrak{h}_1^* \to \mathbb{C}$ be a holomorphic function satisfying $(f|_k\gamma) = \chi(\gamma)f$ for every $\gamma \in \Gamma$. Then, we say that f is an *elliptic modular form of character χ, level Γ, and weight k*. Furthermore, if f vanishes at every cusp, we say that f is a *cusp form*.

Note, we may also say that f is of level N when the congruence subgroup is clear. We denote the space of character χ, level Γ, weight k elliptic modular forms by $M_k^1(\Gamma, \chi)$ and the subspace of cusp forms by $S_k^1(\Gamma, \chi)$. In the setting that $\Gamma = \Gamma_0^1(N)$, we will use the notation $M_k^1(N, \chi) := M_k^1(\Gamma, \chi)$, and similarly for the space of cusp forms. Furthermore, we will often drop the character from the notation when the

form in question is invariant under the slash operator for some congruence subgroup, i.e., we say $f \in M_k(\Gamma)$ if $f|_k\gamma = f$ for all $\gamma \in \Gamma$ and Γ is any congruence subgroup.

Note, as $-I \in \Gamma_0^1(N)$ for all N, it is immediate that $M_k^1(N,\chi) = 0$ if $\chi(-1) \neq (-1)^k$. Furthermore, it is a basic fact that $M_k(N,\chi)$ is a finite-dimensional $\mathbb{C}$-vector space for every k, N, and χ. For example, see Proposition 3 in [85].

The following proposition gives a decomposition of the space $M_k(\Gamma_1^1(N))$. Note, we can drop the character from this notation because $\chi(\gamma) = 1$ for any χ defined modulo N and for all $\gamma \in \Gamma_1^1(N)$.

Proposition 3. *[46, Prop. 28]*

$$M_k(\Gamma_1^1(N)) = \bigoplus_{\chi \,(\mathrm{mod}\ N)} M_k^1(N,\chi),$$

where the direct sum is over all Dirichlet character modulo N.

Due to this proposition, we will frequently restrict our attention to the spaces $M_k^1(N,\chi)$.

Furthermore, from the transformation property, for $f \in M_{k_1}^1(N,\chi)$ and $g \in M_{k_2}^1(N,\chi)$, we have the product fg is in $M_{k_1+k_2}^1(N,\chi)$, i.e., the sum

$$\bigoplus_{k\geq 2} M_k^1(N,\chi),$$

forms a graded $\mathbb{C}$-algebra.

Example 4. We give the simplest non-trivial example of a modular form. Let $k \geq 4$ be an even integer and consider the following series

$$G_k(\tau) = \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k}, \tag{2.1}$$

where the summation is over all pairs of integers m, n not both zero. As $k \geq 4$, the summation is absolutely and uniformly convergent on compact subsets of $\mathfrak{h}_1$. Hence, the series is holomorphic on $\mathfrak{h}_1$. If we set

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then one can easily show that $G_k|_k S = G_k$ and $G_k|_k T = G_k$. As S and T generate $\mathrm{SL}_2(\mathbb{Z})$, we conclude that G_k transforms like a modular form of weight k under the action of $\mathrm{SL}_2(\mathbb{Z})$. The last condition that must be checked is the holomorphicity of G_k at the cusp $\{\infty\}$. In order to show this, we can simply write down the Fourier expansion of G_k and notice that it has no negative terms. By Proposition III.6 in [46], we have

$$G_k(\tau) = 2\zeta(k)\left(1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)\exp(n\tau)\right),$$

where B_k denotes the k^{th} Bernoulli number and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. Thus, $G_k \in M_k^1(\mathrm{SL}_2(\mathbb{Z}))$, i.e., G_k is a modular form of level 1 and weight k. As there is only one character of conductor 1, we need not specify characters of level 1 forms.

For reasons which will become clear later in the section, we will be more interested in the Eisenstein series,

$$E_k(\tau) = \frac{1}{2\zeta(k)}G_k(\tau),$$

which is just a normalization of G_k so that $a_{E_k}(0) = 1$. Furthermore, one can show

that we have an alternate expression for $E_k(\tau)$ which is reminiscent of Equation 2.1,

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{m,n\mathbb{Z} \\ \gcd(m,n)=1}} \frac{1}{(m\tau+n)^k}.$$

This example will be important for us in the later chapters.

We will now proceed to an important family of linear operators, known as Hecke operators, which act on the space of modular forms. First, we will need a few preliminaries.

Let $\Gamma_1, \Gamma_2 \subset \mathrm{SL}_2(\mathbb{Z})$ be congruence subgroups, and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. The set

$$\Gamma_1 \alpha \Gamma_2 = \{\gamma_1 \alpha \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}$$

is called a double coset in $\mathrm{GL}_2^+(\mathbb{Q})$. Note, the group Γ_1 acts on the double coset $\Gamma_1 \alpha \Gamma_2$ by left multiplication. This action gives a decomposition into finitely many cosets, i.e.,

$$\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2 = \bigcup_i \Gamma_1 \beta_i.$$

Using this decomposition, we have the following definition.

Definition 5. Let $\Gamma_1, \Gamma_2, \alpha$ be as above. Let $f \in M_k^1(\Gamma_1)$. We define the *double coset operator*, denoted $\Gamma_1 \alpha \Gamma_2$, to be

$$f[\Gamma_1 \alpha \Gamma_2]_k = \sum_i f|_k \beta_i.$$

We list a few properties of this operator, all of which are easy to show.

1. $[\Gamma_1 \alpha \Gamma_2]_k : M_k^1(\Gamma_1) \to M_k^1(\Gamma_2)$, and furthermore maps the subspace of cusp forms to itself.

2. Suppose $\Gamma_2 \subset \Gamma_1$, and take $\alpha = I_2$, the 2×2 identity matrix. Then, $f[\Gamma_1 \alpha \Gamma_2]_k = f$ is the natural inclusion of $M_k^1(\Gamma_1)$ in $M_k^1(\Gamma_2)$.

3. Suppose $\Gamma_2 = \alpha^{-1}\Gamma_1\alpha$. Then, $f[\Gamma_1\alpha\Gamma_2]_k = f|_k\alpha$.

4. Suppose $\Gamma_1 \subset \Gamma_2$, and take $\alpha = I_2$. Let $\{\gamma_i\}$ be a set of coset representatives for $\Gamma_1\backslash\Gamma_2$. Then,

$$f[\Gamma_1\alpha\Gamma_2]_k = \sum_i f|_k\gamma_i.$$

This is known as the trace map, which we will discuss in more detail in Section 3.3.

We are now prepared to introduce Hecke operators, which are special types of double coset operators. Throughout the discussion on Hecke operators we set $\Gamma = \Gamma_1^1(N)$, and let $f \in M_k^1(\Gamma)$. Note, by Property 1 from above we have that $f[\Gamma\alpha\Gamma]_k \in M_k^1(\Gamma)$.

The first type of Hecke operator which we will consider is sometimes called the "diamond operator." This is given by letting α be any element of $\Gamma_0^1(N)$. It is not hard to show that $\Gamma \lhd \Gamma_0^1(N)$, and we can apply Property 3 from above to obtain $f[\Gamma\alpha\Gamma]_k = f|_k\alpha$. Thus, we have an action of $\Gamma_0^1(N)$ on the space $M_k(\Gamma)$. As this space is invariant under the action of Γ by definition, we have that the action of α is completely determined by its coset in $\Gamma\backslash\Gamma_0^1(N)$. One can show that this coset group is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times$, and that $\delta \in (\mathbb{Z}/N\mathbb{Z})^\times$ acts by

$$\langle\delta\rangle f = f|_k\alpha, \text{ for any } \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^1(N) \text{ with } d \equiv \delta \pmod{N}.$$

Recall, we have the decomposition

$$M_k^1(\Gamma) = \bigoplus_{\chi \,(\mathrm{mod}\ N)} M_k^1(N, \chi).$$

It is immediate that each space in this decomposition is an eigenspace for $\langle d \rangle$ with eigenvalue $\chi(d)$ for all $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, i.e., the diamond operator respects this decomposition. As we will typically be restricting ourselves to one of these subspaces, the action of the diamond operator will be completely determined by the corresponding character.

The second type of Hecke operator which we will consider is given by setting

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \text{ for a prime } p.$$

We will denote this operator as $T(p)f = f[\Gamma\alpha\Gamma]_k$. We can give the action of this operator more explicitly by using the coset representatives for $\Gamma\backslash\Gamma\alpha\Gamma$. If $p|N$, then a complete set of coset representatives is given by the set

$$\left\{ \begin{pmatrix} 1 & u \\ 0 & p \end{pmatrix} : u = 0, \ldots, p-1 \right\}.$$

If $p \nmid N$ then we have the previous set along with the following additional representative

$$\begin{pmatrix} x & y \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \text{ where } px - Ny = 1.$$

Using these representatives, one immediately obtains the following proposition which describes the explicit action of the Hecke operators on the Fourier expansion of f.

Proposition 6. *[23, Prop. 5.2.2] Let* $f(\tau) = \sum a_f(n)\exp(n\tau) \in M_k^1(N,\chi)$. *Then,*

$$a_{T(p)f}(n) = a_f(np) + \chi(p)p^{k-1}a_f(n/p),$$

where $a_f(n/p) = 0$ *if* $n/p \notin \mathbb{Z}$.

As a corollary to this proposition we have the following commutativity property.

Corollary 7. *[23, Prop. 5.2.4] Let* p, q *be distinct primes and let* d, e *be distinct elements of* $(\mathbb{Z}/N\mathbb{Z})^\times$. *Then,*

1. $\langle de\rangle = \langle d\rangle\langle e\rangle = \langle e\rangle\langle d\rangle$,
2. $\langle d\rangle T(p) = T(p)\langle d\rangle$,
3. $T(p)T(q) = T(q)T(p)$.

The Hecke operators $T(p)$ can be extended to $T(n)$ for any positive integer n by setting $T(pq) = T(p)T(q)$ for p, q distinct primes, and $T(p^r) = T(p)T(p^{r-1}) - p^{k-1}\langle p\rangle T(p^{r-2})$ for $r \geq 2$. Note, $T(1)$ is the identity map. This construction makes the collection of all Hecke operators into a $\mathbb{Z}$-algebra. For completeness we present the following proposition, which is analogous to Proposition 6 for the operators $T(n)$.

Proposition 8. *[23, Prop 5.3.1] Let* $f(\tau) = \sum a_f(n)\exp(n\tau) \in M_k^1(N,\chi)$. *Then,*

$$a_{T(n)f}(m) = \sum_{d|(m,n)} \chi(d)d^{k-1}a_f(mn/d^2).$$

In order to take full advantage of the structure of the Hecke operators, we will need the following definition.

Definition 9. Let Γ be a congruence subgroup. The *Petersson inner product* is a map

$$\langle \cdot, \cdot \rangle_\Gamma : S_k^1(\Gamma) \times S_k^1(\Gamma) \to \mathbb{C},$$

which is given by

$$\langle f, g \rangle_\Gamma = \frac{1}{V_\Gamma} \int_{\Gamma \backslash \mathfrak{h}_1^*} f(\tau) \overline{g(\tau)} y^k \frac{dxdy}{y^2},$$

where $V_\Gamma = \frac{\pi}{3}[\mathrm{SL}_2(\mathbb{Z}) : \{\pm 1_2\}\,\Gamma]$.

This definition makes the space of cusp forms into an inner product space. Furthermore, one can show that for this inner product to converge, it is sufficient for fg to vanish at each cusp. In particular, either f or g is allowed to be a modular form which is not necessarily a cusp form. For our purposes we will only need this inner product to state the following theorem.

Theorem 10. *[23, Thm. 5.5.3] In the space $S_k^1(\Gamma_1^1(N))$, the Hecke operators $\langle p \rangle$ and $T(p)$ for $p \nmid N$ have adjoints*

$$\langle p \rangle^* = \langle p \rangle^{-1}, \text{ and } T^*(p) = \langle p \rangle^{-1} T(p).$$

Combining this with Corollary 7 we see that the Hecke operators $T(p)$ and $\langle p \rangle$ on the space $S_k^1(\Gamma_1^1(N))$ are normal when $p \nmid N$. Hence, by applying the spectral theorem for normal operators, we have the Hecke operators are simultaneously diagonalizable, i.e., we can find a basis for $S_k^1(\Gamma_1^1(N))$ which consists of simultaneous eigenvectors of $T(p)$ and $\langle p \rangle$ for all $p \nmid N$. We will refer to these eigenvectors as *eigenforms*.

Let $f \in S_k^1(\Gamma_1^1(N))$ be an eigenform. We denote the eigenvalues of $\langle d \rangle$ by d_f then we have that the map $d \mapsto d_f$ defines a Dirichlet character χ modulo N, hence $f \in S_k^1(N, \chi)$. Furthermore, we can apply Proposition 8 to obtain $a_{T(n)f}(1) = a_f(n)$

for all n. If we denote the eigenvalue of f with respect to $T(n)$ by $\lambda_f(n)$, we also have $a_f(n) = \lambda_f(n)a_f(1)$ when $(n, N) = 1$.

This implies that if $a_f(1) = 0$, then $a_f(n) = 0$ for all $(n, N) = 1$. Suppose this is not the case, i.e., suppose $a_f(1) \neq 0$. Then we can normalize f so that $a_f(1) = 1$ by just dividing through by $a_f(1)$. We call such a form a *normalized eigenform*, and we continue to denote this normalization by f. Then, by Proposition 8, we have that f is an eigenform for $T(n)$ for all n. Furthermore, the $T(n)$-eigenvalue of f is given by the corresponding Fourier coefficient.

Finally, for a normalized eigenform f, we define the Hecke field of f by

$$\mathbb{Q}(f) := \mathbb{Q}\left(\left\{a_f(n) : n \in \mathbb{Z}^+\right\}\right).$$

We will need the following proposition for the main result of Chapter 3.

Proposition 11. *Let $f \in S_k(N, \chi)$ be a normalized eigenform. Then, $[\mathbb{Q}(f) : \mathbb{Q}] < \infty$, i.e., $\mathbb{Q}(f)$ is a number field.*

Proof. We follow the proof of Corollary 5.3.2 in [34].

Recall that the Hecke operators $\langle n\rangle$ and $T(n)$ generate a $\mathbb{Z}$-algebra, which we denote by $\mathcal{H}^{\mathbb{Z}}$. By extending scalars, we can consider this as an algebra generated by $\langle n\rangle$ and $T(n)$ over $\mathbb{Q}$, which we will denote by $\mathcal{H}^{\mathbb{Q}} = \mathcal{H}^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Using f, we can form a $\mathbb{Q}$-algebra homomorphism

$$\lambda_f : \mathcal{H}^{\mathbb{Q}} \to \mathbb{C},$$

given by $\lambda_f(T(n)) = a_f(n)$ and $\lambda_f(\langle n\rangle) = \chi(n)$. By Theorem 3.51 in [65] we have that $\mathcal{H}^{\mathbb{Q}}$ is finitely generated as a $\mathbb{Q}$-algebra. Hence, $\lambda_f(\mathcal{H}^{\mathbb{Q}})$ is an algebra of finite dimension over $\mathbb{Q}$, and the result follows. □

Finally, to any modular form $f \in M_k(N, \chi)$, we can associate an L-function

by setting

$$L(s,f) = \sum_{n=1}^{\infty} a_f(n) n^{-s},$$

where $s \in \mathbb{C}$ and $\Re(s) > k/2 + 1$. Typically, when one has an L-function, there are certain natural properties to desire. For example, the L-function should converge in some right half plane, have an Euler product expansion, and satisfy some functional equation. Of course, the prototypical example is the Riemann zeta function, which satisfies all three of these properties. With regards to these properties for the L-functions of interest to us, we have the following theorem.

Theorem 12. *[23, Prop. 5.9.1, Prop. 5.9.2, Thm. 5.10.2]*

1. *If f is a cusp form of weight k, then $L(s,f)$ converges absolutely for all s satisfying $\Re(s) > k/2 + 1$. If f is not a cusp form, then $L(s,f)$ converges absolutely for all s satisfying $\Re(s) > k$.*

2. *If f is a cusp form of level N and weight k, then*

$$L(k-s,f) = \pm \frac{N^{(2s-k)/2}\Gamma(s)}{(2\pi)^{k+2s}\Gamma(k-s)} L(s,f),$$

where $\Re(s) > k/2 + 1$ and the sign is determined by the eigenvalue of f with respect to the operator W_N where

$$f|W_N = i^k N^{1-k/2} f|_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

3. *If f is a normalized eigenform of character χ and weight k then*

$$L(s,f) = \prod_p L_p(p^{-s}, f)^{-1},$$

where $L_p(X, f) = (1 - a_f(p)X + \chi(p)p^{k-1}X^2)$ *and the product is taken over all primes.*

These L-functions will be useful in explaining some applications to Galois representations.

2.2 Jacobi forms

In this section, we introduce Jacobi forms, which will be needed in the proof of our main result. As we will only need a few basic facts concerning Jacobi forms, we do not give a complete treatment here. For further details on this topic, the standard reference is [24].

Let $\phi : \mathfrak{h}_1 \times \mathbb{C} \to \mathbb{C}$ be a holomorphic function. We say that ϕ is a Jacobi form, roughly speaking, if ϕ behaves like a modular form when restricted to $\mathfrak{h}_1$ and like an elliptic function when restricted to $\mathbb{C}$. In order to make this precise, we will first need to define an appropriate group action on $\mathfrak{h}_1 \times \mathbb{C}$.

We define an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{h}_1 \times \mathbb{C}$ by

$$\gamma \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right). \tag{2.2}$$

It is not difficult to check that this is a group action. Second, we define an action of $\mathbb{Z}^2$ on $\mathfrak{h}_1 \times \mathbb{C}$ by

$$(\lambda, \mu) \cdot (\tau, z) = (\tau, z + \lambda\tau + \mu). \tag{2.3}$$

It is also not difficult to verify that this is a group action, where the group law on $\mathbb{Z}^2$ is simply component-wise addition. In fact, combining these two actions, we have a group action of the semidirect product $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, i.e., the Cartesian product of

$\mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{Z}^2$ with group law

$$(\gamma, (\lambda, \mu))(\gamma', (\lambda', \mu')) = (\gamma\gamma', (\lambda, \mu)\gamma' + (\lambda', \mu')).$$

Note, this action still makes sense if we replace $\mathrm{SL}_2(\mathbb{Z})$ by a subgroup, i.e., we have an action on $\mathfrak{h}_1 \times \mathbb{C}$ by $\Gamma \ltimes \mathbb{Z}^2$ for any $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$.

Let k and m be positive integers. Using this group action we define the index m, weight k slash operator on ϕ by

$$(\phi|_{k,m}\gamma)(\tau, z) = (c\tau + d)^{-k} \exp\left(\frac{-cmz^2}{c\tau + d}\right) \phi(\gamma \cdot (\tau, z)),$$

and

$$(\phi|_{k,m}(\lambda, \mu))(\tau, z) = \exp(m(\lambda^2\tau + 2\lambda z))\phi((\lambda, \mu) \cdot (\tau, z)).$$

One can check that this operator satisfies

$$(\phi|_{k,m}\gamma)|_{k,m}\gamma' = \phi(\tau, z)|_{k,m}(\gamma\gamma'),$$

$$(\phi|_{k,m}(\lambda, \mu))|_{k,m}(\lambda', \mu') = \phi|_{k,m}(\lambda + \lambda', \mu + \mu'),$$

$$(\phi|_{k,m}\gamma)|_{k,m}(\lambda, \mu)\gamma = (\phi|_{k,m}(\lambda, \mu))|_{k,m}\gamma,$$

for all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$ and $(\lambda, \mu), (\lambda', \mu') \in \mathbb{Z}^2$.

Given this slash operator, we are prepared to give the definition of a Jacobi form.

Definition 13. Let k, m, N be positive integers, and let χ be a Dirichlet character modulo N. Let ϕ be as above. Suppose ϕ satisfies:

1.

$$(\phi|_{k,m}\gamma) = \chi(\gamma)\phi, \text{ for all } \gamma \in \Gamma_0^1(N);$$

2.

$$(\phi|_{k,m}(\lambda,\mu)) = \phi, \text{ for all } (\lambda,\mu) \in \mathbb{Z}^2;$$

3. for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $\phi|_{k,m}\gamma$ has a Fourier expansion of the form

$$(\phi|_{k,m}\gamma)(\tau,z) = \sum_{n=0}^{\infty} \sum_{\substack{r\in\mathbb{Z} \\ r^2\leq 4nm}} c_{\phi|\gamma}(n,r)\exp(n\tau + rz).$$

Then, we say that ϕ is a Jacobi form of character χ, index m, level N, and weight k. We denote the space of these forms by $J_{k,m}(N,\chi)$. Furthermore, if $c_{\phi|\gamma}(n,r) = 0$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and whenever $r^2 = 4nm$, we say that ϕ is a Jacobi cusp form. We denote this space by $J_{k,m}^{\mathrm{cusp}}(N,\chi)$.

We assume that that the level is $\Gamma_0^1(N)$ in the definition for simplicity. If one were to consider general congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, then it is necessary to sum n and r over the rationals with bounded denominators in the Fourier expansion.

Example 14. Just as in the previous section, the simplest non-trivial example of a Jacobi form is given by an Eisenstein series. In particular, for $k \geq 4$ we set

$$E_{k,m}^J(\tau,z) = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ \gcd(c,d)=1}} \sum_{\lambda\in\mathbb{Z}} \frac{\exp\left(\frac{m\lambda^2(a\tau+b)}{c\tau+d} + \frac{2m\lambda z}{c\tau+d} - \frac{mcz^2}{c\tau+d}\right)}{(c\tau+d)^k}.$$

Then, $E_{k,m}^J(\tau,z) \in J_{k,m}(1)$ by Theorem 2.1 in [24]. We will need this Eisenstein series in Section 4.2.

For our purposes, we will not require the theory of Hecke operators for Jacobi

forms. However, we will need a certain index raising operator. In particular, for a positive integer t and $\phi(\tau, z) \in J_{k,m}(N, \chi)$, we define

$$(\phi|_{k,m}V_t)(\tau, z) = t^{k-1} \sum_{\substack{\gamma \in \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{M}_2(\mathbb{Z}) \\ \det \gamma = t}} (c\tau + d)^{-k} \exp\left(\frac{-mtcz^2}{c\tau + d}\right) \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{tz}{c\tau + d}\right).$$

By Theorem 4.1 of [24] in the level 1 case and Lemma 3.1 of [35] for arbitrary level, we have that this operator is well-defined, i.e., is independent of coset representative choice, and moreover that $\phi|_{k,m}V_t \in J_{k,mt}(N, \chi)$. Finally, for our main result, we will need to express the action of this operator in terms of the Fourier coefficients of ϕ. To this end, we have the following theorem.

Theorem 15. *([24, Thm. 4.2],[35, §3]) Let* $\phi(\tau, z) = \sum_{n,r} c(n, r) \exp(n\tau + rz)$. *Then,*

$$(\phi|_{k,m}V_t)(\tau, z) = \sum_{n,r} \left(\sum_{a | \gcd(n,r,t)} a^{k-1} \chi(a) c\left(\frac{nt}{a^2}, \frac{r}{a}\right) \right) \exp(n\tau + rz).$$

2.3 Siegel modular forms

In this section, we give an introduction to the theory of Siegel modular forms. For clarity, we will follow the basic framework which was set up in Section 2.1. For more details, the interested reader is referred to [75] for a more complete treatment.

Define the genus n Siegel upper half plane as

$$\mathfrak{h}_n = \left\{ Z \in \mathrm{M}_n(\mathbb{C}) : {}^T Z = Z, \mathrm{Im}(Z) > 0 \right\},$$

where $\mathrm{Im}(Z) > 0$ means that the imaginary part of Z is strictly positive definite. We have an action on $\mathfrak{h}_n$ by the group of $2n \times 2n$ symplectic matrices with real entries

and positive similitude factor, i.e., by the group

$$\mathrm{GSp}_{2n}^{+}(\mathbb{R}) = \left\{\gamma \in \mathrm{M}_{2n}(\mathbb{R}) : {}^{T}\gamma J_n \gamma = \mu(\gamma) J_n, \mu(\gamma) > 0\right\},$$

where $J_n = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$ with 0_n and I_n denoting the additive and multiplicative identities of $\mathrm{M}_n(\mathbb{R})$, respectively. This action is given explicitly by

$$\gamma \cdot Z = (aZ + b)(cZ + d)^{-1}.$$

In Section 2.1, we saw that elliptic modular forms are, roughly speaking, complex valued functions which are transformed by $(c\tau + d)^k$ when acted on by elements of certain discrete subgroups of $\mathrm{GL}_2^{+}(\mathbb{R})$. This $(c\tau+d)^k$ is sometimes referred to as the "automorphy factor". We will need to generalize this notion of "automorphy" in order to define Siegel modular forms. To this end, consider an irreducible representation,

$$\rho : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(V),$$

with V some finite dimensional $\mathbb{C}$-vector space. Representations of this type have been completely classified and are, in fact, in bijective correspondence with tuples of the form $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ with $k_1 \geq k_2 \geq \cdots \geq k_n$ by Proposition 15.47 in [28]. This correspondence is obtained as follows. For each irreducible V, there exists a unique one-dimensional subspace spanned by v_ρ such that

$$\rho(\mathrm{diag}(a_1, \ldots, a_n)) \cdot v_\rho = \prod_{i=1}^{n} a_i^{k_i} \cdot v_\rho.$$

We call $(k_1, \ldots, k_n)$ the highest weight vector of ρ.

Example 16. Let $(k_1, \ldots, k_n) = (1, 0, \ldots, 0)$. Then, ρ is the standard or tautological representation on $\mathbb{C}^n$, i.e.,

$$\rho(\gamma) \cdot v = \gamma \cdot v.$$

Example 17. Let $(k_1, \ldots, k_n) = (1, \ldots, 1)$. Then, ρ is the determinant representation, i.e.,

$$\rho(\gamma) \cdot v = \det \gamma \cdot v.$$

Example 18. Let $V = \mathbb{C}x_1 \oplus \mathbb{C}x_2$ be the standard representation of $\mathrm{GL}_2(\mathbb{C})$. Then, the highest weight vector (k_1, k_2) corresponds to the representation $\mathrm{Sym}^{k_1-k_2}(V) \otimes \det^{k_2}(V)$, where $\mathrm{Sym}^k(V)$ is the k^{th} symmetric power of V, which we can identify with the space of degree $k_1 - k_2$ homogeneous polynomials in $\mathbb{C}[x_1, x_2]$.

If $(k_1, \ldots, k_n)$ and $(k'_1, \ldots, k'_n)$ are the highest weight vectors of ρ and ρ', respectively, then the highest weight vector of $\rho \otimes \rho'$ is $(k_1 + k'_1, \ldots, k_n + k'_n)$. For more details regarding the representation theory of $\mathrm{GL}_n(\mathbb{C})$ the reader is referred to [28].

Let $F : \mathfrak{h}_n \to V$ be a holomorphic function. Then, for $\gamma \in \mathrm{GSp}_{2n}^+(\mathbb{R})$, we define the weight ρ slash operator by

$$(F|_\rho \gamma)(Z) = \rho(cZ + d)^{-1} F(\gamma \cdot Z).$$

In the setting that the highest weight vector of ρ is of the form $(k, \ldots, k)$, then we denote the slash operator by $|_k$ and we have

$$(F|_k \gamma)(Z) = \det(cZ + d)^{-k} F(\gamma \cdot Z).$$

We call $|_k$ the weight k slash operator. In this setting, the representation ρ is a

one-dimensional representation, so we think of F as a map into $\mathbb{C}$.

Just as in the previous section we will be interested in functions which are invariant under the action of certain subgroups of $\mathrm{GSp}_{2n}^+(\mathbb{R})$ by the slash operator. In particular, we define $\mathrm{Sp}_{2n}(\mathbb{Z})$ to be elements of $\mathrm{GSp}_{2n}^+(\mathbb{R})$ which have integral entries and lie within the kernel of the similitude factor μ. This group serves as the analogue to the group $\mathrm{SL}_2(\mathbb{Z})$ in the setting of elliptic modular forms, and, in fact, agrees with $\mathrm{SL}_2(\mathbb{Z})$ when $n = 1$. We also have the analogues of the level N congruence subgroups in this setting, i.e., the subgroups

$$\Gamma_0^n(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Z}) : c \equiv 0_n \pmod{N} \right\},$$

$$\Gamma_1^n(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^n(N) : a \equiv d \equiv 1_n \pmod{N} \right\},$$

$$\Gamma^n(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1^n(N) : b \equiv 0_n \pmod{N} \right\},$$

where we are writing the entries as $n \times n$ blocks. When $n = 1$, this agrees with the congruence subgroups defined in Section 2.1.

We are now prepared to define Siegel modular forms.

Definition 19. Let N be a positive integer and let χ be a Dirichlet character modulo N. Let $F : \mathfrak{h}_n \to V$ be a holomorphic function and $\rho : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(V)$ be an irreducible representation. Then, we say that F is a Siegel modular form of character χ, genus n, level N, and weight ρ if

$$F|_\rho \gamma = \chi(\gamma) F, \text{ for all } \gamma \in \Gamma_0^n(N),$$

where we define $\chi(\gamma) = \chi(\det d)$. Note, if $n = 1$, we must also require holomorphicity at the cusps as in that case we are in the setting of elliptic modular forms. We denote the space of all such functions as $M_\rho^n(N, \chi)$. Furthermore, if the highest weight vector of ρ is given by $(k, \ldots, k)$ then when it is important to make the distinction we say that F is weight k instead of weight ρ, and we denote the corresponding space by $M_k^n(N, \chi)$.

If $\dim_{\mathbb{C}}(V) > 1$ then the modular forms in the definition above are typically referred to as vector-valued Siegel modular forms in the literature, and if $\dim_{\mathbb{C}}(V) = 1$ then they are typically called classical Siegel modular forms.

Similar to the elliptic setting, from results in [81], we have that for $F \in M_{\rho_1}^n(N, \chi)$ and $G \in M_{\rho_2}^n(N, \chi)$, the product $F(Z)G(Z) := F(Z) \otimes_{\mathbb{C}} G(Z)$ is in $M_{\rho_1 \otimes \rho_2}^n(N, \chi)$, and hence

$$\bigoplus_{\rho} M_\rho^n(N, \chi)$$

is a graded $\mathbb{C}$-algebra, where the sum is taken over all irreducible representations of $\mathrm{GL}_n(\mathbb{C})$.

Let $F \in M_\rho^n(N, \chi)$. Then, by the transformation property satisfied by F we have that $F(Z + S) = F(Z)$ for all symmetric $S \in \mathrm{M}_n(\mathbb{Z})$. Hence, F admits a Fourier expansion of the form

$$F(Z) = \sum_{T \in \Lambda_n} a_F(T) \exp(\mathrm{Tr}(TZ)) \text{ with } a_F(T) \in V,$$

where Λ_n denotes the set of all half-integral symmetric matrices, i.e., $2T$ is an integral matrix with even diagonal entries and $\mathrm{Tr}(TZ)$ is the trace of the matrix TZ. Note, as was mentioned in Definition 19 for $n = 1$, we have to make the restriction that F is holomorphic at the cusps, which was defined in terms of the Fourier expansion

of F. The following theorem, referred to as the "Koecher Principle", gives that this restriction is not necessary when $n > 1$.

Theorem 20. *[75, Thm. 2] Let $n > 1$ and suppose $F \in M_\rho^n(N, \chi)$. Then, $a_F(T) = 0$ if T is not positive semi-definite. In other words, F has a Fourier expansion of the form*

$$F(Z) = \sum_{\substack{T \geq 0 \\ T \in \Lambda_n}} a_F(T) \exp(\mathrm{Tr}(TZ)),$$

where we use $T \geq 0$ to mean that T is positive semi-definite.

Example 21. Once again, we have that the simplest non-trivial example of a genus n Siegel modular form is given by an Eisenstein series. In particular, for even $k > n+1$ we set

$$E_k^n(Z) = \sum_{P_{2n} \backslash \mathrm{Sp}_{2n}(\mathbb{Z})} \det(cZ + d)^{-k},$$

where P_{2n} is the Siegel parabolic subgroup consisting of all elements of $\mathrm{Sp}_{2n}(\mathbb{Z})$ with $c = 0_n$. Then, E_k^n is a genus n, level 1, weight k Siegel modular form, referred to as the Siegel Eisenstein series.

In this setting, we also have the notion of cusp forms. In order to define a cusp form properly, we introduce the following operator on the space $M_\rho^n(N, \chi)$,

$$\Phi F(Z') = \lim_{t \to \infty} F\left(\begin{pmatrix} Z' & 0 \\ 0 & it \end{pmatrix}\right),$$

where $F \in M_\rho^n(N, \chi)$, $Z' \in \mathfrak{h}_{n-1}$, and $t \in \mathbb{R}$. In fact, if ρ has highest weight vector $(k_1, \ldots, k_n)$, then $\Phi F \in M_{\rho'}^{n-1}(N, \chi)$ where ρ' has highest weight vector $(k_1, \ldots, k_{n-1})$. This brings us to the definition of a cusp form.

Definition 22. We say that $F \in M_\rho^n(N, \chi)$ is a *cusp form* if $\Phi(F|_\rho \gamma) = 0$ for all

$\gamma \in \mathrm{Sp}_{2n}(\mathbb{Z})$. We denote the subspace of cusp forms by $S_\rho^n(N,\chi)$.

As F is a holomorphic function, we can distribute this limit over the Fourier expansion to obtain

$$\Phi F(Z') = \sum_{\substack{T' \geq 0 \\ T' \in \Lambda_n}} a_F \left(\begin{pmatrix} T' & 0 \\ 0 & 0 \end{pmatrix} \right) \exp(\mathrm{Tr}(T'Z')).$$

This leads us immediately to the following lemma.

Lemma 23. *Let $F \in S_\rho^n(N,\chi)$. Then, $a_F(T) = 0$ if T is not positive definite.*

Just as in Section 2.1, we introduce the Hecke operators for Siegel modular forms. We begin by introducing the double coset operators. Let $\Gamma \subseteq \mathrm{Sp}_{2n}(\mathbb{Z})$ be any of the three congruence subgroups defined above and let $\alpha \in \mathrm{GSp}_{2n}^+(\mathbb{Q})$. Define the double coset

$$\Gamma\alpha\Gamma = \{\gamma_1\alpha\gamma_2 : \gamma_1 \in \Gamma, \gamma_2 \in \Gamma\}.$$

We have a natural action of Γ on $\Gamma\alpha\Gamma$ which gives the following decomposition into finitely many right cosets, i.e.,

$$\Gamma\backslash\Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i.$$

We denote the collection of these double cosets by $H(\Gamma)$.

Let $F \in M_\rho^n(N,\chi)$, we define the weight ρ double coset operator by

$$F[\Gamma_0^n(N)\alpha\Gamma_0^n(N)]_\rho = \sum_i \chi(\det(a_{\alpha_i}))F|_\rho\alpha_i,$$

where the summation runs over a complete set of representatives for $\Gamma\backslash\Gamma\alpha\Gamma$. We have

a natural multiplication of these double coset operators given by

$$F[(\Gamma_0^n(N)\alpha\Gamma_0^n(N))\cdot(\Gamma_0^n(N)\beta\Gamma_0^n(N))]_\rho = \sum_{i,j}\chi(\det(a_{\alpha_i\beta_j}))F|_\rho\alpha_i\beta_j,$$

which makes the collection of double coset operators into an algebra over $\mathbb{Q}$, which is called the Hecke algebra. The following proposition is quite helpful in working with elements of $H(\Gamma_0^n(N))$.

Proposition 24. *[75, Prop. 4] Let $\alpha \in \mathrm{GSp}_{2n}^+(\mathbb{Q}) \cap \mathrm{M}_{2n}(\mathbb{Z})$. Then, the double coset $\Gamma_0^n(N)\alpha\Gamma_0^n(N)$ has a unique representative of the form $\gamma = \mathrm{diag}(a_1,\ldots,a_n,d_1,\ldots,d_n)$ with integers a_j, d_j satisfying $a_j > 0, a_jd_j = \mu(\gamma)$ for all j, and $a_n|d_n, a_j|a_{j+1}$ for $j = 1,\ldots,n-1$.*

If we define H_p to be the subring of double cosets in $H(\Gamma_0^n(N))$ whose representatives have only powers of p in the denominators of the entries, then this proposition gives us that any element of $H(\Gamma_0^n(N))$ can be written as a finite product of elements, each coming from a distinct $H_p(\Gamma_0^n(N))$. In other words, we have a decomposition $H(\Gamma_0^n(N)) = \otimes_p' H_p(\Gamma_0^n(N))$, where $\otimes_p'$ is called the restricted tensor product, and means that all but finitely many elements of the product should be the identity. We will also use $H_p^{\mathbb{Z}}(\Gamma_0^n(N))$ to denote the subring of $H_p(\Gamma_0^n(N))$ whose representatives have only integral entries. We call $H_p^{\mathbb{Z}}(\Gamma_0^n(N))$ the local Hecke algebra at p. Let $H^{\mathbb{Z}}(\Gamma_0^n(N)) = \otimes_p' H_p^{\mathbb{Z}}(\Gamma_0^n(N))$. Concerning the generators of $H_p^{\mathbb{Z}}(\Gamma_0^n(N))$, we have the following theorem.

Theorem 25. *[75, Thm. 9] $H_p^{\mathbb{Z}}(\Gamma_0^n(N))$ for $p \nmid N$ is a $\mathbb{Z}$-algebra generated by the following elements*

$$T(p) = \Gamma_0^n(N)\begin{pmatrix} I_n & 0_n \\ 0_n & pI_n \end{pmatrix}\Gamma_0^n(N),$$

and,

$$T_i(p^2) = \Gamma_0^n(N) \begin{pmatrix} I_{n-i} & 0 & 0 & 0 \\ 0 & p1_i & 0 & 0 \\ 0 & 0 & p^2 I_{n-i} & 0 \\ 0 & 0 & 0 & pI_i \end{pmatrix} \Gamma_0^n(N),$$

for $1 \leq i \leq n$. *Furthermore,* $H_p(\Gamma_0^n(N)) = H_p^{\mathbb{Z}}(\Gamma_0^n(N))[1/T_n(p^2)]$.

Note, from Lemma 4.2 in [4] we have that the spaces $M_k^n(N,\chi)$, $S_k^n(N,\chi)$ are stable under the action of the Hecke operators, and it is not difficult to see that this proof extends to arbitrary weight ρ.

Suppose that $n = 2$. Recall from Example 18, we can identify the representation space V with the homogeneous polynomials $\mathbb{C}[x_1, x_2]$ of degree $k_1 - k_2$, where (k_1, k_2) is the highest weight vector of ρ. For any subring $R \subset \mathbb{C}$, let V_R denote the homogeneous polynomials in $R[x_1, x_2]$ of degree $k_1 - k_2$. Let $S_\rho^2(N,\chi)_R$ denote the subset of $S_\rho^2(N,\chi)$ whose elements have Fourier coefficients in V_R at each cusp. Note, in [36], it is shown that vector-valued modular forms satisfy a "q-expansion principle," i.e., if the Fourier coefficients at one cusp lie in V_R then so do the Fourier coefficients at all of the other cusps.

Corollary 26. *Let* $F \in S_\rho^2(N,\chi)_{\mathbb{Q}(\chi)}$. *Then,* $TF \in S_\rho^2(N,\chi)_{\mathbb{Q}(\chi)}$, *for any* $T \in H_N^{\mathbb{Z}}(\Gamma_0^2(N)) := \otimes'_{p\nmid N} H_p^{\mathbb{Z}}(\Gamma_0^2(N))$, *where* $\mathbb{Q}(\chi)$ *is defined to be the number field obtained by adjoining all of the values of* χ *to* $\mathbb{Q}$.

Proof. This result follows immediately from the formulas in Theorem 89. □

For completeness, we mention that similar results have been obtained in [36] using techniques from arithmetic geometry.

Beyond this collection of linear operators, the space $S_\rho^n(N,\chi)$ also comes equipped with an inner product, known as the Petersson inner product. We will

give the precise formulation for the genus 2 case. The reader is referred to [68] for the formulation in the case of genus n vector-valued Siegel modular forms, where one needs to change the domain integrated over in the case of non-trivial level.

Let $V = \mathbb{C}x_1 \oplus \mathbb{C}x_2$ be the standard representation of $\mathrm{GL}_2(\mathbb{C})$. This space comes with a natural inner product given by

$$\langle a_1x_1 + a_2x_2, b_1x_1 + b_2x_2\rangle = a_1\overline{b}_1 + a_2\overline{b}_2.$$

This induces an inner product on $\mathrm{Sym}^{k_1-k_2}(V)$ given by

$$\langle v_1 \dots v_{k_1-k_2}, w_1 \dots w_{k_1-k_2}\rangle = \frac{1}{(k_1-k_2)!}\sum_{\sigma\in S_{k_1-k_2}}\prod_{j=1}^{k_1-k_2}\langle v_{\sigma(j)}, w_j\rangle,$$

where $v_i, w_i \in V$ for all i. From [68] we have that this inner product satisfies

1. $\langle v, w\rangle = \overline{\langle w, v\rangle}$, for all $v, w \in \mathrm{Sym}^{k_1-k_2}(V)$.
2. $\langle \rho(\gamma_1)v, \rho(\gamma_2)w\rangle = \langle \rho({}^{T}\overline{\gamma_2}\gamma_1)v, w\rangle$, for all $\gamma_1, \gamma_2 \in \mathrm{GL}_2(\mathbb{C})$, $v, w \in \mathrm{Sym}^{k_1-k_2}(V)$, where
$$\rho : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}^{k_1-k_2}(V))$$
has highest weight vector (k_1, k_2).

Using this inner product, we define the Petersson inner product of $F, G \in M^2_\rho(N, \chi)$ with at least one a cusp form to be

$$\langle F, G\rangle_{\Gamma^2_1(N)} = \frac{1}{[\mathrm{Sp}_4(\mathbb{Z}) : \{\pm I_4\}\,\Gamma^2_1(N)]}\int_{\Gamma^2_1(N)\backslash\mathfrak{h}_2}\langle \rho(Z)F(Z), G(Z)\rangle \det(\mathrm{Im}(Z))^{-3}dZ,$$

where $\Gamma^2_1(N)\backslash\mathfrak{h}_2$ is a fundamental domain for $\Gamma^2_1(N)$.

From [6] we have that the Hecke operators are self-adjoint with respect to this

inner product in the level 1, arbitrary genus case. Furthermore, using the formulas derived in Theorem 89, this can be shown to hold for level N and genus 2 for all Hecke operators in $\mathcal{H}_N^{\mathbb{Z}}(\Gamma_0^2(N))$. These formulas are precisely the same, regardless of the level, so the self-adjointness follows immediately. From this, it follows that $S_\rho^2(N,\chi)$ has an orthogonal basis which consists of simultaneous eigenvectors for $T(p)$ and $T_i(p^2)$ for $1 \leq i \leq n$ and for all $p \nmid N$. We refer to such an eigenvector as an eigenform. By our definition of modular forms, any element of $M_\rho^2(N,\chi)$ is automatically an eigenvector for the Hecke operators $T_2(p^2)$ for $p \nmid N$ and has eigenvalue given by $\chi(p)$ up to some normalization factor. In the genus 1 setting, this is the operator from Section 2.1 which was referred to as the diamond operator and denoted by $\langle p \rangle$.

As we will eventually be interested in producing congruences, we will require an analogue to Proposition 11 for the genus 2 case. This leads us to the following theorem.

Theorem 27. *Let $F \in S_\rho^2(N,\chi)$ be an eigenform. Define $\mathbb{Q}(\lambda_F)$ to be the field generated by adjoining all of the eigenvalues of F with respect to the Hecke operators $T(p)$ and $T_i(p^2)$ for $1 \leq i \leq 2$ and $p \nmid N$. Then, $\mathbb{Q}(\lambda_F)/\mathbb{Q}$ is a totally real finite extension.*

Proof. Define $H_N^{\mathbb{Z}}(\Gamma_0^2(N)) = \otimes'_{p \nmid N} H_p^{\mathbb{Z}}(\Gamma_0^n(N))$. For any $t \in H_N^{\mathbb{Z}}(\Gamma_0^2(N))$, let $\lambda(t)$ satisfy $tF = \lambda(t)F$. Note, $\lambda(t)$ is algebraic as it is the root of the characteristic polynomial of t, and as t is self-adjoint, we have that $\lambda(t)$ is totally real.

To obtain that $\mathbb{Q}(\lambda_F)/\mathbb{Q}$ is a finite extension, we proceed as in the proofs of Theorem 1 in [49] where this lemma is proven for classical Siegel modular forms of arbitrary genus and of level 1 and Theorem 1 in [69] where this lemma is proven for vector valued Siegel modular forms of genus 2 and level 1.

By Lemma 2.1 in [71], we have that

$$S_\rho^2(N,\chi)_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathbb{C} = S_\rho^2(N,\chi),$$

where $\mathcal{O}_K$ is the ring of integers of some finite abelian extension $K/\mathbb{Q}$. Without loss of generality, we assume that $\mathbb{Q}(\chi) \subseteq K$.

Let $\mathrm{Aut}(\mathbb{C}/K)$ denote the field automorphisms of $\mathbb{C}$ which fix elements of K. Let $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$. We define

$$F^\sigma(Z) = \sum_T \sigma(a_F(T)) \exp(\mathrm{Tr}(TZ)),$$

and σ acts on $a_F(T)$ by considering $a_F(T) \in \mathbb{C}[x_1, x_2]$ and acting on the the coefficients, i.e., for $a_F(T) = \sum_{i,j} a_{ij} x_1^i x_2^j$ we have $\sigma(a_F(T)) := \sum_{i,j} \sigma(a_{ij}) x_1^i x_2^j$.

We can decompose F as the sum

$$F = \sum_n c_n (F_n \otimes z_n),$$

where $c_n \in \mathcal{O}_K$, $z_n \in \mathbb{C}$, and $F_n \in S_\rho^2(N,\chi)_{\mathcal{O}_K}$. Recall, by Corollary 26, we have that $tF_n \in S_\rho^2(N,\chi)_{\mathcal{O}_K}$ for any $t \in H_N^{\mathbb{Z}}(\Gamma_0^2(N))$. Furthermore, for any $t \in H_N^{\mathbb{Z}}(\Gamma_0^2(N))$, we have

$$tF = \sum_n c_n (tF_n \otimes z_n).$$

It follows that $(tF)^\sigma = t(F^\sigma)$ for any $t \in H_N^{\mathbb{Z}}(\Gamma_0^2(N))$. In particular, $tF^\sigma = \sigma(\lambda_F(t))F^\sigma$. We notice from this that $F^\sigma \in S_\rho^2(N, \sigma \circ \chi)$ and that $\mathbb{Q}(\lambda_{F^\sigma}) = \sigma(\mathbb{Q}(\lambda_F))$.

Let $\mathcal{B}_\chi$ denote a basis of eigenforms for $S_\rho^2(N,\chi)$ and set

$$\mathcal{B} := \bigcup_{\chi \,(\mathrm{mod}\ N)} \mathcal{B}_\chi,$$

where the union is over all Dirichlet characters modulo N. Note, $\mathcal{B}$ is a finite set. From the discussion above, we have a map

$$\mathrm{Aut}(\mathbb{C}/K) \to S_{|\mathcal{B}|},$$

where $S_{|\mathcal{B}|}$ is the symmetric group on $|\mathcal{B}|$ letters. Thus, the action of $\mathrm{Aut}(\mathbb{C}/K)$ on each the direct sum over χ of all $S^2_\rho(N,\chi)$ factors through a finite quotient. Hence, $\mathbb{Q}(\lambda_F)/\mathbb{Q}$ is a finite extension. □

Just as in Section 2.1, we can associate an L-function to a Siegel modular form. As we will only be concerned with the L-functions associated to genus 2 modular forms, we restrict our attention to that setting here. In fact, we will also assume that $F \in S^2_\rho(N,\chi)$ is an eigenform, with ρ having highest weight vector (k_1,k_2). Then, the associated L-function is given by

$$L(s,F) = \prod_{p\nmid N} L_p(p^{-s},F)^{-1} \prod_{p|N} (1-\lambda_F(p)p^{-s})^{-1},$$

with

$$\begin{aligned} L_p(X,F) &= 1-\lambda_F(p)X + (\lambda_F(p)^2 - \lambda_F(p^2;1) - \chi(p^2)p^{k_1+k_2-4})X^2 \\ &- \chi(p^2)\lambda_F(p)p^{k_1+k_2-3}X^3 + \chi(p^4)p^{2k_1+2k_2-6}X^4, \end{aligned}$$

where $T(p)F = \lambda_F(p)F$ and $T_1(p^2)F = \lambda_F(p^2;1)F$. Note, there are actually two distinct L-functions associated to F, however, the L-function presented above, referred to as the spinor L-function, is all we will be concerned with. By Theorem 1 in [2], it is known that this L-function is absolutely convergent in some right half plane and satisfies a functional equation in the scalar weight case.

Up to this point, our introduction to the theory of Siegel modular forms of higher genus has progressed, more or less, parallel to our introduction to the theory of elliptic modular forms. However, while it is possible to normalize elliptic eigenforms so that the Fourier coefficients are the Hecke eigenvalues, such a normalization does not make sense for vector valued Siegel eigenforms. While there are results which provide certain relationships between Hecke eigenvalues and Fourier coefficients ([32],[61]) in the scalar weight case, the known relationships certainly are not as satisfying as in the elliptic setting. With this in mind, we see that, in some sense, the Fourier coefficients of a Siegel eigenform do not provide as much information as in the genus 1 setting. However, there is an "alternate" Fourier expansion which can be used to provide further insights into the properties of Siegel modular forms.

We restrict our discussion to the genus 2, weight k setting, as this will avoid significant technical difficulties and will also be sufficient for our purposes. Let

$$F(Z) = \sum_{T} a_F(T) \exp(\mathrm{Tr}(TZ))$$

be an element of $M_k^2(N, \chi)$. As $Z \in \mathfrak{h}_2$ we can write

$$Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix},$$

where $\tau, \tau' \in \mathfrak{h}_1$, $z \in \mathbb{C}$, and $\mathrm{Im}(z)^2 < \mathrm{Im}(\tau)\,\mathrm{Im}(\tau')$. Furthermore, we can take any positive definite element of Λ_2 and write it in the form

$$T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix},$$

where $n, m \geq 0$, $r \in \mathbb{Z}$, and $4nm - r^2 \geq 0$. Combining, we can rewrite the Fourier expansion of F as

$$F(\tau, z, \tau') = \sum_{\substack{n,m,4nm-r^2 \geq 0 \\ n,m,r \in \mathbb{Z}}} a_F(n, r, m) \exp(n\tau + rz + m\tau').$$

We can rearrange the terms in this summation to obtain

$$F(\tau, z, \tau') = \sum_{m \geq 0} \phi_m(\tau, z) \exp(m\tau'). \tag{2.4}$$

Regarding the coefficients, $\phi_m(\tau, z)$, we have the following theorem.

Theorem 28. *[24, Thm. 6.1] Let $F(Z) \in M_k^2(N, \chi)$ with series expansion as in Equation 2.4. Then, $\phi_m(\tau, z)$ is a character χ, index m, level N, weight k Jacobi form, i.e., $\phi_m \in J_{k,m}(N, \chi)$ for every $m \geq 0$.*

Note, the technique for proving this theorem is to note that for $\gamma \in \Gamma_0^1(N)$ and $(\lambda, \mu) \in \mathbb{Z}^2$, the two matrices

$$\begin{pmatrix} a_\gamma & 0 & b_\gamma & 0 \\ 0 & 1 & 0 & 0 \\ c_\gamma & 0 & d_\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

belong to $\Gamma_0^2(N)$ and act on (τ, z) in precisely the same way as Equations 2.2 and 2.3. The desired transformation properties follow from the fact that F is a Siegel modular form.

The summation in Equation 2.4 is called the Fourier-Jacobi expansion of F. The benefit of considering the Fourier expansion in this way is that in many cases

one can reduce problems of Siegel modular forms to problems of Jacobi forms, which can be easier to work with.

Another benefit of this type of expansion is that it gives a natural construction of Siegel modular forms from Jacobi forms. To demonstrate this construction, we let $\phi \in J_{k,1}(N,\chi)$. Recall the index raising operator, V_t, from the end of Section 2.2. Using this operator, we form the series

$$(\mathcal{M}\phi)(\tau,z,\tau') = \sum_{m=1}^{\infty} (\phi|_{k,1}V_m)(\tau,z)\exp(m\tau').$$

We refer to $\mathcal{M}\phi$ as the Maass lift of ϕ. Regarding the properties of this lift, we have the following theorem.

Theorem 29. *[35, Thm. 3.2 and Thm. 3.6] The lifting defined above gives an injective linear map $\mathcal{M} : J_{k,1}(N,\chi) \to M_k^2(N,\chi)$. Furthermore, this map respects the space of cusp forms, i.e., $\mathcal{M} : J_{k,1}^{\text{cusp}}(N,\chi) \to S_k^2(N,\chi)$.*

Note, this map constitutes part of the Saito-Kurokawa lifting, which maps elliptic eigenforms to Siegel modular forms of genus 2.

Chapter 3

Level stripping for elliptic modular forms

In this chapter we present a result of Ribet which we will generalize to the case of genus two Siegel modular forms in the next chapter. Throughout the chapter we let ℓ be an odd rational prime.

3.1 Congruences of elliptic modular forms

In this short section, we simply introduce some notation and define what it means for two eigenforms to be congruent.

Let $f \in S_k(N, \chi)$ be a normalized eigenform and recall the Hecke field

$$\mathbb{Q}(f) := \mathbb{Q}\left(\left\{\lambda_f(n) : n \in \mathbb{Z}^+\right\}\right).$$

Let $g \in S_{k'}(M, \psi)$ be any other normalized eigenform. Let K be the compositum of $\mathbb{Q}(f)$ and $\mathbb{Q}(g)$ and let ν be any prime lying above ℓ in K. We say f is

congruent to g modulo ν if $a_f(n) \equiv a_g(n) \pmod{\nu}$ for all n and we denote this by $f \equiv g \pmod{\ell}$. Furthermore, if we set Σ to be some set of rational primes, then we use $f \equiv_\Sigma g \pmod{\ell}$ to mean that $a_f(n) \equiv a_g(n) \pmod{\nu}$ for all n pairwise coprime with the elements of Σ.

3.2 On a certain Eisenstein series

In this section, we recall the level 1 Eisenstein series and introduce an Eisenstein series with a specific associated character, both of which satisfy a certain congruence property modulo ℓ.

To begin, recall the normalized level 1 Eisenstein series from Example 4,

$$E_k(\tau) = \frac{1}{2\zeta(k)} G_k(\tau).$$

We consider the series obtained by reducing the Fourier coefficients of $E_k(\tau)$ modulo ℓ for $k = \ell - 1$ when $\ell > 3$ and $k = 4$ for $\ell = 3$. In the case that $\ell > 3$, we have

$$E_{\ell-1}(\tau) = 1 - \frac{2(\ell-1)}{B_{\ell-1}} \sum_{n=0}^{\infty} \sigma_{\ell-1}(n) \exp(n\tau).$$

We have a corollary to the theorem of Clausen and von-Staudt in [38, page 233] which gives $|\ell B_j|_\ell = 1$, when $(\ell - 1)|j$. Thus,

$$\frac{2(\ell-1)}{B_{\ell-1}} \sigma_{\ell-1}(n) \equiv 0 \pmod{\ell},$$

i.e.,

$$E_{\ell-1} \equiv 1 \pmod{\ell}. \tag{3.1}$$

The argument for $\ell = 3$ is similiar and we have that $E_4(\tau) \equiv 1 \pmod{\ell}$.

To produce the desired Eisenstein series, we begin by introducing an Eisenstein series of level $\Gamma^1(N)$ for some positive integer N. Let $v \in (\mathbb{Z}/N\mathbb{Z})^2$, and let $k > 2$ be an integer. Define the following series

$$G_k^v(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\equiv v \,(\mathrm{mod}\ N)}} \frac{1}{(m\tau + n)^k}, \tag{3.2}$$

where we remove $(0, 0)$ from the summation, if necessary. The following proposition gives us that Equation 3.2 gives a modular form.

Proposition 30. *[46, Prop. 3.3.21] Let k and N be as above and let $v = (v_1, v_2) \in (\mathbb{Z}/N\mathbb{Z})^2$. Then, $G_k^v(\tau) \in M_k(\Gamma^1(N))$. Furthermore, if $v_1 \equiv 0 \pmod{N}$, then $G_k^v(\tau) \in M_k(\Gamma_1^1(N))$.*

We will forego giving the Fourier expansion of this particular Eisenstein series, as we require a different Eisenstein series for the main result in this chapter. In fact, the Eisenstein series we are interested in can be expressed as a certain linear combination of the $G_k^v(\tau)$.

Let χ, ψ be Dirichlet characters modulo s, t respectively, where $st = N$. We require $\chi\psi(-1) = (-1)^k$, and for ψ to be primitive modulo t. We consider both χ and ψ as characters modulo N, so that it makes sense to take their product. For $k > 2$, define the following Eisenstein series

$$G_k^{\chi,\psi}(\tau) = \sum_{c=0}^{s-1} \sum_{d=0}^{t-1} \sum_{e=0}^{s-1} \chi(c)\overline{\psi}(d) G_k^{\overline{(ct,d+et)}}(\tau),$$

where $\overline{(ct, d + et)}$ is the mod N reduction of $(ct, d + et)$.

Proposition 31. *[23, Page 127] Let χ, ψ, k, N be as above. Then,*

$$G_k^{\chi,\psi}(\tau) \in M_k(N, \chi\psi).$$

Define

$$E_k^{\chi,\psi}(\tau) := \delta(\chi) + \frac{2}{L(1-k,\psi)} \sum_{n=1}^{\infty} \sigma_{k-1}^{\chi,\psi}(n) \exp(n\tau), \tag{3.3}$$

where

$$\sigma_{k-1}^{\chi,\psi}(n) = \sum_{d|n} \chi(n/d)\psi(d)d^{k-1},$$

and $\delta(\chi)$ is 1 if χ is trivial and 0 otherwise. The following theorem gives the relationship between $E_k^{\chi,\psi}(\tau)$ and $G_k^{\chi,\psi}(\tau)$.

Theorem 32. *[23, Thm. 4.5.1]*

$$G_k^{\chi,\psi}(\tau) = \frac{(-2\pi i)^k g_{\overline{\psi}}}{t^k (k-1)! L(1-k,\psi)} E_k^{\chi,\psi}(\tau),$$

where $g_{\overline{\psi}}$ is the Gauss sum of $\overline{\psi}$.

We will now specialize this Eisenstein series in order to obtain the desired congruence. Let ω denote the Teichmüller character, i.e., the unique homomorphism

$$\omega : \mathbb{F}_\ell^\times \to \mathbb{Z}_\ell,$$

given by mapping $a \in \mathbb{F}_\ell^\times$ to the unique $(\ell-1)^{\text{th}}$ root of unity which is congruent to a modulo ℓ. Then,

$$E_k^{\omega^k}(\tau) := E_k^{1,\omega^k}(\tau) = 1 + \frac{2}{L(1-k,\omega^k)} \sum_{n=1}^{\infty} \sigma_{k-1}^{1,\omega^k}(n) \exp(n\tau) \in M_k(\ell, \omega^k). \tag{3.4}$$

Our goal is to show

$$\frac{2}{L(1-k,\omega^k)}\sigma_{k-1}^{1,\omega^k}(n) \equiv 0 \pmod{\ell} \text{ for all } n > 0.$$

We will first need the following definition.

Definition 33. Let χ be a Dirichlet character of conductor M. Define the *generalized Bernoulli numbers*, denoted $B_{m,\chi}$, by

$$\sum_{a=1}^{M} \frac{\chi(a)te^{at}}{e^{\ell t}-1} = \sum_{m=0}^{\infty} B_{m,\chi}\frac{t^n}{n!}.$$

Using the generalized Bernoulli numbers we have the following expression for $L(1-k,\omega^k)$, which will be needed to prove the desired congruence.

Theorem 34. *[39, Thm. 3.4.2] For $k \geq 1$, we have*

$$L(1-k,\omega^k) = \frac{-B_{k,\omega^{-k}}}{k}.$$

By Proposition 4.1 and the preceding discussion in [76] we can expand the generalized Bernoulli number out in terms of Bernoulli polynomials, denoted $B_k(X)$, and then into Bernoulli numbers, i.e.,

$$\begin{aligned} B_{k,\omega^{-k}} &= \ell^{k-1}\sum_{a=1}^{\ell-1}\omega^{-k}(a)B_k(a/\ell) \\ &= \ell^{k-1}\sum_{a=1}^{\ell-1}\omega^{-k}(a)\sum_{j=0}^{k}\binom{k}{j}B_j a^{k-j}\ell^{j-1} \\ &= \sum_{a=1}^{\ell-1}\omega^{-k}(a)\left(a^k\ell^{-1}+\frac{k}{2}a^{k-1}+\sum_{j=2}^{k}\binom{k}{j}B_j a^{i-j}\ell^{j-1}\right). \end{aligned}$$

A direct application of the theorem of Clausen and von-Staudt yields

$$\left|\sum_{a=1}^{\ell-1}\omega^{-k}(a)\left(a^k\ell^{-1}+\frac{k}{2}a^{k-1}+\sum_{j=2}^{k}\binom{k}{j}B_j a^{i-j}\ell^{j-1}\right)\right|_\ell > 1.$$

Thus,

$$\frac{2}{L(1-k,\omega^k)}\sigma_{k-1}^{1,\omega^k}(n)\equiv 0 \pmod{\ell} \text{ for all } n>0,$$

and it follows that

$$E_k^{\omega^k}\equiv 1 \pmod{\ell}. \tag{3.5}$$

The congruences in Equations 3.1 and 3.5 will be required for the main result of this chapter.

3.3 The trace operator

In this section, we present the trace operator from §3.2 in [63]. However, we have adapted the operator to match the setting in which it will be applied.

Let N be a positive integer with $\ell \nmid N$. Let $f \in S_k(N\ell,\chi)$ with χ having conductor dividing N. Note, as ℓ does not divide the conductor of χ, it is equivalent to consider f as being invariant under the action of the group $\Gamma_0^1(\ell)\cap\Gamma_1^1(N)$. The idea behind the trace operator is to sum f over a complete set of coset representatives for $\Gamma\backslash\Gamma_1^1(N)$ to obtain a form in $S_k(N,\chi)$, where Γ is some group related to $\Gamma_0^1(\ell)\cap\Gamma_1^1(N)$ by a matrix transformation.

Before introducing the explicit set of coset representatives which will be needed, we first introduce a certain "Atkin-Lehner" operator. Note, this operator was used in the level ℓ case by Serre in [63], and has been adapted to the level $N\ell$ case by Li

in [52]. Set

$$W = \begin{pmatrix} \ell x & y \\ N\ell z & \ell \end{pmatrix},$$

where $x, y, z \in \mathbb{Z}$ are chosen so that $\ell x - Nyz = 1$. Note, this is the operator denoted $V_\ell^{N\ell}$ in [52]. It was shown by Li that this operator normalizes the group $\Gamma_0^1(\ell) \cap \Gamma_1^1(N)$, so we have that the action of W fixes the space $S_k(\Gamma_0^1(\ell) \cap \Gamma_1^1(N))$. Furthermore, in Lemma 2 of [52] it is shown that $f|_k W|_k W = \chi(\ell) f$.

With this operator, we define the trace of f to be

$$\mathrm{Tr}(f) = f + \chi^{-1}(\ell)\ell^{1-\frac{k}{2}} T_\ell(f|_k W).$$

The fact that $\mathrm{Tr}(f) \in S_k(\Gamma_1(N))$ follows by applying Lemma 3 in [52] to the function $f|_k W$. By following through the proof of this lemma, we see that the result is obtained by first translating f to a function on the congruence subgroup

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, c \equiv 0 \pmod{N}, b \equiv 0 \pmod{\ell} \right\},$$

and then summing over a complete set of coset representatives for $\Gamma \backslash \Gamma_1^1(N)$, thus it makes sense to refer to this as the trace of f. Note that while $\mathrm{Tr}(f)$ will have the appropriate level, it will generally not be congruent to f.

Before stating the main theorem of this section, we will need another matrix operator, and a certain function which is congruent to 0 modulo some power of ℓ. Set

$$\eta_\ell = \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}.$$

We have that this maps modular forms of level $\mathrm{SL}_2(\mathbb{Z})$ to modular forms for the congruence subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : b \equiv 0 \pmod{\ell} \right\}.$$

To see this, it is enough to note that if $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ then $f|_k\gamma \in S_k(\gamma^{-1}\,\mathrm{SL}_2(\mathbb{Z})\gamma)$ for any $\gamma \in \mathrm{GL}_2(\mathbb{Q})$. Furthermore, we can also view η_ℓ as a map on Fourier expansions via

$$\eta_\ell : \sum a(n)\exp(n) \mapsto \sum a(n)\exp(\ell n).$$

To construct the desired function we let E denote the normalized level 1 Eisenstein series from the previous section which is congruent to 1 $\pmod{\ell}$. Let a denote the weight of this Eisenstein series. As E has level 1, we see that a is even. We follow [63] and define

$$g = E - \ell^{\frac{a}{2}} E|_a W.$$

It is immediate that this function satisfies $g \equiv 1 \pmod{\ell}$. We also need a certain congruence satisfied for the function $g|_a W$. We can rewrite

$$g|_a W = E|_a W - \ell^{\frac{a}{2}} E = \ell^{\frac{a}{2}}(E|_a \eta_\ell - E).$$

It is clear from the action of η_ℓ on Fourier expansions that $E|\eta_\ell \equiv 1 \pmod{\ell}$, and it follows that $g|W \equiv 0 \pmod{\ell^{1+\frac{a}{2}}}$.

The remainder of this section will be devoted to the proof of the following theorem, which will provide one of the steps in the proof of the main result of this chapter.

Theorem 35. *Let $f \in S_k(N\ell, \chi)$ with χ having conductor dividing N and let $g =$*

$E - \ell^{\frac{a}{2}} E|_a W$. *Then,* $\mathrm{Tr}(fg^{\ell^m}) \equiv f \pmod{\ell}$ *for some sufficiently large* m.

Proof. The proof of this theorem follows precisely the techniques employed in §3.2 of [63].

For any modular form $F(\tau) = \sum a_F(n) \exp(n\tau)$, define

$$\mathrm{ord}_\ell(F) = \inf \mathrm{ord}_\nu(a_F(n)),$$

where ν is a prime lying above ℓ in $\mathbb{Q}(F)$. In order to show the desired congruence, we must show $\lim_{m\to\infty} \mathrm{ord}_\ell(\mathrm{Tr}(fg^{\ell^m}) - f) = \infty$. In order to prove this, we will prove

$$\mathrm{ord}_\ell(\mathrm{Tr}(fg^{\ell^m}) - f) \geq \min\left\{m + 1 + \mathrm{ord}_\ell(f), \ell^m + 1 - \frac{k_m}{2} + \mathrm{ord}_\ell(f|_k W)\right\}, \quad (3.6)$$

where k_m is the weight of fg^{ℓ^m}. Note, the right hand side of the inequality clearly increases without bound as m increases.

Before beginning, we note that $\mathrm{ord}_\ell(f) > -\infty$, i.e., that the powers of ℓ appearing in the denominators of the Fourier coefficients of f are bounded above. This follows directly from the finite dimensionality of $S_k(\Gamma_1^1(N\ell))$ and the fact that we can find a basis for $S_k(\Gamma_1^1(N\ell))$ where each basis element has rational Fourier coefficients (see Theorem 3.52 of [65]). Furthermore, since $\mathrm{ord}_\ell(f) > -\infty$ we have $\mathrm{ord}_\ell(f|_k W) > -\infty$.

Now, we are prepared to prove Equation (3.6). Begin by rewriting $\mathrm{Tr}(fg^{\ell^m}) - f = (\mathrm{Tr}(fg^{\ell^m}) - fg^{\ell^m}) + f(g^{\ell^m} - 1)$. From the discussion about g above, we have that $\mathrm{ord}_\ell(f(g^{\ell^m} - 1)) \geq m + 1 + \mathrm{ord}_\ell(f)$. We also know that $\mathrm{Tr}(fg^{\ell^m}) - fg^{\ell^m} =$

$\chi^{-1}(\ell)\ell^{1-\frac{k_m}{2}}T_\ell(fg^{\ell^m}|_{k_m}W)$. This gives us

$$\begin{aligned}
\operatorname{ord}_\ell(\operatorname{Tr}(fg^{\ell^m}) - fg^{\ell^m}) &\geq 1 - \frac{k_m}{2} + \operatorname{ord}_\ell(T_\ell(fg^{\ell^m}|_{k_m}W)) \\
&\geq 1 - \frac{k_m}{2} + \operatorname{ord}_\ell(fg^{\ell^m}|_{k_m}W) \\
&= 1 - \frac{k_m}{2} + \operatorname{ord}_\ell(f|_kW) + \ell^m \operatorname{ord}_\ell(g|_aW) \\
&\geq 1 - \frac{k_m}{2} + \operatorname{ord}_\ell(f|_kW) + \ell^m(1 + \frac{a}{2}) \\
&= 1 + \ell^m - \frac{k}{2} + \operatorname{ord}_\ell(f|_kW),
\end{aligned}$$

and so completes the proof. □

3.4 Ribet's level stripping

In this section we prove the following theorem, which will serve as the analog in the elliptic modular setting to our main result.

Theorem 36. *[60, Thm 2.1] Let $f \in S_k(N\ell^r, \chi)$ be a normalized eigenform with $\ell \nmid N$ and $r > 0$. Then, for some positive integer k' and character χ', there exists a normalized eigenform $g \in S_{k'}(N, \chi')$ such that $f \equiv_\Sigma g \pmod{\ell}$, where Σ is the set of all primes dividing ℓN.*

Before beginning the proof, we note that this statement has been adapted to match the language of modular forms. The original theorem is stated in terms of Galois representations. For a discussion of that interpretation of the theorem the reader is referred to Section 5.2.

Proof. We begin by showing that there is a cuspidal eigenform of level $N\ell^r$ with associated character of conductor coprime to ℓ which is congruent to f. We accomplish this in two steps.

First, we suppose that $r > 1$ and that χ is defined modulo $N\ell^\alpha$, for some $1 < \alpha \leq r$. If this is not the case, then one can simply skip to the next step of the proof. We can factor χ into $\epsilon\eta\omega^i$, where ϵ is defined modulo N, η has ℓ-power order and ℓ-power conductor, and ω^i is the i^{th} power of the Teichmüller character mentioned in Section 3.2 for $1 \leq i \leq \ell - 1$. As η has odd order, we can find some character ξ which satisfies $\eta = \xi^{-2}$. Using this character, we define the twist of f by ξ to be

$$f_\xi(\tau) = \sum_{n=1}^{\infty} \xi(n) a_f(n) \exp(n\tau).$$

By Proposition III.7 in [46], we have that f_ξ is a cuspidal eigenform of level $N\ell^{r'}$ and associated character $\epsilon\omega^i$ for some r'. Furthermore, by adjoining the values of ξ to $\mathbb{Q}(f)$ and taking a prime ν lying above ℓ in this finite extension, it follows from Corollary 10.4 in [57] that $\xi(n) \equiv 1 \pmod{\nu}$ for all n with $\ell \nmid n$. It follows immediately that $f_\xi \equiv_\Sigma f \pmod{\ell}$.

Second, we simply multiply f_ξ by the Eisenstein series $E_i^{\omega^{-i}}$ defined in Equation 3.4, where we may need to choose a different i congruent to the original modulo $\ell - 1$ and satisfying $i > 2$. Due to the congruence in Equation 3.5, we have that $f_\xi E_i^{\omega^{-i}} \equiv_\Sigma f \pmod{\ell}$, where we may need to take another finite degree number field extension so that this congruence makes sense. Furthermore, it is also clear that $f_\xi E_i^{\omega^{-i}}$ has associated character ϵ, which is defined modulo N as desired. Finally, we must apply the Deligne-Serre Lifting Lemma, in fact the special case stated in Corollary 91, to $f_\xi E_i^{\omega^{-i}}$ to obtain an eigenform of level $N\ell^r$ and character ϵ. Note, as this lifting lemma is a standard tool when working with congruences of modular forms, and will also be needed for the main result of Chapter 4, we have included a detailed discussion and proof in Appendix B.

Now that we have an eigenform with the appropropriate character, all that

remains is to remove the powers of ℓ from the level. This is also achieved in two steps.

First, we will find a cuspidal eigenform of level $N\ell$ which is congruent to f. Denote the eigenform of level $N\ell^r$ and character ϵ constructed above by f'. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(f')/\mathbb{Q})$ be a Frobenius element for ν, i.e., $\sigma x \equiv x^\ell \pmod{\nu}$ for all $x \in \mathcal{O}_{\mathbb{Q}(f')}$. By Corollary 2 in [34], we have that the series

$$\sigma^{-1} f'(\tau) = \sum_{n=1}^{\infty} \sigma^{-1}(a_{f'}(n)) \exp(n\tau),$$

defines an eigenform of the same character, level, and weight as f'. We now set $g' = T(\ell)(\sigma^{-1} f')^\ell$. Note, in this case, the operator $T(\ell)$ reduces to the map on the Fourier expansion which sends $a_{\sigma^{-1}f'}(n) \mapsto a_{\sigma^{-1}f'}(\ell n)$. To see that g' is congruent to f' we observe

$$\begin{aligned} T(\ell)\left(\sum_{n=1}^{\infty} \sigma^{-1}(a_{f'}(n)) \exp(n\tau)\right)^\ell &\equiv T(\ell) \sum_{n=1}^{\infty} \sigma^{-1}(a_{f'}(n\tau))^\ell \exp(\ell n\tau) \pmod{\nu} \\ &= \sum_{n=1}^{\infty} \sigma^{-1}(a_{f'}(n))^\ell \exp(n\tau) \\ &\equiv \sum_{n=1}^{\infty} a_{f'}(n) \exp(n\tau) \pmod{\nu}. \end{aligned}$$

Then, Lemma 1 in [52] gives that g' has level $N\ell^{r-1}$. Applying this process $r-1$ times, we obtain an eigenform of level $N\ell$ which is congruent to f.

To move from our eigenform of level $N\ell$ to a congruent cusp form of level N, we simply apply Theorem 35. To complete the proof, we apply Corollary 91 a second time to lift this cusp form to an eigenform of level N.

□

3.5 Examples

In this section we a concrete example of Ribet's level stripping. All computations in this section were completed in SAGE.

Before presenting the example we give the following theorem which is a generalization of a classical result of Sturm in [67].

Theorem 37. *[17, Cor. 2] Let $N \geq 3$ and let $f_1 \in S_{k_1}(N, \chi_1)$ and $f_2 \in S_{k_2}(N, \chi_2)$ with Fourier coefficients lying in some number field K. Let $\nu \nmid N$ be a prime ideal in K lying above an odd prime ℓ. Suppose that $a_f(p) \equiv a_g(p) \pmod{\nu}$ for all primes $p \nmid \ell N$ satisfying $p \leq \frac{\max\{k_1,k_2\}[\mathrm{SL}_2(\mathbb{Z}):\Gamma_1^1(N')]}{12}$ with*

$$N' = \begin{cases} N\ell^2 \prod\limits_{\substack{q|N \\ q \text{ prime}}} q & \text{for} \quad \ell \nmid N \\ N \prod\limits_{\substack{q|N \\ q \text{ prime}}} q & \text{for} \quad \ell | N \end{cases},$$

and suppose that $k_1 \equiv k_2 \pmod{\ell - 1}$. Then, $a_{f_1}(p) \equiv a_{f_2}(p) \pmod{\nu}$ for all $p \nmid \ell N$.

From this theorem we see that in order to show that two cusp forms of congruent, we need only check a finite number of Fourier coefficients. We are now prepared to present our example.

Example 38. We begin by considering the one-dimensional space of cusp forms $S_4(\Gamma_1^1(5))$. Let $f \in S_4(\Gamma_1^1(5))$ be the unique normalized eigenform in this space. Using SAGE we verify that f has trivial associated character and has Fourier coefficients in $\mathbb{Z}$.

From Theorem 35, we know that our desired form lies in $S_{20+5^m 4}(\mathrm{SL}_2(\mathbb{Z}))$ for some sufficiently large $m \geq 0$. A search in Sage yields a form $g \in S_{40}(\mathrm{SL}_2(\mathbb{Z}))$ which

has Fourier coefficients in the number field $\mathbb{Q}(\alpha)$, where α is a root of the quadratic polynomial $x^3 - 548856x^2 - 810051757056x + 213542160549543936$. We note that the prime 5 splits in $\mathbb{Q}(\alpha)$. We choose ν lying above 5 to be the prime in $\mathbb{Q}(\alpha)$ generated by 5 and $\alpha^2/37739520 - 523\alpha/104832 - 7305104/455$.

In order to apply Theorem 37 in this setting, we must show that $4 \equiv 40 \pmod 4$ which is obvious and that

$$a_f(p) \equiv a_g(p) \pmod{\nu}$$

for all primes not equal to 5 which satisfy $p \leq 2000$. The verification of this second part is easily done in SAGE. We present a few of the calculations here.

p	$a_f(p)$	$a_g(p)$	$a_f(p) - a_g(p) \pmod{\nu}$
2	-4	α	0
3	2	$\frac{\alpha^2}{168} - \frac{6501\alpha}{7} - \frac{22907296044}{7}$	0
7	6	$\frac{-174647\alpha^2}{4}$ -20667766734α -25745559079234808	0
11	32	$\frac{-563570865\alpha^2}{8}$ - 114890280778625α + 288741682132974143172	0

Chapter 4

Level stripping for genus 2 Siegel modular forms

In this chapter, we present our main result. Throughout, we let $\ell > 3$ denote an odd rational prime.

4.1 Congruences of genus 2 Siegel modular forms

In this section we define two distinct notions of congruences between genus 2 Siegel modular forms. We then show a relationship between these two notions.

Let F and G be genus 2 eigenforms of level N and M respectively. For any prime $p \nmid MN$, we let $\lambda_F(p), \lambda_F(p^2;i), \lambda_G(p), \lambda_G(p^2;i)$ denote the eigenvalues of F and G with respect to $T(p)$ and $T_i(p^2)$ for $i = 1, 2$, i.e.,

$$T(p)F = \lambda_F(p)F,\ T_i(p^2)F = \lambda_F(p^2;i)F,$$

$$T(p)G = \lambda_G(p)G,\ T_i(p^2)G = \lambda_G(p^2;i)G.$$

We let $\mathbb{Q}(\lambda_F, \lambda_G)$ denote the compositum of $\mathbb{Q}(\lambda_F)$ and $\mathbb{Q}(\lambda_G)$, where $\mathbb{Q}(\lambda_F)$ and $\mathbb{Q}(\lambda_G)$ were defined in Lemma 27. By Lemma 27, $\mathbb{Q}(\lambda_F, \lambda_G)$ is a totally real number field. Let Σ denote a finite set of primes. Then, we write $F \equiv_\Sigma G \pmod{\ell}$ if for all primes $p \notin \Sigma$ we have

$$\lambda_F(p) \equiv \lambda_G(p) \pmod{\nu},\ \lambda_F(p^2; i) \equiv \lambda_G(p^2; i) \pmod{\nu} \text{ for } i = 1, 2,$$

where ν is a prime lying above ℓ in $\mathbb{Q}(\lambda_F, \lambda_G)$. This is referred to as a congruence of eigenvalues.

Our second notion will be the congruence of Fourier coefficients, which we define as in [9]. Before we can make sense of this notion, we will need the following lemma.

Lemma 39. *Let $F \in S^2_\rho(N, \chi)$ be an eigenform and let K denote $\mathbb{Q}(\lambda_F, \chi)$, i.e., the field obtained by adjoining all of the values of χ to $\mathbb{Q}(\lambda_F)$. Set*

$$S^2_\rho(N, \chi; F) = \left\{ G \in S^2_\rho(N, \chi) : \lambda_G(t) = \lambda_F(t) \text{ for all } t \in H^{\mathbb{Z}}_N(\Gamma^2_0(N)) \right\}.$$

Then,

$$S^2_\rho(N, \chi; F) = S^2_\rho(N, \chi; F)_{\mathcal{O}_{KL}} \otimes_{\mathcal{O}_{KL}} \mathbb{C},$$

where $\mathcal{O}_{KL}$ is the ring of integers of the compositum of K and L where $L/\mathbb{Q}$ is some finite extension.

Proof. Recall, by Lemma 2.1 in [71] we have

$$S^2_\rho(N, \chi) = S^2_\rho(N, \chi)_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} \mathbb{C},$$

where we are using the same notation which was defined before Corollary 26 and

$L/\mathbb{Q}$ is a finite abelian extension. We assume that L contains the values of χ. Let $\{F_1, \ldots, F_r\}$ be a $\mathcal{O}_L$-basis for $S^2_\rho(N,\chi)_{\mathcal{O}_L}$. By Theorem 26, we have that

$$tF_i = \sum_{j=1}^{r} c_{ij}(t)F_j \text{ for all } t \in H^{\mathbb{Z}}_N(\Gamma^2_0(N)),$$

where $c_{ij}(t) \in \mathcal{O}_L$.

For each $z = (z_1, \ldots, z_r) \in \mathbb{C}^r$ we put

$$f(z) = \sum_{i=1}^{r} z_i F_i.$$

We set $V(F) = \{z \in \mathbb{C}^r : f(z) \in S^2_\rho(N,\chi;F)\}$. Note, $V(F)$ is a finite dimensional $\mathbb{C}$-vector space and we denote the dimension by d. It is clear that f defines a $\mathbb{C}$-linear isomorphism

$$f : V(F) \to S^2_\rho(N,\chi;F).$$

Take S to be a generating set for $\mathcal{H}^{\mathbb{Z}}_N$ as a $\mathbb{Z}$-algebra , which we know is finite because $\mathcal{H}^{\mathbb{Z}}_N \hookrightarrow \mathrm{End}_{\mathbb{C}}(S^2_\rho(N,\chi))$. For $z \in V(F)$ it is clear that $tf(z) = \lambda_F(t)f(z)$ for all $t \in S$, i.e.,

$$\sum_{i=1}^{r} c_{ij}(t)z_i = \lambda_F(t)z_i.$$

Since the coefficients $\lambda_F(t), c_{ij}(t)$ are in KL, there exists a basis $\{v_1, \ldots, v_d\}$ of $V(F)$ such that $v_j \in (KL)^r$. Take a non-zero $\gamma_j \in \mathcal{O}_{KL}$ such that $v'_j = \gamma_j v_j \in \mathcal{O}^r_{KL}$. Then, $f(v'_j) \in S^n_k(N,\chi;F)_{\mathcal{O}_{KL}}$ and $V(F) = \bigoplus_{i=1}^{d} \mathbb{C}v'_i$. □

Define the following field,

$$\mathbb{Q}(F) = \prod_{T \in \Lambda_2} \mathbb{Q}(a_F(T)),$$

where

$$\mathbb{Q}(a_F(T)) := \mathbb{Q}\left(\left\{a_{ij} : a_F(T) = \sum_{i,j} a_{ij} x_1^i x_2^j\right\}\right).$$

As in Section 2.3, we have identified V with the homogeneous polynomials of degree $k_2 - k_1$ in $\mathbb{C}[x_1, x_2]$, where (k_1, k_2) is the highest weight vector of ρ. Then, the previous lemma gives that after some normalization, we may assume that $\mathbb{Q}(F)$ is a finite extension. We make the same assumption for the field $\mathbb{Q}(G)$.

We are now prepared to define the congruence of Fourier coefficients. Define the ℓ-adic valuation of F as

$$\mathrm{ord}_\ell(F) = \inf_{T \in \Lambda_2} \{\mathrm{ord}_\nu(a_F(T))\},$$

where

$$\mathrm{ord}_\nu(a_F(T)) = \min_{i,j}\left\{\mathrm{ord}_\nu(a_{ij}) : a_F(T) = \sum_{i,j} a_{ij} x_1^i x_2^j\right\},$$

and ν is prime lying above ℓ in $\mathbb{Q}(F)$. Using this, we say that F and G have congruent Fourier coefficients, denoted $F \equiv_{fc} G \pmod{\ell^r}$, if $\mathrm{ord}_\ell(F - G) \geq r$.

For the genus 1 case, it is clear that these two notions of congruence are equivalent, as the Fourier coefficients of a normalized elliptic eigenform are precisely the eigenvalues. This equivalence is not necessarily true for any higher genus. However, we do have the following lemma, which gives that a congruence of Fourier coefficients implies a congruence of eigenvalues.

Lemma 40. *Let F, G, Σ be as defined above. If $F \equiv_{fc} G \pmod{\ell}$ then $F \equiv_\Sigma G \pmod{\ell}$.*

Proof. This proof follows the same argument as in Theorem A.1 in [56], however we include it here to emphasize that this result works for vector-valued forms of arbitrary level, not just the classical forms of level one case as was proven in [56].

Define K to be the compositum of $\mathbb{Q}(F)$ and $\mathbb{Q}(G)$. Also, we adjoin the values of the characters of F and G if necessary and continue to denote this field by K. Let $c \in K$ so that at least one component of one Fourier coefficient of cF is an ℓ-unit, i.e., for some $T \in \Lambda_2$ and $i, j \in \mathbb{N}$ we have that $\mathrm{ord}_\nu(a_{ij}) = 0$, where $a_F(T) = \sum_{i,j} a_{ij} x_1^i x_2^j$ and ν is a prime lying above ℓ in K. Without loss of generality, we replace F and G by cF and cG, respectively. Denote this component by $a_F(T)_{ij}$. Let $t \in H_N^{\mathbb{Z}}(\Gamma_0^2(N))$ with $tF = \lambda_F(t)F$ and $tG = \lambda_G(t)G$. Define the form $H = F - G$. Then,

$$\lambda_F(t)F - \lambda_G(t)G = t(F - G) = tH.$$

By Theorem 26, we have that $\mathbb{Q}(tH) \subseteq K$. Hence,

$$\lambda_F(t)a_F(T)_{ij} \equiv \lambda_G(t)a_G(T)_{ij} \pmod{\nu},$$

where ν is a prime lying above ℓ in K. Since $a_F(T)_{ij}$ is an ℓ-unit and $a_F(T)_{ij} \equiv a_G(T)_{ij} \pmod{\nu}$, we have that $\lambda_F(t) \equiv \lambda_G(t) \pmod{\nu}$, which completes the proof. □

4.2 A certain Eisenstein series

In this section, we give the construction of a degree 2 Eisenstein series which is congruent to 1 (mod ℓ). Note, this construction is due to Kikuta, see [45]. We include the details of this construction not only because the techniques used are interesting in their own right, but also because the proof of this theorem may provide a method for generalizing our main result, which we will mention in later sections.

In fact, our main goal of this section is to provide a proof of the following theorem.

Theorem 41. *[45, Thm. 1.2] Let $1 \leq i \leq \ell$. There exists a sequence $\{G_{k_m} \in M^2_{k_m}(\ell, \omega^i)\}$ such that*

$$\lim_{m\to\infty} G_{k_m} = 1,$$

where the limit is taken ℓ-adically with respect to the Fourier coefficients, ω is Teichmüller character, and the sequence $\{k_m\}$ will be defined later in this section.

We also give the following corollary which is what we will need for our main result.

Corollary 42. *For an integer k from the sequence $\{k_m\}$, there exists a $G \in M^2_k(\ell, \omega^i)$ such that $G \equiv 1 \pmod{\ell}$, where ω is the Teichmüller character and i is an integer modulo $\ell - 1$.*

Proof. Note, as the sequence $\{G_{k_m}\}$ converges ℓ-adically to 1 as $m \to \infty$, we have that for every $\epsilon > 0$, there is some M such that $|G_{k_m} - 1|_\ell < \epsilon$ for every $m > M$. By definition of the ℓ-adic absolute value this means that $\ell^{-\operatorname{ord}_\ell(G_{k_m}-1)} < \epsilon$ for every $m > M$. As the choice of ϵ is arbitrary, we can find some G_{k_m} such that $\operatorname{ord}_\ell(G_{k_m} - 1) > 1$, i.e., $G_{k_m} \equiv 1 \pmod{\ell}$. □

Before beginning the proof we fix an embedding

$$\mathbb{Q}(\mu_{\ell-1}) \hookrightarrow \mathbb{Q}_\ell,$$

where $\mu_{\ell-1}$ is the group of $\ell - 1$ roots of unity. To accomplish this we fix a primitive element of $\mu_{\ell-1}$ which we will denote by $\zeta_{\ell-1}$. We want to consider the factorization of ℓ in $\mathbb{Z}[\zeta_{\ell-1}]$. Let $\Phi(x)$ denote the cyclotomic polynomial which has $\zeta_{\ell-1}$ as a root. Then

$$\Phi(x) \equiv p_1(x) \dots p_r(x) \pmod{\ell},$$

where $r = \phi(\ell - 1)$ and the $p_i(x)$ are all distinct degree 1 polynomials. Then, ℓ decomposes as a product of r prime ideals $\lambda_i = (p_i(\zeta_{\ell-1}), \ell)$. If we write $p_i(x) = x - d_i$, then we obtain an embedding $\sigma_i : \mathbb{Q}(\zeta_{\ell-1}) \hookrightarrow \mathbb{Q}_\ell$ associated to λ_i by setting $\zeta_{\ell-1}^{\sigma_i} = \omega(d_i)$. In the following proof we fix one such embedding and denote it by σ.

We also set $X = \mathbb{Z}_\ell \times \mathbb{Z}/(\ell - 1)\mathbb{Z}$, which we refer to as the group of weights, as defined in [63]. Roughly speaking, if we have a sequence of elliptic modular forms which converge ℓ-adically, then the corresponding weights converge to an element of X. For a more precise statement of this result, the reader is referred to Théorème 1.4.2 in [63].

Proof of Theorem 41. Let $E_{k,1}^J$ denote the normalized Jacobi Eisenstein series of weight k and index 1, which was introduced in Example 14. This $E_{k,1}^J$ has Fourier coefficients in $\mathbb{Q}$. Furthermore, we let E_k and E_{k,ω^i} be defined as in Example 4 and Equation 3.4, respectively. From Chapter 2, we know that E_k has Fourier coefficients in $\mathbb{Q}$ and E_{k,ω^i} has Fourier coefficients in $\mathbb{Q}(\mu_{\ell-1})$.

Define the sequence $\{k_m = a\ell^{m+1}\}$ for $0 < a \in \mathbb{Z}$, which satisfy $a \equiv -i \pmod{\ell - 1}$.

If we set

$$\phi_{k_m}(\tau, z) = E^1_{a(\ell-2),\omega^i}(\tau) E^1_{a\ell(\ell^m-1)}(\tau) E^J_{2a,1}(\tau, z),$$

then ϕ_{k_m} is Jacobi form of index 1, level ℓ and weight k_m. It is clear that in the Fourier expansion

$$\phi_{k_m}(\tau, z) = \sum_{n,r} c(n, r) \exp(n\tau + rz),$$

we have $c(n, r) \in \mathbb{Q}(\mu_{\ell-1})$ and $c(0, 0) = 1$.

We now need the following lemma.

Lemma 43. *[45, Lemma 3.1] The sequence $\{\phi_{k_m}\}$ converges uniformly in the formal power series ring $\mathbb{Q}_\ell[\zeta, \zeta^{-1}][[q]]$.*

Recall the Maass lift $F_{k_m} := \mathcal{M}\phi_{k_m} \in M^2_{k_m}(\ell, \omega^i)$ from Section 2.3. Applying Theorem 15 we have the following Fourier expansion

$$F_{k_m}(\tau, z, \tau') = \frac{1}{2}L(1-k_m, \omega^i) + \sum_{n=1}^{\infty} \sum_{\substack{d|n \\ (\ell,d)=1}} \omega^i(d) d^{k_m-1} \exp(n\tau)$$
$$+ \sum_{s=1}^{\infty} \sum_{4ns-r^2\geq 0} \sum_{\substack{d|(n,r,s) \\ (\ell,d)=1}} \omega^i(d) d^{k_m-1} c\left(\frac{ns}{d^2}, \frac{r}{d}\right) \exp(n\tau + rz + s\tau').$$

If we apply σ to the Fourier expansion then the s^{th} Fourier coefficient is

$$\sum_{4ns-r^2\geq 0} \sum_{\substack{d|(n,r,s) \\ (\ell,d)=1}} d^{k_m+\alpha-1} c\left(\frac{ns}{d^2}, \frac{r}{d}\right)^{\sigma} \exp(n\tau + rz).$$

Furthermore, as the constant term of F_{k_m} is given by $E^1_{k_m,\omega^i}$, then by applying σ to the constant term and using an argument as in §1.6 of [63] we obtain

$$\frac{1}{2}L_\ell(1-k_m, \omega^{k_m+\alpha}) + \sum_{n=1}^{\infty} \sum_{\substack{d|n \\ (\ell,d)=1}} d^{k_m+\alpha-1} \exp(n\tau),$$

where $L_\ell(1-k_m, \omega^{k_m+\alpha})$ is the ℓ-adic L-function as defined in Section 3.5 of [34].

Setting $G_{k_m} = 2L(1-k_m,\omega^i)^{-1}F_{k_m}$ we have that

$$G^{\sigma}_{k_m}(\tau,z,\tau') = 1 + \frac{2}{L_\ell(1-k_m,\omega^{k_m+\alpha})}\sum_{n=1}^{\infty}\sum_{\substack{d|n\\(\ell,d)=1}} d^{k_m+\alpha-1}\exp(n\tau)$$
$$+ \frac{2}{L_\ell(1-k_m,\omega^{k_m+\alpha})}\sum_{s=1}^{\infty}\sum_{4ns-r^2\geq 0}\sum_{\substack{d|(n,r,s)\\(\ell,d)=1}} d^{k_m+\alpha-1}c\left(\frac{ns}{d^2},\frac{r}{d}\right)^{\sigma}\exp(n\tau+rz+s\tau'). \tag{4.1}$$

As k_m tends to $(0,-\alpha)$ in X, we have $(1-k_m,k_m+\alpha)$ tends to $(1,0)$ in X. Note, $L_\ell(s,\omega^u)$ has a simple pole at $(s,u)=(1,0)$. Combining this with the Lemma 43 we have the proof. □

4.3 The $U(\ell)$ operator

In this short section, we introduce a certain operator on the space of Siegel modular forms which is analogous to the U_ℓ^N operator in [52] and then give the relevant properties which will be important for our purposes.

We define the operator $U(\ell)$ by its action on Fourier expansions,

$$U(\ell):\sum_{0\leq T\in\Lambda_2} a_F(T)\exp(\mathrm{Tr}(TZ))\mapsto\sum_{0\leq T\in\Lambda_2} a_F(\ell T)\exp(\mathrm{Tr}(TZ)).$$

For our main result we will need the following two properties of the $U(\ell)$ operator.

Lemma 44. *[7, Thm 3.1]*
If $\ell||M$, the operator $U(\ell)$ is an injective map from $M^2_\rho(M,\chi)$ to itself.

Proof. We give a sketch of the proof here, as the result is only shown for the scalar weight case in [7].

Let $F \in M_\rho^2(M,\chi)$ with $\ell||M$. Following d) in Remark 1 of [7], we consider the operator

$$tF = F| \sum_{\substack{M \in \mathrm{M}_2(\mathbb{F}_\ell) \\ M = {}^T M}} \begin{pmatrix} 0 & -I_2 \\ I_2 & M \end{pmatrix}.$$

Note that this is the operator denoted $\tau(1,n)$ in [7]. This operator is invertible by Proposition 2.1 in [7].

From Equation 3.2 in [7], we can decompose t as follows

$$\begin{aligned} tF &= F| \sum_{\substack{M \in \mathrm{M}_2(\mathbb{F}_\ell) \\ M = {}^T M}} \begin{pmatrix} 0 & -I_2 \\ I_2 & M \end{pmatrix} \\ &= p^{3-k} F|W_\ell|U(\ell), \end{aligned}$$

where

$$W_\ell = \begin{pmatrix} 0_2 & -I_2 \\ \ell I_2 & 0_2 \end{pmatrix}.$$

Note, W_ℓ is an involution. Furthermore, W_ℓ normalizes the group $\Gamma_0^2(M)$, which gives that $F|W_\ell \in M_\rho^2(M,\chi)$. Combining this with the invertibility of t, we have that $U(\ell)$ is injective. □

Lemma 45. *If $\ell^2|M$ and χ is defined modulo $\frac{M}{\ell}$, the operator $U(\ell)$ maps $M_\rho^2(M,\chi)$ to $M_\rho^2(M/\ell,\chi)$.*

Proof. Here we have adapted a proof of Andrianov from [3].
Let $F \in M_\rho^2(M,\chi)$. From [7] we have that the operator $U(\ell)$ is given by,

$$U(\ell)F = \ell^3 \sum_S F| \begin{pmatrix} 1 & S \\ 0 & \ell \end{pmatrix},$$

where the summation runs over all symmetric matrices in $\mathrm{M}_2(\mathbb{Z}/\ell\mathbb{Z})$. We have

$$U(\ell)F = \ell^3 \sum_S F| \begin{pmatrix} 1 & S \\ 0 & \ell \end{pmatrix} = \ell^3 F| \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \sum_S \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}.$$

Define the following subgroup of $\Gamma_0^2(M/\ell)$,

$$\Gamma(M/\ell, \ell) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^2(M/\ell) : B \equiv 0 \pmod{\ell} \right\}.$$

Then, for $\gamma \in \Gamma(M/\ell, \ell)$ we have

$$\begin{aligned} F| \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} | \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} &= F| \begin{pmatrix} a_\gamma & b_\gamma \\ \ell c_\gamma & \ell d_\gamma \end{pmatrix} \\ &= F| \begin{pmatrix} a_\gamma & \frac{b_\gamma}{\ell} \\ \ell c_\gamma & d_\gamma \end{pmatrix} | \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \\ &= \chi(\gamma) F| \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix}. \end{aligned}$$

Note, a complete set of right coset representatives for

$$\Gamma(M/\ell, \ell) \backslash \Gamma_0^2(M/\ell)$$

is given by

$$\left\{ \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} : {}^T S = S, S \in \mathrm{M}_2(\mathbb{Z}/\ell\mathbb{Z}) \right\}.$$

Let $\gamma \in \Gamma_0(M/\ell)$, and let $S \in \mathrm{M}_2(\mathbb{Z}/\ell\mathbb{Z})$ be symmetric. Set S' to be the unique

symmetric matrix in $\mathrm{M}_2(\mathbb{Z}/\ell\mathbb{Z})$ which is congruent to $(a_\gamma + Sc_\gamma)^{-1}(b_\gamma + Sd_\gamma) \pmod{\ell}$. Then, from Lemma 13 in [3], there exists $\gamma_S \in \Gamma(M/\ell, \ell)$ such that

$$\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \gamma = \gamma_S \begin{pmatrix} 1 & S' \\ 0 & 1 \end{pmatrix}.$$

Note, such a γ_S also satisfies $\chi(\gamma) = \chi(\gamma_S)$. Thus,

$$\begin{aligned} U(\ell)F|\gamma &= \ell^3 \sum_S F| \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \gamma \\ &= \ell^3 \sum_S F| \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \gamma_S \begin{pmatrix} 1 & S' \\ 0 & 1 \end{pmatrix} \\ &= \ell^3 \chi(\gamma_S) F| \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \sum_{S'} \begin{pmatrix} 1 & S' \\ 0 & 1 \end{pmatrix} \\ &= \chi(\gamma) U(\ell) F. \end{aligned}$$

This completes the proof. □

Corollary 46. *Let $F \in S^2_\rho(N\ell^r, \chi)$ be an eigenform with χ defined modulo N, $r > 1$, and $\ell \nmid N$. Then, for some ρ' and some χ' defined modulo N, there is a form $G \in S^2_{\rho'}(N\ell^{r-1}, \chi')$ satisfying*

$$F \equiv_{fc} G \pmod{\ell}.$$

Proof. We begin by letting $\sigma \in \mathrm{Gal}(\mathbb{Q}(F)/\mathbb{Q})$ be a Frobenius element for ν a prime over ℓ in $\mathbb{Q}(F)$, i.e., $\sigma x \equiv x^\ell \pmod{\nu}$ for all $x \in \mathcal{O}_{\mathbb{Q}(F)}$. By realizing σ as an element of $\mathrm{Aut}(\mathbb{C})$, we can apply Theorem 1 in [69] to see that $F^{\sigma^{-1}}$, as defined in the proof

of Lemma 27, is an eigenform in $S^2_\rho(N\ell^r, \sigma^{-1} \circ \chi)$. Define a form $G = U(\ell)(F^{\sigma^{-1}})^\ell$. Then, just as in the proof of Theorem 36, we have

$$\begin{aligned} U(\ell)\left(\sum_{\substack{T>0\\ T\in\Lambda_2}} \sigma^{-1}(a_F(T))\exp(\mathrm{Tr}(TZ))\right)^\ell &\equiv U(\ell)\sum_{\substack{T>0\\ T\in\Lambda_2}} \sigma^{-1}(a_F(T))^\ell \exp(\ell\,\mathrm{Tr}(TZ)) \pmod{\nu} \\ &= \sum_{\substack{T>0\\ T\in\Lambda_2}} \sigma^{-1}(a_F(T))^\ell \exp(\mathrm{Tr}(TZ)) \\ &\equiv \sum_{\substack{T>0\\ T\in\Lambda_2}} a_F(T)\exp(\mathrm{Tr}(TZ)) \pmod{\nu}. \end{aligned}$$

Thus, G is congruent in Fourier coefficients to F. Moreover, by Lemma 45, $G \in S^2_{\rho'}(N\ell^{r-1}, \chi')$ for some ρ' and χ'. □

4.4 Theta series

In this section, we introduce a certain family of theta series which satisfy a certain congruence. We follow [10], [11] throughout this section.
We begin with some preliminary definitions.

Definition 47. Let V be a vector space and let Q be a quadratic form defined on V. Then, we call (V, Q) a *quadratic space*. Furthermore, we say that (V, Q) is *positive definite* if $Q(v) > 0$ for all $v \in V$.

We assume throughout that our quadratic spaces are defined over $\mathbb{Q}$ of dimension d, i.e., V is a vector space over $\mathbb{Q}$ of dimension d.

Associated to any quadratic form we have a natural bilinear form given by

$$(v_1, v_2) = Q(v_1 + v_2) - Q(v_1) - Q(v_2).$$

It is clear that such a bilinear form is symmetric.

Definition 48. Let (V, Q) be a vector space. Let $L \subset V$ be a lattice. Then, we say that L is *integral* if $(v_1, v_2) \in \mathbb{Z}$ for all $v_1, v_2 \in L$. Furthermore, we say L is an *even lattice* if $(v, v) \in 2\mathbb{Z}$ for all $v \in L$ and we say L is *odd* if it is not even.

Note that if $L \subset V$ is an even lattice, then it is immediate that $Q(v) \in \mathbb{Z}$ for all $v \in L$.

Definition 49. Let L be a lattice in a quadratic space (V, Q). We define the *dual lattice of* L to be

$$\widehat{L} := \{v \in \mathbb{Q}^d : (v, v') \in \mathbb{Z} \text{ for all } v' \in L\}.$$

Note, if L is integral, then by definition we have $L \subseteq \widehat{L}$.

Definition 50. We say that a lattice L is *level* N if $Q(v) \in \frac{1}{N}\mathbb{Z}$ for all $v \in \hat{L}$.

We will frequently identify V with $\mathbb{Q}^d$ by choosing a basis, say $v_1, \ldots, v_d$, and hence consider L as being a subset of $\mathbb{Q}^d$.

Definition 51. Associated to L, we have a $d \times d$ matrix G, called the *Gram matrix*, by defining $G_{i,j} = (v_i, v_j)$. We define the determinant of L to be $\det G$.

Note, this definition of determinant is well-defined because any two bases of L vary by an integral unimodular matrix.

It is immediate that the Gram matrix is symmetric, since $(\cdot, \cdot)$ is symmetric. Furthermore, the Gram matrix satisfies $Q(v) = {}^T vGv$ for all $v \in V$.

We now introduce lattices which will be of particular interest to us.

Definition 52. Let L be an integral lattice. We say that L is *ℓ-special* if there exists a group of automorphisms of L, denoted C_ℓ, of order ℓ which acts freely on $L \backslash \{0\}$.

We will present a construction of such a lattice after mentioning the properties in which we are interested.

Suppose d is even. Let L be an even ℓ-special lattice and let $T \in \Lambda_2$ be non-zero. Define the following finite set,

$$\mathcal{A}(G,T) := \{X \in \mathrm{M}_{d\times 2}(\mathbb{Z}) : TXGX = 2T\}.$$

By choosing an appropriate basis, we can identify L with $\mathbb{Z}^d$, and hence we obtain a natural action of C_ℓ on $\mathcal{A}(G,T)$. As the action of C_ℓ on L is free, so is the action on $\mathcal{A}(G,T)$. Through this action we can decompose $\mathcal{A}(G,T)$ into C_ℓ-orbits with representatives $X_1,\ldots,X_r$. Then, each orbit is isomorphic to $C_\ell/\operatorname{Stab}_i$, where Stab_i is the subgroup of C_ℓ which stabilizes X_i. As the order of C_ℓ is prime and C_ℓ acts freely, we have that Stab_i is trivial for every $1 \le i \le r$. Thus, the order of $\mathcal{A}(G,T)$, denoted $A(G,T)$, is divisible by ℓ.

We associate a genus 2 theta series to L by

$$\Theta_L^2(Z) := \sum_{X\in \mathrm{M}_{d\times 2}(\mathbb{Z})} \exp(\pi i \operatorname{Tr}({}^T XGXZ)) = \sum_{T\in\Lambda_2} A(G,T)\exp(\operatorname{Tr}(TZ)).$$

From [1] and [27] we have that such a series is a modular form of character $\chi(\gamma) = \left(\frac{(-1)^{d/2}\det L}{\det d_\gamma}\right)$, level $\Gamma_0^2(\ell)$, and weight $d/2$. By construction, we have that $\Theta_L^2 \equiv 1 \pmod{\ell}$ since $\ell | A(G,T)$ for every non-zero $T \in \Lambda_2$ and $A(G,0_2) = 1$.

We now proceed to the construction of an ℓ-special lattice. To accomplish this, we consider the root lattice $A_{\ell-1}$ given by

$$A_{\ell-1} := \{x = {}^T(x_1,\ldots,x_\ell) \in \mathbb{Z}^\ell : \sum_i x_i = 0\}.$$

From §6 in [19], we have that this is an even lattice of rank $\ell - 1$, level ℓ, and determinant ℓ. The symmetric group, S_ℓ, acts on $A_{\ell-1}$ by permuting the coordinates.

Suppose that $\sigma \in S_\ell$ has a nontrivial fixed point $x \in A_{\ell-1}$. Divide the indices of x into two sets by defining $M^+ := \{i : x_i \geq 0\}$ and $M^- := \{i : x_i < 0\}$. Then, M^+ and M^- are stable under the action of σ, i.e., we can view σ as an element of $S_r \times S_{\ell-r}$, where the order of M^+ is r. In particular, σ cannot be an element of order ℓ. Thus, we have that any ℓ-Sylow subgroup of S_ℓ acts freely on $R_{\ell-1}\backslash\{0\}$. Therefore, $A_{\ell-1}$ is an ℓ-special lattice.

For our purposes, we will need the lattice $L = A_{\ell-1} \bigoplus A_{\ell-1}$, which is ℓ-special when equipped with the diagonal action of a ℓ-Sylow subgroup of S_ℓ. Note, L has rank $2\ell - 2$, level ℓ, and determinant ℓ^2. Thus, we have construced a form

$$\Theta_L^2 \in M_{\ell-1}^2(\ell, 1_\ell), \text{ with } \Theta_L^2 \equiv 1 \pmod{\ell}, \tag{4.2}$$

where we use 1_ℓ to denote the trivial character.

4.5 The trace operator

In this section, we introduce the trace operator from [8] in the genus 2 setting.

Let $F \in S_\rho^2(N, \chi)$ be a Siegel modular form of genus 2, and let $\Gamma_1 \subset \Gamma_2$ be level N congruence subgroups of the type listed in Section 2.3. Then, we define the trace of F to be

$$\mathrm{Tr}_{\Gamma_1}^{\Gamma_2}(F) := \frac{1}{[\Gamma_2 : \Gamma_1]} \sum_{\gamma \in \Gamma_1 \backslash \Gamma_2} \chi^{-1}(\gamma) F|\gamma,$$

where the summation is taken over a complete set of coset representatives. Note, when the levels are clear from context we will simply write $\mathrm{Tr}(F)$. The following proposition gives us an important result on the level of $\mathrm{Tr}(F)$.

Proposition 53. *[8, Prop. 2.1]*

Let $F \in S_\rho^2(N,\chi)$. Then,

1. *if the conductor of χ does not divide N we have*

$$\mathrm{Tr}_{\Gamma_1^2(N)}^{\Gamma_1^2(N\ell)}(F) = 0.$$

2. *if the conductor of χ divides N we have*

$$\mathrm{Tr}_{\Gamma_1^2(N)}^{\Gamma_1^2(N\ell)}(F) \in M_\rho^2(N,\chi) \text{ and } \mathrm{Tr}_{\Gamma_0^2(N)}^{\Gamma_0^2(N\ell)}(F) = \mathrm{Tr}_{\Gamma_1^2(N)}^{\Gamma_1^2(N\ell)}(F).$$

For our main result we will need an explicit set of representatives for $\Gamma_0^2(N\ell)\backslash\Gamma_0^2(N)$. We recall the following construction given in the proof of Theorem 4.6 in [8].

Let

$$P := \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{F}_\ell) \right\}$$

be the Siegel parabolic subgroup of $\mathrm{Sp}_4(\mathbb{F}_\ell)$, and define

$$\omega_j := \begin{pmatrix} 1_{2-j} & 0 & 0_{2-j} & 0 \\ 0 & 0_j & 0 & -1_j \\ 0_{2-j} & 0 & 1_{2-j} & 0 \\ 0 & 1_j & 0 & 0_j \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{F}_\ell),$$

for $0 \leq j \leq 2$. Using these matrices we have the Bruhat decomposition (see Section 14 in [12] for details)

$$\mathrm{Sp}_4(\mathbb{F}_\ell) = \bigsqcup_{j=0}^{2} P\omega_j P.$$

Furthermore, we have the Levi decomposition $P = MN$, where the Levi factor is

given by

$$M := \left\{ m(A) := \begin{pmatrix} A & 0 \\ 0 & {}^{T}A^{-1} \end{pmatrix} : A \in \mathrm{GL}_2(\mathbb{F}_\ell) \right\},$$

and the unipotent radical is given by

$$N := \left\{ n(B) := \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} : {}^{T}B = B, B \in \mathrm{M}_2(\mathbb{F}_\ell) \right\}.$$

Combining these we have that a complete set of representatives of $P\backslash P\omega_j P$ is given by

$$\left\{ \omega_j n(B_j) m(A) : {}^{T}B_j = B_j,\ B_j \in \mathrm{M}_j(\mathbb{F}_\ell),\ A \in P_{2,j}(\mathbb{F}_\ell)\backslash \mathrm{GL}_2(\mathbb{F}_\ell) \right\},$$

where $\mathrm{M}_j(\mathbb{F}_\ell)$ is embedded into $\mathrm{M}_2(\mathbb{F}_\ell)$ by $B_j \mapsto \begin{pmatrix} 0 & 0 \\ 0 & B_j \end{pmatrix}$ and

$$P_{2,j}(\mathbb{F}_\ell) := \left\{ \gamma \in \mathrm{GL}_2(\mathbb{F}_\ell) : \gamma = \begin{pmatrix} * & * \\ 0_{j,2-j} & * \end{pmatrix} \right\}.$$

Note that $P_{2,j}(\mathbb{F}_\ell) = \mathrm{GL}_2(\mathbb{F}_\ell)$ when $j \neq 1$. We can lift these representatives to representatives of $\Gamma_0^2(N\ell)\backslash\Gamma_0^2(N)$ using strong approximation, where we identify the lifts with their image modulo ℓ. Thus, for ω_j satisfying

$$\omega_j \equiv 1_4 \pmod{N} \text{ and } \omega_j \equiv \begin{pmatrix} 1_{2-j} & 0 & 0_{2-j} & 0 \\ 0 & 0_j & 0 & -1_j \\ 0_{2-j} & 0 & 1_{2-j} & 0 \\ 0 & 1_j & 0 & 0_j \end{pmatrix} \pmod{\ell},$$

we have that

$$\{\omega_j n(B_j) m(A) : 0 \leq j \leq 2\}$$

is a complete set of representatives for $\Gamma_0^2(N\ell)\backslash\Gamma_0^2(N)$. Furthermore, we may assume for our lifted $m(A)$ that $\det A = 1$. This gives us that χ^{-1} is trivial on our set of representatives. Using these representatives we rewrite

$$\mathrm{Tr}(F) = F + \sum_{b=1}^{\ell-1} \sum_{A} F|\omega_1 n(b) m(A) + \sum_{\substack{{}^T B = B \\ B \,(\mathrm{mod}\ \ell)}} F|\omega_2 n(B).$$

To complete this section, we give a more explicit expression for the last term in the trace. Note that since F is a cusp form, we have that $F|\omega_2$ is also a cusp form. In particular, we know that the Fourier expansion is of the following form,

$$(F|\omega_2)(Z) = \sum_{T \in \frac{1}{\ell}\Lambda_2} a(T) \exp(\mathrm{Tr}(TZ)).$$

From [9] we have that

$$\sum_{\substack{{}^T B = B \\ B \,(\mathrm{mod}\ \ell)}} (F|\omega_2 n(B))(Z) = \ell^3 \sum_{T \in \Lambda_2} a(T) \exp(\mathrm{Tr}(TZ)).$$

Using this equality we obtain

$$\sum_{\substack{{}^T B = B \\ B \,(\mathrm{mod}\ \ell)}} F|\omega_2 n(B) = \ell^3 F|\omega_2 \begin{pmatrix} \ell_2 & 0 \\ 0 & 1_2 \end{pmatrix} U(\ell).$$

We will need the following lemma, which is from [9], for the proof of our main result. We prove it here for completeness.

Lemma 54. *Let $F \in S^2_\rho(N\ell, \chi)$ be an eigenform with associated character χ defined modulo N. Then, for some ρ' there exists $G \in S^2_{\rho'}(N, \chi)$ such that $F \equiv_{fc} G \pmod{\ell}$.*

Proof. As we are only proving this for degree 2 Siegel modular forms, we will drop the 2 from the superscript for the remainder of this proof.

By Lemma 44 we have that the Hecke operator $U(\ell)$ is an injective map from $S_\rho(N\ell, \chi)$ into itself. As $S_\rho(N\ell, \chi)$ is a finite dimensional vector space, this means that $U(\ell)$ is surjective as well. Thus, we can find a $G' \in S_\rho(N\ell, \chi)$ such that $G'|U(\ell) = F$. Using this G', define the following form

$$G = G'| \begin{pmatrix} \frac{1}{\ell} \cdot 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} |\omega_2^{-1}.$$

Let $\mathcal{K}_{\ell-1} := \ell^2 \Theta_L^2$, where Θ_L^2 is the theta series from Equation 4.2. Then, from [11], we have that $\mathcal{K}_{\ell-1}$ satisfies

$$\mathcal{K}_{\ell-1} \equiv_{fc} \mathcal{K}_{\ell-1}|\omega_1 \equiv_{fc} 0 \pmod{\ell},$$

$$\mathcal{K}_{\ell-1}|\omega_2 \equiv_{fc} 1 \pmod{\ell}.$$

Furthermore, $\mathcal{K}_{\ell-1}$ has integral Fourier coefficients. Applying our formula for the trace we may write

$$\begin{aligned} \mathrm{Tr}(G\mathcal{K}_{\ell-1}^{\ell^m}) = G\mathcal{K}_{\ell-1}^{\ell^m} &+ \sum_{b=1}^{\ell-1} \sum_{A} (G\mathcal{K}_{\ell-1}^{\ell^m})|\omega_1 n(b) m(A) \\ &+ \sum_{\substack{{}^T B = B \\ B \,(\mathrm{mod}\ \ell)}} (G\mathcal{K}_{\ell-1}^{\ell^m})|\omega_2 n(B), \end{aligned}$$

for an arbitrary constant m. Our goal is to show that

$$\mathrm{ord}_\ell(\mathrm{Tr}(G\mathcal{K}_{\ell-1}^{\ell^m})) \to \infty \text{ as } m \to \infty.$$

To this end we will examine each piece of the sum separately and use the fact that $\mathrm{ord}_\ell(G|\omega_i) > -\infty$ for $i = 0, 1, 2$. This fact comes from Proposition 3.3 in [9].

First,

$$\mathrm{ord}_\ell(G\mathcal{K}_{\ell-1}^{\ell^m}) = \mathrm{ord}_\ell(G) + \ell^m \,\mathrm{ord}_\ell(\mathcal{K}_{\ell-1}),$$

which becomes arbitrarily large with m.

Second,

$$\mathrm{ord}_\ell\left(\sum_{b=1}^{\ell-1}\sum_{A}(G\mathcal{K}_{\ell-1}^{\ell^m})|\omega_1 n(b)m(A)\right) \quad \geq \quad 1 + \mathrm{ord}_\ell(G|\omega_1) + \ell^m \,\mathrm{ord}_\ell(\mathcal{K}_{\ell-1}|\omega_1),$$

which also becomes arbitrarily large with m.

Third, to examine the last term of the summation we rewrite $\mathcal{K}_{\ell-1}^{\ell^m} = 1 + \ell^{m+1}X$, where X is a Fourier series with integral Fourier coefficients. Then,

$$\begin{aligned}\sum_{\substack{{}^TB=B \\ B \,(\mathrm{mod}\ \ell)}} (G\mathcal{K}_{\ell-1}^{\ell^m})|\omega_2 n(B) &= G|\omega_2| \begin{pmatrix} \ell \cdot 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} U(\ell) \\ &\quad + \ell^{m+1}(G|\omega_2 \cdot X)| \begin{pmatrix} \ell \cdot 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} U(\ell) \\ &= F + \ell^{m+1}(G|\omega_2 \cdot X)| \begin{pmatrix} \ell \cdot 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} U(\ell).\end{aligned}$$

Note that $\mathrm{ord}_\ell(X) \geq 0$. Combining we have,

$$\mathrm{ord}_\ell(F - \mathrm{Tr}(G\mathcal{K}_{\ell-1}^{\ell^m})) \quad \geq \quad m + 1 + \mathrm{ord}_\ell(G|\omega_2).$$

Which goes to infinity as m does. Thus, for large enough m, we have that $\mathrm{Tr}(G\mathcal{K}_{\ell-1}^{\ell^m}) \equiv_{fc} F \pmod{\ell}$. □

4.6 Main result

In this section, we will prove the following theorem, which constitutes our main result. Note, the corresponding result for scalar valued forms can be found in [42].

Theorem 55. *Let $F \in S_\rho^2(\ell^r N, \chi)$ be an eigenform with the highest weight vector of ρ satisfying $k_2 \geq 3$ and χ defined modulo ℓN with $\ell \nmid N$. Let Σ be the set of rational primes which divide ℓN. Then, for some χ' and ρ', there exists an eigenform $G \in S_{\rho'}^2(N, \chi')$ such that $F \equiv_\Sigma G \pmod{\ell}$.*

Proof. Throughout we are working with degree 2 Siegel modular forms, so we will drop the superscript. Furthermore, throughout the proof we will not be explicit about the weights of the intermediate forms, but we will make a note about the final weight ρ' at the end. Finally, we will take finite extensions of $\mathbb{Q}$ as needed.

As χ is a character modulo ℓN we obtain a factorization $\chi = \omega^i \kappa$, where ω is the unique character of conductor ℓ and order $\ell - 1$, i.e., the Teichmüller character, and κ is a character modulo N.

Let $E \in M_k(\ell, \omega^{-i})$ be a form from the sequence in Theorem 41 such that $E \equiv_{fc} 1 \pmod{\ell}$. Consider the product of Siegel modular forms FE.

We first want to show that this product transforms correctly under the action of $\Gamma_0(\ell^r) \cap \Gamma_1(N)$. Let $\gamma \in \Gamma_0(\ell^r) \cap \Gamma_1(N)$. Then,

$$\begin{aligned}(F(Z)E(Z))|\gamma &= \kappa\omega^i(\gamma)\omega^{-i}(\gamma)\det(cZ+d)^{-k}\rho(cZ+d)^{-1}F(\gamma Z)E(\gamma Z) \\ &= F(Z)E(Z).\end{aligned}$$

Thus, the product is a form of the desired level and of character κ. We will denote the weight of this form by ρ'. Furthermore, as $E \equiv_{fc} 1 \pmod{\ell}$ we have that

$$FE \equiv_{fc} F \pmod{\ell}.$$

Thus, FE is an eigenform when reduced modulo ν for a prime ν lying above ℓ in $\mathbb{Q}(F)$, and Lemma 40 gives us

$$FE \equiv_{\Sigma} F \pmod{\ell}.$$

Let $\mathcal{O}_\nu$ be an extension of $\mathbb{Z}_\ell$ which has ν as its maximal ideal. As $S_{\rho'}(N\ell^r, \kappa)$ is a finite, free $\mathcal{O}_\nu$ module, we can apply the Deligne-Serre lifting lemma as stated in Corollary 91 to obtain an eigenform $F_1 \in S_{\rho'}(N\ell^r, \kappa)$ such that

$$F_1 \equiv_{\Sigma} F \pmod{\ell}.$$

We can now apply Corollary 46 $(r-1)$ times to F_1 in order to obtain a form $F_2 \in S_{\rho'}(N\ell, \chi')$ for some ρ' and χ', which is congruent in Fourier coefficients modulo ℓ to F. By the same argument used above we can find an eigenform in $S_{\rho'}(N\ell, \chi')$ satisfying this same congruence.

We now apply Lemma 54 to F_2 to obtain a form $F_3 \in S_{\rho'}(N, \chi')$ which is

congruent in Fourier coefficients to F modulo ν. Just as before, this yields the desired eigenform G.

Finally, with regards to the weight ρ' of G, if we let the highest weight vector of ρ be (k_1, k_2), then the highest weight vector of ρ' is

$$(\ell(k_1 + i\ell^{m_1} + \ell^{m_2-1}(\ell-1)), \ell(k_2 + i\ell^{m_1} + \ell^{m_2-1}(\ell-1)),$$

where m_1 and m_2 are both sufficiently large integers. In particular, we have that

$$(k_1', k_2') \equiv (k_1 + i, k_2 + i) \pmod{\ell - 1},$$

where (k_1', k_2') is the highest weight vector of ρ'. □

4.7 Examples

We conclude this chapter by giving a computational example of the result in Theorem 55.

In order to construct the example we will use the Saito-Kurokawa lift. We suppress the details of this lift here but the reader is referred to [53], [54], and [58] for classical and representation theoretic treatments of the Saito-Kurokawa lift.

Before giving our examples, we will present some necessary facts concerning this lifting. Let $f \in S^1_{2k-2}(N, \chi)$ be a normalized eigenform with $\chi = \psi^2$ for some Dirichlet character ψ defined modulo N. Let $F_f \in S^2_k(N, \psi)$ denote a Saito-Kurokawa lift of f. Then, from Theorem 5.2 in [54] we have that the L-function associated to

F_f factors as

$$L(s,F) = \left(\prod_{p|N} (1-p^{k-1-s})(1-p^{k-2-s}) \right) \zeta(s-k+1)\zeta(s-k+2)L(s,f).$$

From this factorization we can deduce the following relationship between the eigenvalues of F_f and f for $p \nmid N$

$$\begin{aligned} \lambda_{F_f}(p;S) &= \lambda_f(p) + \chi(p)p^{k-2}(p+1), \\ \lambda_{F_f}(p^2;S) &= \chi(p)^2 p^{2k-6}(p^2-1) + \chi(p)\lambda_f(p)p^{k-3}(p+1), \end{aligned} \tag{4.3}$$

where $\lambda_{F_f}(p;S)$ and $\lambda_{F_f}(p^2;S)$ are eigenvalues with respect to certain Hecke operators $T_S(p)$ and $T_S(p^2)$ which are defined in [4]. It should be noted that $T_S(p)$ and $T_S(p^2)$ also generate the local Hecke algebra H_p, so it is sufficient to work with these operators.

Example 56. We construct our example by setting $f \in S_4(\Gamma_1^1(5))$ and $g \in S_{40}(\mathrm{SL}_2(\mathbb{Z}))$ be the forms given in Example 38.

From the above discussion, we have that the Saito-Kurokawa lift of f and of g, which we denote by F_f and F_g, respectively, satisfy $F_f \in S_3(\Gamma_1^2(5))$ and $F_g \in S_2 1(\mathrm{Sp}_4(\mathbb{Z}))$. Furthermore, F_f and F_g are eigenforms with eigenvalues satisfying Equation 4.3.

Due to the equations given in 4.3 and the fact that f and g are congruent modulo 5, it is elementary to check that the eigenvalues of F_f and F_g are congruent modulo 5.

Chapter 5

Applications to Galois representations

In this chapter, we give a brief introduction to Galois representations and provide some applications of the previous level stripping arguments to Galois representations. Throughout, we will use $G_{\mathbb{Q}}$ to denote $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and after fixing an odd prime ℓ, we fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

5.1 Galois representations

In this section, we introduce the basic objects of study and a few examples. We begin by recalling the definition of a profinite group.

Definition 57. We say that a topological group G is a *profinite group* if it can be expressed as the inverse limit of finite groups equipped with the discrete topology. We refer to the topology of G as the *profinite topology*.

Note, if $G = \varprojlim_i G_i$, where the G_i are finite groups equipped with the discrete topology, then the topology of G is simply given as the subspace topology of the

product topology on $\prod_i G_i$. We consider two more explicit examples of this, both of which will be used throughout the remainder of the chapter.

Example 58. 1. Consider the collection of finite groups $\{\mathbb{Z}/\ell^n\mathbb{Z} : n \in \mathbb{Z}^+\}$, each equipped with the discrete topology. We take the inverse limit of this collection with respect to the reduction homomorphism $\mathbb{Z}/\ell^n\mathbb{Z} \to \mathbb{Z}/\ell^{n-1}\mathbb{Z}$ to obtain the ℓ-adic integers

$$\mathbb{Z}_\ell = \varprojlim_n \mathbb{Z}/\ell^n\mathbb{Z}.$$

It is well known that $\mathbb{Z}_\ell$ comes equipped with the absolute value

$$|a|_\ell = \ell^{-\operatorname{ord}_\ell(a)}.$$

One can check that the topology induced by this absolute value agrees with the profinite topology described above.

2. Consider the group $G_\mathbb{Q}$. We have that

$$G_\mathbb{Q} = \varprojlim_K \operatorname{Gal}(K/\mathbb{Q}),$$

where each $K/\mathbb{Q}$ is a finite Galois extension and the inverse limit is taken with respect to the restriction homomorphism $\operatorname{Gal}(K_1/\mathbb{Q}) \to \operatorname{Gal}(K_2/\mathbb{Q})$ when $K_2 \subseteq K_1$. We have a topology on $G_\mathbb{Q}$ given by the following basis of open sets,

$$\left\{\sigma \operatorname{Gal}(\overline{\mathbb{Q}}/K) : [K : \mathbb{Q}] < \infty, \sigma \in G_\mathbb{Q}\right\}.$$

This topology is referred to as the Krull topology, and once again, one can check that this topology is equivalent to the profinite topology defined above.

We are now prepared to define the objects of primary interest.

Definition 59. Let d be a positive integer and ℓ an odd prime. Let K be either the complex numbers or an extension of $\mathbb{Q}_\ell$. A d-dimensional Galois representation is a continuous homomorphism

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}(V),$$

where V is a d-dimensional K vector space and the topology on V is induced from the topology on K. In the case that $K = \mathbb{C}$, we call ρ an Artin representation, and in the case that $\mathbb{Q}_\ell \subseteq K$, we call ρ an ℓ-adic representation. Note, after choosing a basis, we can assume that

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_d(K).$$

As $G_{\mathbb{Q}}$ is equipped with the profinite topology and a finite dimensional $\mathbb{C}$ vector space is equipped with the Euclidean topology, the continuity condition ensures that the image of any Artin representation is finite. To see this, note that the identity $I_d \in \mathrm{GL}_d(\mathbb{C})$ has an open neighborhood, denoted U, which contains no non-trivial subgroup. By continuity, the pullback of U is an open neighborhood of the identity in $G_{\mathbb{Q}}$, and since $G_{\mathbb{Q}}$ is profinite the pullback contains a subgroup of finite index, say U_G. As ρ is a homomorphism, we have that $\rho(U_G)$ is a subgroup of $\mathrm{GL}_d(\mathbb{C})$, and hence $U_G \subseteq \ker(\rho)$. Thus, ρ factors through a subgroup of finite index, so we must have that the image of ρ is finite. We can think of this as implying that the topologies of $G_{\mathbb{Q}}$ and $\mathbb{C}$ are incompatible in some sense.

Due to this, we will be interested in ℓ-adic Galois representations, and throughout the remainder of the chapter, it should be assumed that all Galois representations are ℓ-adic. Note, in this case, $\mathrm{GL}_d(K)$ comes equipped with the profinite topology as well, and hence the continuity condition is not so restrictive in this setting as it is for Artin representations.

We begin by giving some basic definitions and results concerning ℓ-adic Galois

representations which will be useful to us.

Definition 60. If we let $c \in G_{\mathbb{Q}}$ denote complex conjugation, then we say that ρ is *odd* if

$$\det \rho(c) = -1.$$

Definition 61. We say that ρ is *irreducible* if ρ has no nontrivial invariant subspaces.

Recall the following subgroups of $G_{\mathbb{Q}}$ from algebraic number theory. Let $\mathfrak{p}$ be a prime in $\overline{\mathbb{Z}}$, the ring of integers of $\overline{\mathbb{Q}}$. Then, the decomposition group of $\mathfrak{p}$ is

$$D_{\mathfrak{p}} = \{\sigma \in G_{\mathbb{Q}} : \sigma(\mathfrak{p}) = \mathfrak{p}\},$$

and the inertia group of $\mathfrak{p}$ is given by

$$I_{\mathfrak{p}} = \left\{\sigma \in D_{\mathfrak{p}} : \sigma(x) \equiv x \pmod{\mathfrak{p}} \text{ for all } x \in \overline{\mathbb{Z}}\right\}.$$

Using the inertia group we are prepared to give the following definition.

Definition 62. Let $p \neq \ell$ be a rational prime. We say ρ is *unramified at* p if for any maximal ideal $\mathfrak{p} \subset \overline{\mathbb{Z}}$ lying over p, $I_{\mathfrak{p}}$ is contained in the kernel of ρ.

Finally, we have the following standard definition from representation theory.

Definition 63. We say that $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_d(K)$ is *semi-simple* if K^d can be written as a direct sum of simple $G_{\mathbb{Q}}$-modules K_i, i.e., modules $G_{\mathbb{Q}}$-modules K_i such that K_i has no proper submodule.

To expand on this last definition just a bit, we know that K^d has a composition series of ρ-invariant subspaces, i.e.,

$$K^d = K_0 \supset K_1 \supset \cdots \supset K_n = 0,$$

such that each K_i/K_{i+1} is simple. Set

$$K' = \bigoplus_{i=0}^{n-1} K_i/K_{i+1},$$

and use ρ to define a new representation

$$\rho^{ss} : G_{\mathbb{Q}} \to \bigoplus_{i=0}^{n-1} \mathrm{GL}(K_i/K_{i+1}).$$

We call ρ^{ss} the *semi-simplification* of ρ.

We also have the notion of isomorphic Galois representations. If ρ and ρ' are both d-dimensional Galois representations and there is some $M \in \mathrm{GL}_d(K)$ which satisfies

$$\rho'(\sigma) = M^{-1}\rho(\sigma)M \text{ for all } \sigma \in G_{\mathbb{Q}},$$

then we say that ρ and ρ' are isomorphic. We will need the following proposition, as we will be interested in reducing the image of a Galois representation modulo a prime.

Proposition 64. *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_d(K)$ be an ℓ-adic Galois representation. Then, there exists a $\rho' : G_{\mathbb{Q}} \to \mathrm{GL}_d(\mathcal{O}_K)$ such that ρ and ρ' are isomorphic.*

Proof. Let $\Lambda = \mathcal{O}_K^d$. As Λ is a lattice of K^d, we have that Λ is compact in K^d. Since ρ is continuous, we have that the map

$$F_\rho : K^d \times G_{\mathbb{Q}} \to K^d, \text{ given by } (v, \sigma) \mapsto \rho(\sigma) \cdot v$$

is continuous. Hence, $\Lambda' := F_\rho(\Lambda \times G_{\mathbb{Q}})$ is compact in K^d. Then, Λ' lies in $\varpi^{-r}\Lambda$ for some $r \in \mathbb{Z}^+$, where ϖ is a uniformizer of $\mathcal{O}_K$. Since Λ' contains Λ, the rank of Λ' is at least d. Furthermore, as $\mathcal{O}_K$ is an integral domain, we have that Λ' is free, and it

follows that the rank of Λ' is d. Therefore, by choosing an $\mathcal{O}_K$ basis of Λ', we obtain the desired representation

$$\rho' : G_{\mathbb{Q}} \to \mathrm{GL}_d(\mathcal{O}_K).$$

□

To conclude this section we will give a few well-known examples of Galois representations.

Example 65. Consider the following sequence of maps

$$G_{\mathbb{Q}} \xrightarrow{\pi} \mathrm{Gal}\left(\mathbb{Q}(\mu_\ell)/\mathbb{Q}\right) \xrightarrow{\sim} (\mathbb{Z}/\ell\mathbb{Z})^\times \xrightarrow{\omega} \mathbb{Z}_\ell^\times,$$

where μ_ℓ is a primitive ℓ^{th} root of unity, π is the projection to $G_{\mathbb{Q}}/\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_\ell)\right) \cong \mathrm{Gal}\left(\mathbb{Q}(\mu_\ell)/\mathbb{Q}\right)$, and ω is the Teichmüller character. If we take the composition of these maps, then we obtain the cyclotomic character at ℓ, i.e.,

$$\chi_\ell : G_{\mathbb{Q}} \to \mathbb{Z}_\ell^\times.$$

This is a continuous, one-dimensional, odd, ℓ-adic Galois representation. Note, for any rational prime $p \neq \ell$, we have that p is unramified in $\mathbb{Q}(\mu_\ell)$, i.e., for any $\wp$ lying over p in $\mathbb{Q}(\mu_\ell)$ we have $I_\wp$ is trivial. Thus, $I_\wp \subset \ker \chi_\ell$, i.e., χ_ℓ is unramified at p. In fact, for any maximal ideal $\mathfrak{p} \subset \overline{\mathbb{Z}}$ lying over a rational prime $p \neq \ell$, we have

$$\chi_\ell(\mathrm{Frob}_\mathfrak{p}) = p,$$

where $\mathrm{Frob}_\mathfrak{p}$ is the (arithmetic) Frobenius element at $\mathfrak{p}$ in $G_{\mathbb{Q}}$. It is a consequence of

the continuity of χ_ℓ and the Chebotarov density theorem that the set

$$\{\mathrm{Frob}_\mathfrak{p} \in G_\mathbb{Q} : \mathfrak{p} \cap \mathbb{Z} = p, p \neq \ell\}$$

is dense in $G_\mathbb{Q}$, and hence χ_ℓ is completely determined by its image on this set.

Now, we will consider a slightly more involved example.

Example 66. Let $E/\mathbb{Q}$ be an elliptic curve of conductor N with distinguished point $\mathcal{O}$. For m a positive integer, we define the m-torsion subgroup by

$$E[m] = \left\{P \in E(\overline{\mathbb{Q}}) : m \cdot P = \mathcal{O}\right\}.$$

It is a basic fact from the theory of elliptic curves that $E[m] \cong (\mathbb{Z}/m\mathbb{Z})^2$. Since we have an action of $G_\mathbb{Q}$ on the set $E[m]$, we have a mod m representation

$$\overline{\rho}_{m,E} : G_\mathbb{Q} \to \mathrm{Aut}(E[m]) \cong \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z}),$$

where the isomorphism requires a choice of basis. If we equip $\mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$ with the discrete topology, then it is not hard to show that $\overline{\rho}_{m,E}$ is continuous.

Our goal is to use these "mod m" representations to construct ℓ-adic representations. Towards this goal, we have the following definition.

Definition 67. The *ℓ-adic Tate module* of an elliptic curve E is the group

$$T_\ell(E) := \varprojlim_n E[\ell^n],$$

where the inverse limit is taken with respect to the multiplication by ℓ map.

It is clear that $T_\ell(E) \cong \mathbb{Z}_\ell^2$. Furthermore, the action of $G_\mathbb{Q}$ commutes with the

inverse limit, so we obtain an action of $G_{\mathbb{Q}}$ on $T_\ell(E)$. Thus, we have the following ℓ-adic representation,

$$\rho_{\ell,E} : G_{\mathbb{Q}} \to \mathrm{Aut}(T_\ell(E)) \cong \mathrm{GL}_2(\mathbb{Z}_\ell),$$

where the isomorphism requires a choice of basis for $T_\ell(E)$. Note, to show that this representation is continuous, one needs to recall a certain universal property satisfied by the inverse limit, see Proposition IV.2.5 in [57] for example. Furthermore, regarding the other properties of interest to us, we have the following theorem.

Theorem 68. *[23, Thm. 9.4.1] The Galois representation $\rho_{\ell,E}$ is irreducible, odd, and unramified at every prime $p \nmid \ell N$. Furthermore, for any such p, let $\mathfrak{p} \subset \overline{\mathbb{Z}}$ be any maximal ideal lying over p. Then, the characteristic polynomial of $\rho_{\ell,E}(\mathrm{Frob}_{\mathfrak{p}})$ is*

$$x^2 - a_p(E)x + p,$$

where $a_p(E) = p + 1 - |\overline{E}(\mathbb{F}_p)|$ is the aptly named "trace of Frobenius".

Just as in the previous example, we see that giving the image of the Frobenius elements considered in the theorem under the map $\rho_{\ell,E}$ is sufficient to completely determine the map.

Our final two examples are quite complicated compared to the previous two examples, however they are more closely related to the setting of interest for us. To avoid getting too far off course, several results will be stated without proof. For a complete treatment of this construction, the reader is referred to §9.5 in [23].

Example 69. Let ℓ be an odd prime. Let $X_1(N) = \Gamma_1^1(N)\backslash\mathfrak{h}_1^*$ be the compactified modular curve. In fact, $X_1(N)$ is a projective nonsingular algebraic curve defined over

$\mathbb{Q}$, and we will denote its genus by g. If we consider $X_1(N)$ as a being defined over $\mathbb{C}$, and denote this by $X_1(N)_{\mathbb{C}}$, then we can think of $X_1(N)_{\mathbb{C}}$ as a compact Riemann surface. Denote the Jacobian of $X_1(N)_{\mathbb{C}}$ by $J_1(N)$, which is isomorphic to the g dimensional torus $\mathbb{C}^g/\Lambda$ for some lattice Λ. Set $\mathrm{Pic}^0(X_1(N)) = \mathrm{Div}^0(X_1(N))/\mathrm{Prin}(X_1(N))$, where $\mathrm{Div}^0(X_1(N))$ and $\mathrm{Prin}(X_1(N))$ denote the degree 0 divisors of $X_1(N)$ and the principal divisors of $X_1(N)$, respectively. One can identify $\mathrm{Pic}^0(X_1(N))$ with a subgroup of $\mathrm{Pic}^0(X_1(N)_{\mathbb{C}})$, which in turn is isomorphic to $J_1(N)$ by Abel's Theorem. Thus, we obtain the following inclusion map,

$$\mathrm{Pic}^0(X_1(N))[\ell^n] \hookrightarrow \mathrm{Pic}^0(X_1(N)_{\mathbb{C}})[\ell^n] \cong (\mathbb{Z}/\ell^n\mathbb{Z})^{2g},$$

where we are using $[\ell^n]$ to denote the ℓ^n torsion subgroup, just as in the previous example.

Just as before, we define the ℓ-adic Tate module of $X_1(N)$ by

$$T_\ell(\mathrm{Pic}^0(X_1(N))) = \varprojlim_n (\mathrm{Pic}^0(X_1(N))[\ell^n]) \cong \mathbb{Z}_\ell^{2g}.$$

We also have an action of $G_{\mathbb{Q}}$ on $\mathrm{Div}^0(X_1(N))$ given by

$$\Big(\sum n_P(P)\Big)^\sigma = \sum n_P(P^\sigma),$$

for any $\sigma \in G_{\mathbb{Q}}$. This action descends to an action of $G_{\mathbb{Q}}$ on $\mathrm{Pic}^0(X_1(N))$ and also commutes with the inverse limit, i.e., after a choice of basis we obtain a continuous homomorphism

$$\rho_{X_1(N),\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_{2g}(\mathbb{Z}_\ell) \subset \mathrm{GL}_{2g}(\mathbb{Q}_\ell).$$

Analogous to Theorem 68 in the previous example, we have the following theorem

regarding the properties of $\rho_{X_1(N),\ell}$.

Theorem 70. *[23, Thm. 9.5.1] The Galois representation $\rho_{X_1(N),\ell}$ is unramified at every prime $p \nmid \ell N$. Furthermore, for any such p, let $\mathfrak{p} \subset \overline{\mathbb{Z}}$ be any maximal ideal lying over p. Then, $\rho_{\ell,E}(\mathrm{Frob}_{\mathfrak{p}})$ satisfies*

$$x^2 - T_p x + \langle p \rangle p = 0.$$

Example 71. This example can be viewed as a continuation of the previous example to the setting of weight 2 eigenforms, and as such, we will use the same setup as in the previous example.

Let $f \in S_2^1(N, \chi)$ be a normalized eigenform and define the following set

$$I_f = \{T_n : T_n f = 0\} \cup \{\langle n \rangle : \gcd(n, N) > 0\}.$$

In fact, I_f is an ideal of the $\mathbb{Z}$-algebra of Hecke operators. Using this ideal, we define the Abelian variety of f by

$$A_f = J_1(N)/I_f J_1(N),$$

where the quotient makes sense because we have an action of the Hecke operators on $J_1(N)$ given by composition, i.e., $T \cdot [\phi] = [\phi \cdot T]$ for any Hecke operator T and $[\phi] \in J_1(N)$.

It is not difficult to see that the map

$$H^{\mathbb{Z}}/I_f \to \mathbb{Z}\left[\left\{a_f(n) : n \in \mathbb{Z}^+\right\}\right], \text{ where } T_p + I_f \mapsto a_p,$$

is an isomorphism. Thus, we have an action of $\mathbb{Z}[\{a_f(n) : n \in \mathbb{Z}^+\}]$ on A_f. The field

$\mathbb{Q}(f)$ is the field of fractions of $\mathbb{Z}[\{a_f(n) : n \in \mathbb{Z}^+\}]$ and the index $d = [\mathbb{Q}(f) : \mathbb{Q}]$ is the dimension of A_f. Once again, we form the Tate module

$$T_\ell(A_f) = \varprojlim_n A_f[\ell^n] \cong \mathbb{Z}_\ell^{2d},$$

and the action of $\mathbb{Z}[\{a_f(n) : n \in \mathbb{Z}^+\}]$ on A_f extends to $T_\ell(A_f)$. The following lemma also gives the action of $G_\mathbb{Q}$ on $A_f[\ell^n]$.

Lemma 72. *[23, Lemma 9.5.2] The map* $\mathrm{Pic}^0(X_1(N))[\ell^n] \to A_f[\ell^n]$ *is surjective with kernel stable under the action of* $G_\mathbb{Q}$.

Thus, we have an action of $G_\mathbb{Q}$ on $A_f[\ell^n]$ which extends to an action on $T_\ell(A_f)$ and commutes with the action of $\mathbb{Z}[\{a_f(n) : n \in \mathbb{Z}^+\}]$. Thus, by choosing an appropriate basis, we have a Galois representation

$$\rho_{A_f,\ell} : G_\mathbb{Q} \to \mathrm{GL}_{2d}(\mathbb{Q}_\ell).$$

This representation is continuous since $\rho_{X_1(N),\ell}$ is continuous and is unramified at all primes $p \nmid \ell N$ since $\ker(\rho_{A_f,\ell}) \subseteq \ker(\rho_{X_1(N),\ell})$. Furthermore, for any maximal ideal of $\mathfrak{p} \subset \overline{\mathbb{Z}}$ which lies above p we have that $\rho_{A_f,\ell}(\mathrm{Frob}_\mathfrak{p})$ satisfies the polynomial

$$x^2 - a_f(p)x + \chi(p)p = 0,$$

since T_p acts by $a_f(p)$ and $\langle p \rangle$ acts by $\chi(p)$.

Since $T_\ell(A_f)$ is a $\mathbb{Z}[\{a_f(n) : n \in \mathbb{Z}^+\}]$-module, we have that $V_\ell(A_f) = T_\ell(A_f) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is a module over $\mathbb{Z}[\{a_f(n) : n \in \mathbb{Z}^+\}] \otimes_\mathbb{Z} \mathbb{Q}_\ell = \mathbb{Q}(f) \otimes_\mathbb{Q} \mathbb{Q}_\ell$. We have an action of $G_\mathbb{Q}$ on $V_\ell(A_f)$ which is $\mathbb{Q}(f) \otimes_\mathbb{Q} \mathbb{Q}_\ell$ linear and Lemma 9.5.3 in [23] gives that $V_\ell(A_f) \cong (\mathbb{Q}(f) \otimes_\mathbb{Q} \mathbb{Q}_\ell)^2$. Thus, after choosing a basis for $V_\ell(A_f)$ we have a homo-

morphism $G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}(f) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$. We can factor $\mathbb{Q}(f) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ to obtain

$$\mathbb{Q}(f) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = \prod_{\nu|\ell} \mathbb{Q}(f)_\nu,$$

where the product is over all primes in $\mathbb{Q}(f)$ which divide ℓ and $\mathbb{Q}(f)_\nu$ represents the ν-adic completion of $\mathbb{Q}(f)$. After projecting on to any factor in this product, we obtain our desired Galois representation

$$\rho_{f,\nu} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}(f)_\nu).$$

Just as in the previous examples, we have the following theorem giving the properties of this Galois representation.

Theorem 73. *[23, Thm. 9.5.4] The Galois representation*

$$\rho_{f,\nu} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}(f)_\nu)$$

is unramified at all $p \nmid \ell N$ and for any maximal ideal $\mathfrak{p} \subset \overline{\mathbb{Z}}$ lying over p we have that $\rho_{f,\nu}(\mathrm{Frob}_{\mathfrak{p}})$ satisfies

$$\det(X \cdot 1_2 - \rho_{f,\nu}(\mathrm{Frob}_{\mathfrak{p}})) = L_p(X, f),$$

where $L_p(X, f)$ was defined at the end of Section 2.1.

5.2 Serre's conjecture

In this section we state Serre's Conjecture, and use this to provide the proper context for Theorem 36.

Let $f \in S_k^1(N, \chi)$ be a normalized eigenform and let ℓ be an odd prime. We have already seen a method for attaching a Galois representation to f in the case that $k = 2$ in the last example from the previous section. To generalize this construction to arbitrary weight, we have a construction of Deligne. In particular, from [20], we have a family of continuous, irreducible representations

$$\rho_{f,\nu} : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_\nu),$$

where ν is a prime lying above ℓ and K_ν is the ν-adic completion of K. Furthermore, for all primes $p \nmid \ell N$ we have

$$\mathrm{Tr}(\rho_{f,\nu}(\mathrm{Frob}_p)) = a_f(p),$$

$$\det(\rho_{f,\nu}(\mathrm{Frob}_p)) = \chi(p)p^{k-1},$$

which characterizes $\rho_{f,\nu}$ up to isomorphism. Note that just as in the last example of the previous section, this gives us that

$$\det(X \cdot 1_2 - \rho_{f,\nu}(\mathrm{Frob}_{\mathfrak{p}})) = L_p(X, f).$$

In contrast to the techniques surveyed in the last example of the previous section, the techniques used in Deligne's construction are quite a bit more complicated, and would take us too far afield to survey in this dissertation.

Applying Proposition 64, we obtain a representation isomorphic to $\rho_{f,\nu}$, whose image is contained in $\mathrm{GL}_2(\mathcal{O}_{K_\nu})$. We will continue to use $\rho_{f,\nu}$ to denote this isomorphic representation. As the image of $\rho_{f,\nu}$ is contained in $\mathrm{GL}_2(\mathcal{O}_{K_\nu})$, it makes sense to talk

about the reduction of $\rho_{f,\nu}$ modulo ν. This reduction gives a representation

$$\overline{\rho}_{f,\nu} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell),$$

where we have composed with the injection which maps the residue field of K_ν into $\overline{\mathbb{F}}_\ell$. However, there is some ambiguity in this construction as we had to choose a basis in order to make sense of the reduction modulo ν. To remove this ambiguity, we take the semisimplification of $\overline{\rho}_{f,\nu}$, which is independent of basis choice. We refer to this semisimplification as the residual representation and continue to denote it by $\overline{\rho}_{f,\nu}$, as is customary.

Given this construction, we say that any representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2\left(\mathcal{O}_{K_\nu}/\nu\mathcal{O}_{K_\nu}\right) \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ is modular if it is isomorphic to some Galois representation of the form $\overline{\rho}_{f,\nu}$. Given this definition, we are naturally led to ask under what conditions is an arbitrary $\overline{\rho}$ modular? The answer to this question is the content of Serre's conjecture, but before stating this precisely, we will give a little motivation.

Note, we obtain a 1-dimensional Galois representation by composing the following maps

$$\det \rho_{f,\nu} : G_{\mathbb{Q}} \xrightarrow{\rho_{f,\nu}} \mathrm{GL}_2(K_\nu) \xrightarrow{\det} \mathrm{GL}_1(K_\nu).$$

As we have already seen, for any prime $p \nmid \ell N$, we have that $\det \rho_{f,\nu}(\mathrm{Frob}_p) = \chi(p)p^{k-1}$. By the Chebotarev density theorem, this gives that $\det \rho_{f,\nu} = \chi\chi_\ell^{k-1}$, where we have identified χ with a character of $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ where ζ_N is a primitive N^{th} root of unity and χ_ℓ is the ℓ-adic cyclotomic character defined in Example 65. If we let $c \in G_{\mathbb{Q}}$ denote complex conjugation, then we have that $\chi_\ell(c) = -1$. Furthermore, we have that $c : \zeta_N \to \zeta_N^{-1}$, i.e., $\chi(c) = \chi(-1)$. Applying the parity condition

$\chi(-1) = (-1)^k$ we have

$$\det \rho_{f,\nu}(c) = \chi(c)\chi_\ell^{k-1}(c) = -1,$$

and hence $\det \overline{\rho}_{f,\nu}(c) = -1$, i.e., $\overline{\rho}_{f,\nu}$ is an odd representation.

In [64], Serre conjectured that this parity condition is sufficient to determine when a continuous, irreducible representation $\overline{\rho}$ is modular, i.e., we have the following conjecture.

Theorem 74. *[64, 3.2.3?] Let $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be a continuous, irreducible, odd representation. Then, for some k, N, χ, we have an isomorphism $\overline{\rho} \cong \overline{\rho}_{f,\nu}$, where $f \in S_k^1(N, \chi)$ is a normalized eigenform.*

One of the major accomplishments in modern number theory is that this conjecture is now a theorem of Khare and Wintenberger ([43],[44]).

In fact, Serre went further than this conjecture and predicted a specific triplet k, N, χ which depends only on $\overline{\rho}$ and ℓ. Note that the elements of this triplet are sometimes referred to as the Serre weight, level, and character, respectively. Before we can properly state this refined conjecture, we will need to give expressions for the Serre weight, level, and character.

We begin with the Serre level. For any $p \neq \ell$, we let $D_{\mathfrak{p}} \subset G_{\mathbb{Q}}$ denote a decomposition group for any prime $\mathfrak{p}$ lying above p in $\overline{\mathbb{Z}}$. We define certain subgroups of $D_{\mathfrak{p}}$ by

$$G_i := \left\{\sigma \in D_{\mathfrak{p}} : \sigma(x) \equiv x \pmod{\mathfrak{p}^{i+1}}\right\}, \text{ for } i \geq 0.$$

These are known as the higher ramification groups. Note that G_0 is the inertia subgroup defined earlier. It is also clear that the higher ramifications groups give a

filtration

$$G_0 \supset G_1 \supset \cdots \supset G_i \supset \ldots.$$

Let V denote the two dimensional $\overline{\mathbb{F}}_\ell$ vector space associated with the representation $\overline{\rho}$. For each $i \geq 0$, we define a subspace $V_i \subset V$ by letting V_i be the set of elements in V which are fixed by all of the elements in G_i. Using this we define the following constant

$$n_p = \sum_{i \geq 0} \frac{\mathrm{codim}(V_i)}{[G_0 : G_i]},$$

where $\mathrm{codim}(V_i)$ denotes the codimension of V_i as a subspace of V. Then, we define the Serre level for $\overline{\rho}$ to be

$$N = \prod_{p \neq \ell} p^{n_p}.$$

The Serre level is precisely the Artin conductor, as defined in Section 11 of [57], for $\overline{\rho}$ with the prime ℓ removed. For our purposes, it is important to recall that $\ell \nmid N$.

Next, we define the appropriate character for $\overline{\rho}$. Note that this construction of the appropriate character is only valid for $\ell \geq 5$. In fact, when $\ell = 3$, Serre produced counterexamples to this being the appropriate character in an unpublished letter written to Ribet. However, as we are primarily concerned with the level, this will not cause any problems.

We begin by considering the determinant of $\overline{\rho}$, i.e., the map

$$\det \overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_1(\overline{\mathbb{F}}_\ell).$$

The image of this map is a finite cyclic subgroup of order prime to ℓ. To see this it is enough to note that $\det \overline{\rho}$ is a one dimensional Galois representation, and the statement follows from class field theory. Furthermore, by class field theory, we have

that the conductor of $\det \overline{\rho}$ divides the Artin conductor, i.e., divides ℓN. Thus, we can identify $\det \overline{\rho}$ with a homomorphism

$$(\mathbb{Z}/\ell N\mathbb{Z})^{\times} \to \mathrm{GL}_1(\overline{\mathbb{F}}_\ell),$$

which in turn factors into a pair of homomorphisms

$$\phi : (\mathbb{Z}/\ell\mathbb{Z})^{\times} \to \mathrm{GL}_1(\overline{\mathbb{F}}_\ell),$$

$$\chi : (\mathbb{Z}/N\mathbb{Z})^{\times} \to \mathrm{GL}_1(\overline{\mathbb{F}}_\ell).$$

As $(\mathbb{Z}/\ell\mathbb{Z})^{\times}$ is cyclic of order $\ell - 1$, we can identify ϕ with some power of the mod ℓ reduction of the cyclotomic character, say $\phi = \overline{\chi}_\ell^r$.

Finally, we introduce the Serre weight. In this presentation of the Serre weight, we follow Edixhoven's discussion of the minimal weight, which was originally formulated for Katz modular forms, rather than the classical modular forms which we use throughout this dissertation. For details, the interested reader is referred to [82].

As the weight is related to the ramification of $\overline{\rho}$ at ℓ, we may consider the restriction of $\overline{\rho}$ to the decomposition group D_ν, where ν is any prime lying over ℓ in $\overline{\mathbb{Z}}$. As we have already fixed an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, we have an isomorphism $D_\nu \cong G_{\mathbb{Q}_\ell} = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$, i.e., $\overline{\rho}$ restricted to D_ν gives a representation

$$\overline{\rho}_\ell : G_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell).$$

Let $\overline{\rho}_\ell^{ss}$ denote the semisimplification of $\overline{\rho}_\ell$. By Proposition 4 in [62], we have that the wild inertia group, which we denoted by G_1 above, is contained in the kernel of $\overline{\rho}_\ell^{ss}$. Using this fact, we have an action of $I_t = G_0/G_1$ on $(\overline{\mathbb{F}}_\ell^{\times})^2$ via. $\overline{\rho}_\ell^{ss}$. Furthermore, we

have an isomorphism $I_t \cong \varprojlim_r \mathbb{F}_{\ell^r}^\times$, which gives that I_t is Abelian. Hence, the action of I_t on $(\overline{\mathbb{F}}_\ell^\times)^2$ via $\overline{\rho}_\ell^{ss}$ is given by two continuous characters

$$\phi_1, \phi_2 : I_t \to \mathrm{GL}_1(\overline{\mathbb{F}}_\ell).$$

We will need the following definition for the formulation of k.

Definition 75. Let $\psi : I_t \to \mathrm{GL}_1(\overline{\mathbb{F}}_\ell)$ be any continuous character. We say ψ is level m if m is the smallest integer for which ψ factors through $\mathrm{GL}_1(\mathbb{F}_{\ell^m})$. Furthermore, the fundamental characters of level m, denoted ψ_i for $0 \leq i < m$, are the m characters given by the following composition

$$I_t \cong \varprojlim_r \mathbb{F}_{\ell^r}^\times \twoheadrightarrow \mathbb{F}_{\ell^m}^\times \overset{\tau_i}{\hookrightarrow} \overline{\mathbb{F}}_\ell^\times,$$

where τ_i are the m embeddings of $\mathbb{F}_{\ell^m}$ into $\overline{\mathbb{F}}_\ell$.

The fundamental characters of level m are given by $\left\{\psi, \psi^\ell, \psi^{\ell^2}, \ldots, \psi^{\ell^{m-1}}\right\}$ for some fixed fundamental character ψ. This comes from the fact that the embeddings τ_i are simply given by the ℓ-power Frobenius map. Furthermore, any character of I_t of level at most m is equal to ψ^i for a unique $0 \leq i < \ell^{m-1}$. Note that the cyclotomic character, χ_ℓ, is the unique fundamental character of level 1.

Regarding the level of ϕ_1 and ϕ_2 we have the following proposition.

Proposition 76. *[64, Prop. 1] The characters ϕ_1 and ϕ_2 are either both of level 1 or both of level 2. In the latter case, $\phi_1^\ell = \phi_2$, $\phi_2^\ell = \phi_1$, and $\overline{\rho}_\ell$ is irreducible.*

Let $K/\mathbb{Q}_\ell$ be a finite Galois extension. Let K^{tr} and K^{nr} be the maximal tamely ramified and maximal unramified subextensions of K, respectively. Suppose $\mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{nr}}) \cong (\mathbb{Z}/\ell\mathbb{Z})^\times$ and that $\mathrm{Gal}(K/K^{\mathrm{tr}}) \cong (\mathbb{Z}/\ell\mathbb{Z})^r$ for some r. Then,

$K^{\mathrm{tr}} = K^{\mathrm{nr}}(\zeta_\ell)$ for a primitive ℓ^{th} root of unity, and by Kummer theory there are $x_1, \ldots, x_r \in K^{\mathrm{nr}}$ such that $K = K^{\mathrm{tr}}(x_1^{1/\ell}, \ldots, x_r^{1/\ell})$. Using this set up we give the following technical definition.

Definition 77. We say K is *little ramified* if all the x_i can be chosen from the units of K^{nr}. Otherwise, we say K is *very ramified*.

We are now prepared to give the formula for the Serre level. We proceed in several cases.

1. Suppose ϕ_1, ϕ_2 are both of level 2. Then, for unique a, b satisfying $0 \leq a < b \leq \ell - 1$ we can write

   $$\phi_1 = \psi^{a+\ell b}, \text{ and } \phi_2 = \psi^{b+\ell a},$$

   where ψ is the fixed fundamental character mentioned above. We set

   $$k = 1 + \ell a + b.$$

2. Suppose ϕ_1, ϕ_2 are both of level 1.

   (a) Suppose that the wild inertia group acts trivially, i.e., $\overline{\rho}_\ell(G_1) = 0$. In this case we say that $\overline{\rho}_\ell$ is tamely ramified. Then, for unique a, b satisfying $0 \leq a \leq b \leq \ell - 2$, we can write

   $$\phi_1 = \chi_\ell^a, \text{ and } \phi_2 = \chi_\ell^b.$$

   We set

   $$k = 1 + \ell a + b.$$

   (b) Suppose that the wild inertia group does not act trivially. Then, there are

unique integers $0 \leq \alpha \leq \ell - 2$ and $1 \leq \beta \leq \ell - 1$ such that the restriction of $\overline{\rho}_\ell$ satisfies

$$\overline{\rho}_\ell|_{G_0} \cong \begin{pmatrix} \chi_\ell^\beta & * \\ 0 & \chi_\ell^\alpha \end{pmatrix}.$$

Let $a = \min\{\alpha, \beta\}$ and $b = \max\{\alpha, \beta\}$. Let $K/\mathbb{Q}_\ell$ be the extension that satisfies $\mathrm{Gal}(\overline{K}/K) = \ker(\overline{\rho}_\ell)$. Note that K satisfies the criteria from Definition 77. If $\beta = \alpha + 1$ and K is very ramified, we set

$$k = 1 + \ell a + b + (\ell - 1).$$

Otherwise, we set

$$k = 1 + \ell a + b.$$

Given the k, N, χ which we just constructed, we are prepared to state Serre's Refined Conjecture.

Theorem 78. *[64, (3.2.4?)] Let $\overline{\rho} : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be a continuous, irreducible, odd representation. Then, for the k, N, χ constructed above, we have an isomorphism $\overline{\rho} \cong \overline{\rho}_{f,\nu}$, where $f \in S_k^1(N, \chi)$ is a normalized eigenform.*

Throughout the late eighties and the early nineties, a large body of work was produced to show that Theorem 74 and Theorem 78 are, in fact, equivalent statements. To see an overview of this body of work, the interested reader is referred to [22] and [60]. In fact, many cases of this equivalence have also been shown in the case that $\ell = 2$, see [16]. Note, due to this equivalence, and the work Khare and Wintenberger mentioned above, we have that Serre's Refined Conjecture is a theorem.

To show a small piece of this body of work, we begin with a continuous,

irreducible, odd representation

$$\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell),$$

which we assume arises from a normalized eigenform of some character, level, and weight, i.e., we assume $\overline{\rho}$ is modular. Then, Serre's Refined Conjecture says, among other things, that the level of the associated modular form should be prime to ℓ. By Theorem 36, we know that we can remove the prime ℓ from the level of a normalized eigenform, while preserving a congruence of eigenvalues modulo ℓ. Using this fact, we prove the following corollary, which is the precise statement of Theorem 2.1 in [60].

Corollary 79. *Suppose $\overline{\rho}$ is modular of level $N\ell^r$ with $(N, \ell) = 1$. Then, $\overline{\rho}$ is modular of level N.*

Proof. Suppose $\overline{\rho}$ arises from f, an eigenform of level $N\ell^r$. Applying Theorem 36 and constructing the associated residual Galois representation, we obtain a representation $\overline{\rho}'$ of level N which satisfies

$$\mathrm{charpoly}(\overline{\rho}(\mathrm{Frob}_p)) = \mathrm{charpoly}(\overline{\rho}'(\mathrm{Frob}_p)), \text{ for all } p \nmid \ell N.$$

To see this equality, first note that as the eigenvalues of T_p and $\langle p \rangle$ are congruent modulo ν for ν lying above ℓ in some extension and for all $p \nmid \ell N$. Thus, the eigenvalues of T_p and $\langle p \rangle$ are equal in $\overline{\mathbb{F}}_\ell$. Hence, we have equality of characteristic polynomials for all Frob_p with $p \nmid \ell N$. As the representations are continuous, the Chebotarev Density Theorem gives that $\mathrm{charpoly}(\overline{\rho}) = \mathrm{charpoly}(\overline{\rho}')$. Applying the Brauer-Nesbitt Theorem (see Theorem 2.4.6 and the following remarks in [82]), we have that the characteristic polynomial of a representation determines the representation uniquely. Thus, $\overline{\rho} \cong \overline{\rho}'$, which completes the proof. □

5.3 A Serre type conjecture in genus 2

In this section, we present an application of Theorem 55 which provides evidence for a conjecture of Herzig and Tilouine.

We begin with the following result which gives the existence of a Galois representation attached to a cuspidal Siegel eigenform of genus 2 as well as the characteristic polynomial of the images of the Frobenius elements with respect to this representation. Note that this result is stated in [66], however the proof is essentially due to Laumon in [50] and Weissauer in [78],[79]. The last reference is necessary to conclude that the associated Galois representation is symplectic in the case that the Siegel eigenform does not arise as a Saito-Kurokawa lift.

Theorem 80. *Let $F \in S^2_\rho(M, \chi)$ be an eigenform with ρ having highest weight vector (k_1, k_2) which satisfies $k_2 \geq 3$. Let $K = \mathbb{Q}(\lambda_F)$ and let ν be a prime lying above ℓ in K. Then, there exists a continuous, semi-simple Galois representation*

$$\rho_{F,\nu} : G_{\mathbb{Q}} \to \mathrm{GL}_4(\mathcal{O}_{K_\nu})$$

such that for all primes $p \nmid \ell M$ we have

$$\det(X \cdot 1_4 - \rho_{F,\nu}(\mathrm{Frob}_{\mathfrak{p}})) = L_p(X, F).$$

and $\rho_{F,\nu}$ is unramified at p, and we remind the reader that $L_p(X, F)$ is the local factor at p of the spinor L-function as defined in Section 2.3.

Throughout the remainder of the section, we will suppose that F is not a Saito-Kurokawa lift, so that we may assume the image of $\rho_{F,\nu}$ is contained in $\mathrm{GSp}_4(\mathcal{O}_{K_\nu})$. Furthermore, we will denote the weight ρ by its highest weight vector (k_1, k_2) in order

to avoid confusion.

As we have chosen a lattice so that our representation takes values in $\mathrm{GSp}_4(\mathcal{O}_{K_\nu})$, we may form the residual representation of $\rho_{F,\nu}$ at ℓ, i.e., the representation

$$\overline{\rho}_{F,\nu} : G_{\mathbb{Q}} \to \mathrm{GSp}_4\left(\mathcal{O}_{K_\nu}/\nu\mathcal{O}_{K_\nu}\right) \hookrightarrow \mathrm{GSp}_4(\overline{\mathbb{F}}_\ell),$$

by reducing the image of $\rho_{F,\nu}$ modulo ν. Once again, we will take the semisimplification of the residual representation and continue to denote it as $\overline{\rho}_{F,\nu}$

With this in mind, we can ask when is a representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{F}}_\ell)$ modular?

In a partial answer to this question, Herzig and Tilouine have given sufficient conditions under which $\overline{\rho}$ is conjectured be modular. The reason this is a partial answer is that Herzig and Tilouine restrict to the ordinary setting. In order to state precisely the conjecture of Herzig and Tilouine we need a bit of background. For more details the reader is referred to [33].

First, we say that $\overline{\rho}$ is odd if $\mu \circ \overline{\rho}(c) = -1$, where $c \in G_{\mathbb{Q}}$ is complex conjugation and μ is the similitude factor. Note, to see that this is necessary for a representation to be modular, the reader is referred to Section 9 of [73].

Second, we need the following definition.

Definition 81. Let $F \in S^2_{(k_1,k_2)}(M,\chi)$ be an eigenform. We say that F is *ordinary at* ℓ if it satisfies one of the following two equivalent conditions

1. $\mathrm{ord}_\ell(\lambda_F(\ell)) = 0,\ \mathrm{ord}_\ell(\lambda_F(\ell^2;1)) = k_2 - 3.$

2. The roots of the characteristic polynomial of $\rho_{F,\nu}(\mathrm{Frob}_\ell)$, denoted r_1, r_2, r_3, r_4

satisfy

$$\mathrm{ord}_\ell(r_1) = 0, \mathrm{ord}_\ell(r_2) = k_2 - 2, \mathrm{ord}_\ell(r_3) = k_1 - 1, \mathrm{ord}_\ell(r_4) = k_1 + k_2 - 3.$$

Note that the equivalence in the above definition comes directly from the characteristic polynomial in Theorem 80.

Let D_ν be the decomposition group of ℓ in $G_{\mathbb{Q}}$, where ν is any prime lying above ℓ in $\overline{\mathbb{Z}}$. Let χ_ℓ denote the ℓ-adic cyclotomic character and for an ℓ-adic number u, we set $\epsilon(u)$ to be the unramified character of D_ν which sends Frob_ℓ to u. Then, for F ordinary at ℓ as in the definition, from [74] we have

$$\rho_{F,\nu}|_{D_\nu} \sim \begin{pmatrix} \chi_\ell^{k_1+k_2-3}\epsilon\left(\frac{r_4}{\ell^{k_1+k_2-3}}\right) & * & * & * \\ 0 & \chi_\ell^{k_1-1}\epsilon\left(\frac{r_3}{\ell^{k_1-1}}\right) & * & * \\ 0 & 0 & \chi_\ell^{k_2-2}\epsilon\left(\frac{r_2}{\ell^{k_2-2}}\right) & * \\ 0 & 0 & 0 & \epsilon(r_1) \end{pmatrix},$$

where $\sim$ denotes that the representations are isomorphic.

With this in mind, for a representation

$$\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{F}}_\ell),$$

we will say $\overline{\rho}$ is ordinary at ℓ if up to conjugation we have

$$\overline{\rho}|_{D_\nu} \sim \begin{pmatrix} \overline{\chi}_\ell^{e_3}\epsilon(u_3) & * & * & * \\ 0 & \overline{\chi}_\ell^{e_2}\epsilon(u_2) & * & * \\ 0 & 0 & \overline{\chi}_\ell^{e_1}\epsilon(u_1) & * \\ 0 & 0 & 0 & \overline{\chi}_\ell^{e_0}\epsilon(u_0) \end{pmatrix},$$

where $\overline{\chi}_\ell$ is the reduction of χ_ℓ modulo ℓ, the exponents satisfy $e_3 \geq e_2 \geq e_1 \geq e_0$, ϵ is as above, and $u_3, u_2, u_1, u_0 \in \overline{\mathbb{F}}_\ell^\times$. We denote such a representation as $(\overline{\rho}, \{e_j\})$. After twisting by an appropriate power of $\overline{\chi}_\ell$ we may assume $e_0 = 0$ and that $e_j \leq j(\ell-2)$ for $j = 1, 2, 3$. This brings us to the next definition.

Definition 82. For a representation $(\overline{\rho}, \{e_j\})$, as above, we say that the exponents $\{e_j\}$ are ℓ-small if we can twist $\overline{\rho}$ by a power of $\overline{\chi}_\ell$ so that $0 = e_0 \leq e_1 \leq e_2 \leq e_3 < \ell - 1$.

Furthermore, if we can write $e_1 = k_2 - 2$ and $e_2 = k_1 - 1$ for some integers $k_1 \geq k_2 \geq 3$ then we call (k_1, k_2) the modular weight of $(\overline{\rho}, \{e_j\})$.

We are now prepared to state the following conjecture.

Conjecture 83. [33, Conj. 0] Let $(\overline{\rho}, \{e_j\})$ be an irreducible, odd Galois representation which is ordinary at ℓ and has modular weight (k_1, k_2). Suppose further that the exponents $\{e_j\}$ are ℓ-small. Then, $\overline{\rho}$ is modular of level N with $\ell \nmid N$.

As evidence for this conjecture, we can state the following corollary which follows from Theorem 55.

Corollary 84. *Suppose that $\overline{\rho}$ is modular of level $\ell^r N$ and character χ of conductor ℓN with $\ell \nmid N$. Then, $\overline{\rho}$ is modular of level N.*

Proof. Suppose that $\overline{\rho}$ arises from $F \in S_{(k_1,k_2)}(\ell^r N, \chi)$. Then, we can apply Theorem 55 to obtain a representation $\overline{\rho}'$ of level N such that the characteristic polynomials of $\overline{\rho}(\mathrm{Frob}_p)$ and $\overline{\rho}'(\mathrm{Frob}_p)$ are equal for all $p \nmid \ell N$. Thus, the characteristic polynomials of $\overline{\rho}$ and $\overline{\rho}'$ are equal everywhere by the Chebotarev Density Theorem. Just as in the proof of Corollary 79, the Brauer-Nesbitt Theorem gives that $\overline{\rho}$ is isomorphic to $\overline{\rho}'$. □

Finally, while the previous corollary made no restriction on the reducibility of the Galois representations, we also have a slightly different level stripping result

which holds for certain reducible Galois representations. We remark that the reducible representations in this setting have been classified in [66]. In particular, the reader is referred to Section 3.2.

We will assume that the Galois representation

$$\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{F}}_\ell)$$

is modular and arises from a genus 2 Siegel modular form which appears as a Saito-Kurokawa lift or a weak endoscopic lift. For our purposes, it is enough to note that the Saito-Kurokawa lift provides a lifting of elliptic eigenforms to genus 2 Siegel eigenforms in which the associated Galois representation has a one-dimensional invariant subspace, and the weak endoscopic lift provides a lifting from 2 elliptic eigenforms to a genus 2 Siegel eigenform in which the associated Galois representation is reducible but has no one dimensional invariant subspace.

As we gave the necessary facts for our purposes concerning the Saito-Kurokawa lift in Section 4.7, we only need consider the weak endoscopic lifting here. Let $f_1 \in S^1_{r_1}(N, \chi)$ and $f_2 \in S^1_{r_2}(N, \chi)$ be normalized eigenforms with $\chi = \psi^2$ for some Dirichlet character ψ defined modulo N. Let $F_{f_1,f_2} \in S^2_{(k_1,k_2)}(N, \psi)$ be a weak endoscopic lift of f_1 and f_2. Then, for all primes $p \nmid N$, the L-function of F_{f_1,f_2} satisfies

$$L_p(s, F_{f_1,f_2}) = L_p(s, f_1)L_p(s + (r_2 - r_1)/2, f_2),$$

where $L_p(\cdot)$ is the local L-factor defined in Chapter 2. Once again, we use this

factorization to deduce the following relationships

$$\begin{aligned}\lambda_{F_{f_1,f_2}}(p;S) &= \lambda_{f_1}(p) + p^{\frac{r_1-r_2}{2}}\lambda_{f_2}(p),\\ \lambda_{F_{f_1,f_2}}(p^2;S) &= \lambda_{f_1}(p)^2 + p^{r_1-r_2}\lambda_{f_2}(p)^2 + p^{\frac{r_1-r_2}{2}}\lambda_{f_1}(p)\lambda_{f_2}(p) - \chi(p^2)p^{r_1-2}(2p+1),\end{aligned} \tag{5.1}$$

for all primes $p \nmid N$. For more details concerning the weak endoscopic lift, the reader is referred to [77], [78], and [80]. Moreover, as a special case of the weak endoscopic lift, there is the slightly better known Yoshida lift. The interested reader is referred to [40] and [84] for more details. We are now prepared to state and give a proof of the corresponding level stripping result for Galois representations associated to these lifts.

Theorem 85. *Let $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{F}_\ell})$ be a modular Galois representation of level $\ell^r N$ and character having trivial tame ramification. Assume $\overline{\rho}$ is associated to a form arising from one of the two lifts discussed above Then, $\overline{\rho}$ is modular of level N and also arises from a lifted form.*

Before beginning the proof, we simply note the difference between the condition on the character in Corollary 84 and the condition on the character in this theorem. In Corollary 84, we have assumed that the character factored as $\omega^i\kappa$ where ω is the Teichmüller character, κ is a character defined modulo N, and $i \in \mathbb{Z}/(p-1)\mathbb{Z}$. In the statement of Theorem 85, we are assuming that the character factors as $\eta\kappa$ where η has conductor ℓ^α with $\alpha > 1$ and κ is defined modulo N.

Proof. We provide the technique of the proof as the details are completely elementary.

Just as in Corollary 84, we can reduce the proof to a level stripping problem for the associated modular forms. Say $\overline{\rho}$ arises from the lifted form F. Then, we consider

the elliptic modular form(s) from which F is lifted. Applying Theorem 36, we can strip powers of ℓ from these form(s). Then, after lifting, we obtain the desired Siegel modular form. The content of this proof revolves around using the relationships in Equation 4.3 or Equation 5.1, depending on the lift in question, to show that the eigenvalues are in fact congruent. This involves carefully keeping track of the weight in the proof of Theorem 36. It is this congruence which provides the need for the restriction on the character. See [14] for the details. □

Chapter 6

Future work

In this chapter we mention some open problems related to generalizing our result.

6.1 Twisting of Siegel modular forms

Let F be as in Theorem 55, except with corresponding character of conductor $N\ell^r$ for any $r > 1$. Note that in Theorem 55 we required that $r = 1$. Is there a way to construct an eigenform with corresponding character of conductor $N\ell$ whose Fourier coefficients are congruent to those of F away from the level?

An affirmative answer to this question would allow us to relax the restrictions on the character of F in Theorem 55. In the proof of Theorem 36, we saw that this was accomplished by twisting the original form by an appropriate character so that the conductor of the character is lowered. However, this technique is for elliptic modular forms, and the generalization of this twisting to the genus 2 setting is not so easy due to the lack of a nice relationship between Fourier coefficients and Hecke eigenvalues, and also due to the ambiguity arising from having "multiple" Fourier

expansions to consider.

For example, twists of classical Siegel modular forms have been investigated in [5], [37], [47], and [48]. All of these results rely on twists of either the Fourier expansion of F or the Fourier Jacobi expansion of F, which is quite analogous to the twisting of elliptic modular forms. However, in all of these cases, the twisted modular forms only transform with respect to some subgroup of the congruence subgroups which we have considered up to this point. There seems to be no obvious way of relaxing this restriction.

Finally, in the recent preprint [51], the authors have taken the approach of investigating character twists from the representation theoretic perspective. In particular, they consider the properties of a character twist of an automorphic representation which contains a vector fixed by certain paramodular groups. In the sequel to this work, the authors intend to translate this result to the more classical setting of paramodular forms. It remains to be seen if such a twisting will translate to the setting of Siegel modular forms.

6.2 Level stripping for reducible Galois representations

Let E is the Eisenstein series introduced in Section 4.2. If F is a Saito-Kurokawa lift (respectively a weak endoscopic lift), then is the Deligne-Serre lift of the form EF also a Saito-Kurokawa lift (respectively a weak endoscopic lift)?

An affirmative answer to this question would allow us to eliminate the character restriction in the statement of Theorem 85.

If we first consider the case that F is a Saito-Kurokawa lift, then this prob-

lem should be approachable by showing that the resulting form satisfies the Maass relations on the Fourier-Jacobi coefficients. The correct Maass relations, however, must be determined in the case of arbitrary level. Note, this is related to problem (3) in [35]. We also mention an alternate approach to this problem which follows from results in [26]. In this paper, the authors produce local criteria for determining when a given Siegel modular form is a Saito-Kurokawa lift. The benefit being that these criteria involve checking only finitely many conditions, as opposed to showing the Maass relations hold. However, it is once again necessary to determine the correct notion of these criteria for arbitrary level.

In the case that F arises as a weak endoscopic lift, it should be sufficient to determine that the local factors of the spinor L-function associated to the Deligne-Serre lift of FE factor appropriately for almost all factors. However, this sufficient condition is only known for square-free level, as the results concerning weak endoscopic lifts are primarily in the language of representation theory. Once again, the correction conditions remain to be determined for arbitrary level.

6.3 Level stripping for arbitrary genus

Can we find an Eisenstein series of arbitrary genus analogous to the one used in the proof of Theorem 55, i.e., can we find an Eisenstein series, E, of genus n, level ℓ, and character ω^{-i} (as in the proof) such that $E \equiv_{fc} 1 \pmod{\ell}$?

Note that this is closely related to Problem 4.1 from [45]. If one could answer this question affirmatively then we could generalize Theorem 55 to arbitrary genus.

Recently, in [41], the author was able to construct a ℓ-adic Siegel modular form of arbitrary genus, which interpolates the Fourier coefficients of the Siegel Eisenstein series with prescribed character. This result was obtained using techniques from

Hida theory as it applies to Siegel modular forms. This construction naturally yields a family of forms which are close ℓ-adically. However, for this to be sufficient for our purposes, one still requires a calculation of the ℓ-adic valuation of the Fourier coefficients as was done in [45] for the genus 2 setting. Given the explicit nature of the construction in [41], this calculation should be tractable, though tedious.

6.4 Level stripping of automorphic representations

Finally, can one reproduce the proof of Theorem 55 in the language of automorphic representations?

To be more precise about what this question means, we must first deal with the most basic issue of how one reduces an automorphic representation modulo a prime. This was introduced in a very general setting by Gross in [31]. Essentially, this is accomplished by finding a lattice of the representation space which is stable under $\mathbb{Z}_\ell$ linear combinations of the Hecke operators. This allows one to extract an "integral" model for the automorphic representation, which can then be reduced modulo ℓ. At the heart of this process, one must separate the infinite component from the finite components of the automorphic representations and treat them separately. With this in mind, one can try to reproduce the proof of Theorem 55 by calculating the effect on the infinite and finite portion of the corresponding automorphic representations separately.

The reason for desiring a result of this type is for purposes of generalization. While arithmetic questions are typically more approachable in the classical language of modular forms, these typically have a very limited scope as to how they apply in other settings. By translating to the language of automorphic representations, it is expected that a quite general framework for producing congruences will arise,

independent of the underlying group.

Appendices

Appendix A Explicit action of Hecke operators in genus 2

In this section, we provide explicit formulas for the action of Hecke operators on genus 2 Siegel modular forms. We will adapt techniques used by Andrianov in [4] for scalar weight modular forms to the vector valued setting.

First, we derive a basic property of Fourier coefficients, which will help motivate our technique. Let $F \in M^2_\rho(N, \chi)$. As we have seen, the Fourier expansion of F at ∞ is of the form

$$F(Z) = \sum_{T \in \Lambda_2} a_F(T) \exp(\mathrm{Tr}(TZ)) \text{ with } a_F(T) \in V,$$

where $\rho : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}(V)$. Furthermore, each Fourier coefficient is given by the integral

$$\int_{X \pmod 1} F(Z) \exp(-\mathrm{Tr}(TZ)) dX,$$

where we write $Z = X + iY$, dX is the Euclidean volume of the space of X coordinates, and the integral runs over $-1/2 \leq X_{ij} \leq 1/2$ for all i, j. This integral formula allows us to derive the following relationship between the Fourier coefficients of F,

$$\begin{aligned} a_F(MT\,{}^TM) &= \int_{X \pmod 1} F(Z) \exp(-\mathrm{Tr}(MT\,{}^TMZ)) dX \\ &= \int_{X \pmod 1} F(Z) \exp(-\mathrm{Tr}(T\,{}^TMZM)) dX \\ &= \chi(\det(M))\rho(M) \int_{X \pmod 1} F({}^TMZM) \exp(-\mathrm{Tr}(T\,{}^TMZM)) dX \\ &= \chi(\det(M))\rho(M) a_F(T), \end{aligned}$$

where $M \in \mathrm{GL}_2(\mathbb{Z})$. Note, to move from the second line to the third line we use that

$$F(Z) = \chi(\det(M))\rho(M)F({}^T MZM),$$

which follows from the transformation property of F and noticing that

$$\begin{pmatrix} {}^T M & 0 \\ 0 & M^{-1} \end{pmatrix} \in \Gamma_0^2(N).$$

In summary, the desired property of the Fourier coefficients of F is

$$a_F(MT\,{}^T M) = \chi(\det(M))\rho(M)a_F(T), \text{ for all } M \in \mathrm{GL}_2(\mathbb{Z}). \tag{1}$$

With this property in mind, we define a more general space of functions. Let $\mathcal{F}(V)$ denote the space of holomorphic functions $F : \mathfrak{h}_2 \to V$ which have a Fourier expansion of the form

$$F(Z) = \sum_{T \in \Lambda_2} a_F(T) \exp(\mathrm{Tr}(TZ)) \text{ with } a_F(T) \in V.$$

Let ϵ be a character of the group $\mathrm{GL}_2(\mathbb{Z})$. Define a subspace $\mathcal{F}_\epsilon(V) \subset \mathcal{F}(V)$ by considering only functions $F \in \mathcal{F}(V)$ which satisfy

$$\epsilon(M)F(({}^T MZ + M')M) = F(Z), \text{ for all } \begin{pmatrix} {}^T M & M' \\ 0 & M^{-1} \end{pmatrix} \in P_4,$$

where P_4 is the Siegel parabolic subgroup. To summarize, we have defined the space $\mathcal{F}_\epsilon(V)$, to behave like modular forms with respect to the Siegel parabolic subgroup, rather than congruence subgroups. Using an argument as in the preceding paragraph

we have that for $F \in \mathcal{F}_\epsilon(V)$, the Fourier coefficients satisfy

$$a_F(MT\,{}^TM) = \epsilon(M)a_F(T),$$

where $M \in \mathrm{GL}_2(\mathbb{Z})$. Note that by Equation 1, we have that $M^2_\rho(N,\chi) \subseteq \mathcal{F}_\epsilon(V)$ if $\epsilon(M) = \chi(\det(M))\rho(M)$. Throughout, we will fix a ρ, χ and set $\epsilon = \chi\rho$.

As our functions in $\mathcal{F}_\epsilon(V)$ behave like modular forms with respect to the Siegel parabolic subgroup, it makes sense to define the double coset operator in this setting

$$P_4\alpha P_4 : \mathcal{F}_\epsilon(V) \to \mathcal{F}_\epsilon(V),$$

given by

$$F[P_4\alpha P_4]_\epsilon = \sum_i \chi(\alpha_i)F|_\epsilon\alpha_i,$$

where we are summing over a complete set of coset representatives for $P_4\backslash P_4\alpha P_4$, $\alpha \in \mathrm{GSp}_4^+(\mathbb{Q})$ satisfies $c_\alpha = 0$, and the slash operator is defined to be $(F|_\epsilon\gamma)(Z) = \rho(d_\gamma)^{-1}F(\gamma Z)$.

In [4], Andrianov defines a map, ι, from $H^{\mathbb{Z}}(\Gamma_0^2(N))$ to the double coset operators of the type listed above. This map is defined by

$$\iota : \sum_i \Gamma_0^2(N)\alpha_i \mapsto \sum_i P_4\alpha_i.$$

The benefit of this somewhat messy map can be seen in the following lemma, which provides us with a compatibility between the Hecke operators on $M^2_\rho(N,\chi)$ and the double coset operators on $\mathcal{F}_\epsilon(V)$.

Lemma 86. *Let $F \in M^2_\rho(N,\chi)$. Then, $TF = \iota(T)F$, for every $T \in H^{\mathbb{Z}}(\Gamma_0^2(N))$.*

Proof. Note, this is stated as part of Lemma 4.12 from [4], we simply restate it here to

emphasize that we are interested in vector valued modular forms, not just the scalar valued case.

The lemma follows from the fact that we can find coset representatives, $\{\alpha_i\}$ for T which have $c_{\alpha_i} = 0$ for all i. □

With this lemma in mind, we use explicit coset representatives computed for double cosets of the form $P_4 \backslash P_4 \alpha P_4$ to compute formulas for the action of elements of $H^{\mathbb{Z}}(\Gamma_0^2(N))$. In fact, it is enough for our purposes to give coset representatives for ι applied to the generators of $H_p^{\mathbb{Z}}(\Gamma_0^2(N))$ taken from Theorem 25 for each $p \nmid N$. First, we give the image of these generators as double cosets, then we will give their explicit decompositions.

Lemma 87. *[4, Lemma 3.64]*

$$\begin{aligned}
\iota(T(p)) &= [P_4 \operatorname{diag}(p,p,1,1) P_4] + [P_4 \operatorname{diag}(p,1,1,p) P_4] + [P_4 \operatorname{diag}(1,1,p,p) P_4], \\
\iota(T_1(p^2)) &= \frac{1}{p}[P_4 \operatorname{diag}(p,p,1,1) P_4][P_4 \operatorname{diag}(p,1,1,p) P_4] \\
&+ \frac{1}{p}[P_4 \operatorname{diag}(p,1,1,p) P_4][P_4 \operatorname{diag}(1,1,p,p) P_4] \\
&+ \frac{1}{p}[P_4 \operatorname{diag}(p,1,1,p) P_4]^2 - [P_4 \operatorname{diag}(p^2,1,1,p^2) P_4] \\
&- \frac{p+1}{p^3}[P_4 \operatorname{diag}(p,p,1,1) P_4][P_4 \operatorname{diag}(1,1,p,p) P_4], \\
\iota(T_2(p^2)) &= \frac{1}{p^3}[P_4 \operatorname{diag}(p,p,1,1) P_4][P_4 \operatorname{diag}(1,1,p,p) P_4].
\end{aligned}$$

Combining Lemma 3.60 and Proposition 3.61 from [4], we obtain the following left coset decompositions for the double coset operators in the previous lemma,

$$P_4 \backslash P_4 \operatorname{diag}(p,p,1,1) P_4 = P_4 \begin{pmatrix} pI_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix},$$

$$P_4\backslash P_4 \operatorname{diag}(1,1,p,p)P_4 = \bigcup_{B={}^TB\in \mathrm{M}_2(\mathbb{Z})/p\mathbb{Z})} P_4 \begin{pmatrix} I_2 & B \\ 0_2 & pI_2 \end{pmatrix},$$

$$P_4\backslash P_4 \operatorname{diag}(p,1,1,p)P_4 = \bigcup_{\substack{D\in S(p) \\ B(D) \,(\mathrm{mod}\ D)}} P_4 \begin{pmatrix} p\,{}^TD^{-1} & B \\ 0_2 & D \end{pmatrix},$$

$$P_4\backslash P_4 \operatorname{diag}(p^2,1,1,p^2)P_4 = \bigcup_{\substack{D\in S(p^2) \\ B(D) \,(\mathrm{mod}\ D)}} P_4 \begin{pmatrix} p^2\,{}^TD^{-1} & B \\ 0_2 & D \end{pmatrix},$$

where $S(d) = \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z}) \operatorname{diag}(1,d) \mathrm{SL}_2(\mathbb{Z})$, $B(D) = \left\{B : {}^TBD = {}^TDB\right\}$, and $B \equiv B' \pmod{D}$ if $(B-B')D^{-1} \in \mathrm{M}_2(\mathbb{Z})$.

With these left cosets, we are able to compute the action of each of these double cosets on the Fourier coefficients of elements of $M^2_\rho(N,\chi)$. We will only require the action for primes not dividing N.

Lemma 88. *Let $F \in M^2_\rho(N,\chi)$ and let $p \nmid N$ be a prime. Then,*

1. $a_{F[P_4 \operatorname{diag}(p,p,1,1)P_4]_\epsilon}(T) = \chi(p^2) a_F\left(\frac{T}{p}\right)$.
2. $a_{F[P_4\backslash P_4 \operatorname{diag}(1,1,p,p)P_4]_\epsilon}(T) = p^3 \rho(\operatorname{diag}(p,p))^{-1} a_F(pT)$.
3. $a_{F[P_4\backslash P_4 \operatorname{diag}(p,1,1,p)P_4]_\epsilon}(T) = p\chi(p) \displaystyle\sum_{D\in S(p)} \rho(D)^{-1} a_F\left(\frac{DT\,{}^TD}{p}\right)$
4. $a_{F[P_4\backslash P_4 \operatorname{diag}(p,1,1,p)P_4]_\epsilon}(T) = p^2\chi(p^2) \displaystyle\sum_{D\in S(p^2)} \rho(D)^{-1} a_F\left(\frac{DT\,{}^TD}{p^2}\right)$.

We set $a_F(T) = 0$ if $T \notin \Lambda_2$.

Proof. This is essentially the proof of Lemma 4.14 in [4].

Number 1 follows immediately. Number 2 follows by decomposing

$$\begin{pmatrix} I_2 & B \\ 0_2 & pI_2 \end{pmatrix} = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & pI_2 \end{pmatrix} \begin{pmatrix} I_2 & B \\ 0_2 & I_2 \end{pmatrix},$$

applying the definition of the slash operator, and noticing that there are p^3 elements of $\mathrm{M}_2(\mathbb{Z}/p\mathbb{Z})$ which are symmetric.

To show the formula in Number 3, we begin by applying the appropriate left coset representatives to the Fourier expansion to obtain

$$\begin{aligned} & \chi(p) \sum_{\substack{D \in S(p) \\ B(D) \,(\mathrm{mod}\ D)}} \rho(D)^{-1} \sum_{T \in \Lambda_2} a_F(T) \exp(\mathrm{Tr}(T(p\,({}^T D^{-1} Z + B) D^{-1})) \\ = \; & \chi(p) \sum_{\substack{D \in S(p) \\ B(D) \,(\mathrm{mod}\ D)}} \rho(D)^{-1} \sum_{T \in \Lambda_2} a_F\left(\frac{DT\,{}^T D}{p}\right) \exp(\mathrm{Tr}(TZ)) \exp\left(\mathrm{Tr}\left(\frac{DT\,{}^T DBD^{-1}}{p}\right)\right). \end{aligned}$$

Thus, by fixing T, we have the following expression

$$a_{F[P_4 \backslash P_4 \,\mathrm{diag}(1,1,p,p) P_4]_\epsilon}(T) = \chi(p) \sum_{\substack{D \in S(p) \\ B(D) \,(\mathrm{mod}\ D)}} \rho(D)^{-1} a_F\left(\frac{DT\,{}^T D}{p}\right) \exp\left(\mathrm{Tr}\left(\frac{DT\,{}^T DBD^{-1}}{p}\right)\right).$$

Furthermore, in the proof of Lemma 4.14 in [4], it is shown that for any $D \in S(p)$ we have

$$\sum_{B(D) \,(\mathrm{mod}\ D)} \exp\left(\mathrm{Tr}\left(\frac{DT\,{}^T DBD^{-1}}{p}\right)\right) = p.$$

Thus, our expression becomes

$$a_{F[P_4 \backslash P_4 \,\mathrm{diag}(1,1,p,p) P_4]_\epsilon}(T) = p\chi(p) \sum_{D \in S(p)} \rho(D)^{-1} a_F\left(\frac{DT\,{}^T D}{p}\right),$$

as desired. Note, the proof of Number 4 follows precisely the same argument as the

proof of number 3. □

We can combine Lemma 86, Lemma 87, and Lemma 88 to give formulas for the action of the Hecke operators in $H_p^{\mathbb{Z}}(\Gamma_0^2(N))$ on the Fourier coefficients of elements in $M_\rho^2(N,\chi)$ for all $p \nmid N$. Note, we will only be concerned with the action of $T(p)$ and $T_1(p^2)$, as we have already restricted to the eigenspace of $T_2(p^2)$ as was described in Section 2.3.

Theorem 89. *Let $F \in M_\rho^2(N,\chi)$. Then,*

1.

$$a_{T(p)F}(T) = \chi(p^2)a_F\left(\frac{T}{p}\right) + p^3\rho(\mathrm{diag}(p,p))^{-1}a_F(pT) + p\chi(p)\sum_{D\in S(p)}\rho(D)^{-1}a_F\left(\frac{DT\,{}^TD}{p}\right).$$

2.

$$\begin{aligned} a_{T_1(p^2)F}(T) &= \chi(p^2)\sum_{D\in S(p)}\rho(D)^{-1}\left(a_F\left(\frac{DT\,{}^TD}{p^2}\right) + p^3\chi(p)\rho(\mathrm{diag}(p,p))^{-1}a_F(DT\,{}^TD)\right) \\ &+ p\chi(p^2)\left(\left(\sum_{D\in S(p)}\rho(D)^{-1}a_F\left(\frac{DT\,{}^TD}{p}\right)\right)^2 - \sum_{D\in S(p^2)}\rho(D)^{-1}a_F\left(\frac{DT\,{}^TD}{p^2}\right)\right) \\ &- (p+1)\chi(p)\rho(\mathrm{diag}(p,p))^{-1}a_f(T). \end{aligned}$$

Appendix B The Deligne-Serre lifting lemma

In this section, we present a well-known commutative algebra result due to Deligne and Serre. For completeness, we also present a fairly detailed proof of this lemma. We begin by stating the Deligne-Serre lifting lemma and a corollary which was used in the previous sections.

Theorem 90. *[21, Lemme 6.11] Let $\mathcal{D}$ be a discrete valuation ring with field of fractions K, maximal ideal $\mathfrak{m}$, and residue field $k = \mathcal{D}/\mathfrak{m}$. Let M be a free $\mathcal{D}$-module of finite rank and $\mathcal{T}$ a set of commuting $\mathcal{D}$ endomorphisms of M. Let $f \in M/\mathfrak{m}M \cong k \otimes M$ be a non-zero eigenvector for all $T \in \mathcal{T}$ with eigenvalues $a(T) \in k$. Then, there exists a discrete valuation ring $\mathcal{D}'$ containing and finite over $\mathcal{D}$, with maximal ideal $\mathfrak{m}'$ such that $\mathcal{D} \cap \mathfrak{m}' = \mathfrak{m}$, and a non-zero element $f' \in M' = \mathcal{D}' \otimes_{\mathcal{D}} M$ which is an eigenvector for all $T \in \mathcal{T}$ with eigenvalue $a'(T)$ satisfying $a'(T) \equiv a(T) \pmod{\mathfrak{m}'}$.*

Corollary 91. *Let i be either 1 or 2. Let M be either $S^i_\rho(\Gamma^i_0(\ell^r) \cap \Gamma^i_1(N))$ or $S_\rho(\Gamma^i_1(N))$. Let $f \in M/\ell M = \mathbb{F}_\ell \otimes M$ be a non-zero eigenform. Then, there exists a finite extension $\mathcal{O}_\nu \supseteq \mathbb{Z}_\ell$ with maximal ideal ν lying above ℓ and a non-zero eigenform $f' \in \mathcal{O}_\nu \otimes_{\mathbb{Z}_\ell} M$ with eigenvalues congruent to the eigenvalues of f modulo ν.*

This corollary follows immediately from the previous theorem by fixing an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$ which allows us to view M as a $\mathcal{D}$-module which is free and of finite rank, where $\mathcal{D}$ is the ring of integers of some finite extension of $\mathbb{Q}_\ell$, and the set of endomorphisms of M is given by the Hecke operators, i.e., $\mathcal{T} = \mathcal{H}(\Gamma^i_0(N))$.

Before we can prove Theorem 90, we will need a few basic results from commutative algebra. A good reference for this material is [25]. Throughout, we assume all rings to be commutative. First, a couple of results concerning Artinian rings.

Lemma 92. *If R is an Artinian integral domain, then R is a field.*

Proof. Let $x \in R$ be non-zero and consider the descending chain of ideals

$$(x) \supseteq (x^2) \supseteq \cdots .$$

As this chain eventually stabilizes, we have $x^n = ax^{n+1}$ for some positive integer n and some $a \in R$. As R is an integral domain, we conclude that x is a unit, and hence R is a field. □

Corollary 93. *Any prime ideal of an Artinian ring is maximal.*

Proof. Let R be an Artinian ring and $\wp \subseteq R$ be a prime ideal. Then, $R/\wp$ is an Artinian integral domain, and, by the previous lemma, a field. □

Next, we will need a few results concerning associated primes, which we first define.

Definition 94. Let R be a ring and M an R-module. A prime ideal $\wp \subset R$ is said to be *associated* to M if $\wp$ is the annihilator of an element of M.

Lemma 95. *Let M be an R-module. Then, the set of prime ideals of R which are associated primes of M is contained in the set of prime ideals of R which are in the support of M.*

Proof. Let $\wp$ be an associated prime of M. Then, $\wp$ is the annihilator of some $m \in M$. If the localization of M at $\wp$, denoted $M_\wp$, is zero, then there exists $x \in R - \wp$ such that $xm = 0$. This implies that $x \in \wp$, which is a contradiction. Thus, $M_\wp \neq 0$, i.e., $\wp$ is a support prime of M. □

Lemma 96. *Let R be a Noetherian ring, M a non-zero R-module, and $\wp$ a prime in the support of M. Then, $\wp$ contains an associated prime of M.*

Proof. Since $\wp$ is in the support of M, there is some $x \in M$ satisfying $(Rx)_\wp \neq 0$. Since R is Noetherian, Theorem 3.1 in [25] gives that the set of associated primes of $(Rx)_\wp$ is non-empty. The previous lemma gives that the associated primes of $(Rx)_\wp$ is contained in the support of $(Rx)_\wp$. Let $\mathfrak{p}$ be an associated prime of $(Rx)_\wp$. Then, there is a non-zero element $\frac{x'}{y} \in (Rx)_\wp$ with $x' \in Rx$ and $y \in R - \wp$, such that $\mathfrak{p}$ is the annihilator of $\frac{x'}{y}$.

Suppose $z \in \mathfrak{p} - \wp$. Then, $z\frac{x'}{y} = 0$, which implies that $\frac{x'}{y} = 0$ since z is a unit in $(Rx)_\wp$. Hence, no such z exists, i.e., $\mathfrak{p} \subseteq \wp$.

As R is Noetherian, we have that any $\mathfrak{p}$ is finitely generated. Let $\{z_1, \ldots, z_n\}$ be a set of generators for $\mathfrak{p}$. Then, for each z_i, there exists a $t_i \in R - \wp$ such that $z_i t_i x' = 0$. Let $t = t_1 \ldots t_n$. Then, $\mathfrak{p}$ is the annihilator of $tx' \in M$. Thus, $\mathfrak{p}$ is an associated prime of M. □

With these results in hand, we are prepared to prove the main result of this section.

Proof of Theorem 90. Let $\mathcal{H}$ be the $\mathcal{D}$-subalgebra of $\mathrm{End}_{\mathcal{D}}(M)$ generated by $\mathcal{T}$ over $\mathcal{D}$. Since M is free and of rank $r < \infty$ over $\mathcal{D}$, we have that $\mathrm{End}_{\mathcal{D}}(M)$ is a free $\mathcal{D}$-module of rank r^2. Hence, $\mathcal{H}$ is free and of finite rank. Choosing a basis for $\mathcal{H}$, we may assume $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$ is a finite set. It will be sufficient to prove the theorem for these basis elements.

Let m_{T_i} be the minimal polynomial of T_i. By adjoining all roots of m_{T_i} to K for all i, we have a finite field extension $K \subseteq K'$ such that each m_{T_i} splits over K'. The integral closure of $\mathcal{D}$ in K' gives a discrete valuation ring $\mathcal{D}'$ with maximal ideal $\mathfrak{m}'$ satisfying $\mathcal{D} \cap \mathfrak{m}' = \mathfrak{m}$, and residue field k' containing k. Furthermore, we have that $\mathcal{D}' \otimes_{\mathcal{D}} M$ is a $\mathcal{D}'$ module, and we will continue to denote this module by M, and we will also continue to denote the analogous subalgebra of endomorphisms, $\mathcal{D}' \otimes_{\mathcal{D}} \mathcal{H}$,

by $\mathcal{H}$.

Consider the homomorphism of $\mathcal{D}'$-algebras,

$$\pi_f : \mathcal{H} \to k', \text{ given by } \pi_f(T) = a(T) \pmod{\mathfrak{m}'}.$$

Note, we have an obvious injection of $\mathcal{D}'$ into $\mathcal{H}$, which implies that π_f is surjective. By Zorn's lemma, we can choose a minimal non-zero prime ideal $\mathfrak{p} \subset \mathcal{H}$ which is contained in the maximal ideal $\ker \pi_f$.

Claim. The prime ideal $\mathfrak{p}$ is contained in the set of zero divisors of $\mathcal{H}$.

Proof of Claim. Let Z denote the set of zero divisors of $\mathcal{H}$. Let $D = \mathcal{H} - Z$. Note, D is closed under multiplication.

Consider the set $S = \mathcal{H} - \mathfrak{p}$. As $\mathfrak{p}$ is prime, we have that S is closed under multiplication. Furthermore, if S' is a multiplicatively closed set which contains S, then $S' = \mathcal{H}$ or $S' = \mathcal{H} - \mathfrak{p}'$ for some prime ideal $\mathfrak{p}' \subset \mathfrak{p}$. As $\mathfrak{p}$ is minimial, we must have $S' = \mathcal{H}$, and hence S is a maximal multiplicatively closed set. If D is not contained in S, then SD is a multiplicatively closed set strictly containing S, i.e., $SD = \mathcal{H}$, which is impossible since $0 \notin SD$. Thus, $D \subset S$, which implies $\mathfrak{p} \subset Z$. $\square$

The fact that $\mathcal{H}$ is free over $\mathcal{D}'$ and $\mathcal{D}'$ is an integral domain combined with the previous claim gives that $\mathfrak{p} \cap \mathcal{D}' = 0$, where we are again considering $\mathcal{D}'$ as lying in $\mathcal{H}$. Further, note that as $\mathcal{H}$ is finitely generated over $\mathcal{D}'$, so is $\mathcal{H}/\mathfrak{p}\mathcal{H}$, hence $\mathcal{H}/\mathfrak{p}\mathcal{H}$ is a finite integral extension of $\mathcal{D}'$. Let L be the field of fractions of $\mathcal{H}/\mathfrak{p}\mathcal{H}$, and let $\mathcal{D}_L$ denote the integral closure of $\mathcal{D}'$ in L, with $\mathfrak{m}_L$ the corresponding maximal ideal and l the corresponding residue field.

Consider the projection

$$\pi'_f : \mathcal{H} \to \mathcal{H}/\mathfrak{p}\mathcal{H} \hookrightarrow \mathcal{D}_L,$$

where the final injection comes from the fact that $\mathcal{H}/\mathfrak{p}\mathcal{H}$ is an integral extension of $\mathcal{D}'$ and $\mathcal{D}_L$ is the integral closure of $\mathcal{D}'$ in L. Recall, for π_f defined above, we have that $\ker \pi_f$ is a maximal ideal in $\mathcal{H}$. Combining Proposition 4.15 and Corollary 4.17 in [25], we have that $\pi'_f(\ker \pi_f) \subseteq \mathfrak{m}_L$.

Let $\pi'_f(T) = a'(T) \in \mathcal{H}/\mathfrak{p}\mathcal{H} \subset \mathcal{D}_L$, for all $T \in \mathcal{H}$. Then,

$$\pi_f(T - a(T)I) = \pi_f(T) - a(T)\pi_f(1) = a(T) - a(T) = 0,$$

i.e., $T - a(T) \in \ker \pi_f$. It follows that $\pi'_f(T - a(T)I) = a'(T) - a(T) \in \mathfrak{m}_L$, i.e., $a'(T) \equiv a(T) \pmod{\mathfrak{m}_L}$.

Let $\mathfrak{p}'$ be the prime ideal in $K' \otimes_{\mathcal{D}'} \mathcal{H}$ which is generated by $\mathfrak{p}$. Since $\mathfrak{p}$ is minimal, so is $\mathfrak{p}'$. We want to show that $\mathfrak{p}'$ is an associated prime of $K' \otimes_{\mathcal{D}'} M$, i.e., that $\mathfrak{p}'$ is the annihilator of some nonzero element of $K' \otimes_{\mathcal{D}'} M$. As our tensor products will be over $\mathcal{D}'$ for the remainder of the proof, we will drop this from the notation.

First, we need that $K' \otimes \mathcal{H}$ is Artinian and has all maximal ideals isomorphic. This follows simply from the fact that $\mathcal{H}$ is free and of finite rank over $\mathcal{D}'$, which implies that

$$K' \otimes \mathcal{H} \cong \bigoplus_{i=1}^{n} K',$$

for some $n < \infty$. It is clear that all maximal ideals of $\bigoplus_{i=1}^{n} K'$ are isomorphic. Furthermore, $\bigoplus_{i=1}^{n} K'$ is Artinian since K' is.

Note, the annihilator of $K' \otimes M$ in $K' \otimes \mathcal{H}$, denoted $\mathrm{Ann}_{K' \otimes \mathcal{H}}(K' \otimes M)$, is

an ideal in $K' \otimes \mathcal{H}$, and is hence is contained in some maximal ideal. By Corollary 93, we have that $\mathfrak{p}'$ is maximal in $K' \otimes \mathcal{H}$, and since all maximal ideals of $K' \otimes \mathcal{H}$ are isomorphic, we may assume that $\mathrm{Ann}_{K' \otimes \mathcal{H}}(K' \otimes M) \subset \mathfrak{p}'$. By Corollary 2.7 in [25], we have that $\mathfrak{p}'$ is in the support of $K' \otimes M$, which gives that $\mathfrak{p}'$ is an associated prime of $K' \otimes M$ by Lemma 96 and the minimality of $\mathfrak{p}'$.

Thus, there is some $f' \in K' \otimes M$, which is annihilated by $\mathfrak{p}'$. Note, we may assume that $f' \in M$ by simply clearing denominators. As $T - a'(T) \in \mathfrak{p}'$, we have that $Tf' = a'(T)f'$, and as we have already shown that $a'(T) \equiv a(T) \pmod{\mathfrak{m}_L}$, the proof is complete. $\square$

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