Math. Res. Lett. **17** (2010), *no.* 05, 977–991

ON THE CONGRUENCES PRIMES OF SAITO-KUROKAWA LIFTS OF ODD SQUARE-FREE LEVEL

Jim Brown

Abstract. In this paper we extend a conjecture of Katsurada's characterizing the congruence primes of Saito-Kurokawa lifts of weight κ and full level in terms of the divisibility of $L_{\mathrm{alg}}(\kappa, f)$ to the case of odd square-free level. After stating the conjecture, we provide evidence for the conjecture by constructing a congruence.

1. Introduction

Let $\kappa > 2$ be an even integer and $M \geq 1$ an odd square-free integer. It is well known that given a newform $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$, one can associate an eigenform $F_f \in S_\kappa(\Gamma_0^{(2)}(M))$ called the Saito-Kurokawa lift. As the Galois representations attached to Saito-Kurokawa lifts are reducible, it is natural to consider the congruence primes of Saito-Kurokawa lifts. For example, given a congruence between a Saito-Kurokawa lift and a cuspidal eigenform with irreducible Galois representation, one can produce non-trivial torsion elements in an appropriate Shafarevich-Tate group attached to the (twisted) Galois representation associated to f by following the same type of arguments used by Ribet in his proof of the converse of Herbrand's theorem. One can see [4] for such results.

In the case that $M = 1$ Katsurada conjectured a classification of the congruence primes of a Saito-Kurokawa lift in terms of divisibility of $L_{\mathrm{alg}}(\kappa, f)$. We extend this conjecture as follows. Let $S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)})$ be the space of new Maass spezialschar. Let $\mathfrak{p} \subset \mathcal{O}_{\mathbb{Q}(f)}$ be a prime with $\mathfrak{p} \nmid M(2\kappa-1)!$. Then $\mathfrak{p}$ is a congruence prime of F_f with respect to $S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))^\perp$ if and only if $\mathfrak{p} \mid L_{\mathrm{alg}}(\kappa, f)$. We provide evidence for this conjecture by explicitly constructing a congruence between F_f and an eigenform $G \in S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))^\perp$ under suitable hypotheses. This is accomplished by considering the pullback from Sp_8 to $\mathrm{Sp}_4 \times \mathrm{Sp}_4$ of a suitably chosen Siegel Eisenstein series and then using an inner product formula of Shimura to give a spectral expansion of the pullback. One then constructs Hecke operators to "kill off" the forms occurring in the expansion that lie in $S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))$.

One should note that even though $G \in S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))^\perp$, we are not guaranteed that G is not a CAP form. For instance, G could be an "old" Saito-Kurokawa lift if f happens to be congruent to an oldform. One can eliminate such possibilities by putting further requirements on the weight κ or requiring f to have Serre-level M. This is discussed in Corollary 5.10. There is also still the possibility that G arises as

Received by the editors June 3, 2010.
2000 *Mathematics Subject Classification.* Primary 11F33; Secondary 11F46.
Key words and phrases. Saito-Kurokawa correspondence, congruence primes.

a CAP form that is a theta lift twisted by a quadratic character, see [14] for example. We do not address this possibility here. It is possible to show that G is not a weakly endoscopic form by considering the Galois representations of G and F_f. While these considerations are not necessary for the result in this paper, they are of paramount importance in current ongoing work with Mahesh Agarwal described briefly below.

Aside from its intrinsic interest, such results can be used to produce evidence for the Bloch-Kato conjecture for modular forms. Applying Theorem 5.9 and Corollary 5.10 to produce such results is current ongoing joint work with Mahesh Agarwal. By applying the restrictions mentioned above to ensure that the form G produced is not CAP combined with an argument showing that G is not weakly endoscopic, we aim to construct a nontrivial torsion element in the appropriate Shafarevich-Tate group as predicted by the Bloch-Kato conjecture. In particular, let p be a prime, $\mathfrak{p}$ a prime over p in a suitably large finite extension of $\mathbb{Q}$, and let $V_{f,\mathfrak{p}}$ be the 2-dimensional $\mathfrak{p}$-adic Galois representation associated to f. Let $T_{f,\mathfrak{p}}$ be a Galois-stable lattice and let $W_{f,\mathfrak{p}} = V_{f,\mathfrak{p}}/T_{f,\mathfrak{p}}$. If $\mathfrak{p} \mid L_{\mathrm{alg}}(\kappa, f)$, we aim to use the congruence between F_f and G to produce a nontrivial p-torsion element in $\text{Ш}(W_{f,\mathfrak{p}}(k))$ where $W_{f,\mathfrak{p}}(k)$ denotes $W_{f,\mathfrak{p}}$ twisted by the kth power of the p-adic cyclotomic character.

It is also natural to consider a Λ-adic version of the results given in this paper. Analogous theorems in that context would naturally lend themselves to results on main conjectures in Iwasawa theory as formulated by Greenberg by pursuing the natural generalizations of the arguments used by Wiles in his proof of the main conjecture for totally real fields ([19]). This is a topic the author plans to return to in subsequent work.

2. The Saito-Kurokawa Correspondence

In this section we briefly recall the Saito-Kurokawa correspondence and some of its basic properties. The full level Saito-Kurokawa correspondence was established through the work of many mathematicians; see [20] for an account. We are interested in the case of odd square-free level which was established in [12] in the classical setting and in [13] in the language of automorphic forms. We let κ be an even integer throughout this section.

We begin by recalling the relationship between elliptic modular forms of weight $2\kappa-2$ and half-integer weight modular forms of weight $\kappa - 1/2$. Let M be a positive odd square-free integer and D be a fundamental discriminant with $D < 0$. There exists a Shimura lifting ζ_D that maps $S^+_{\kappa-1/2}(\Gamma_0(4M))$ to $S_{2\kappa-2}(\Gamma_0(M))$ and a Shintani lifting $\mathcal{SH}_D$ mapping $S_{2\kappa-2}(\Gamma_0(M))$ to $S^+_{\kappa-1/2}(\Gamma_0(4M))$. These maps are adjoint on cusp forms with respect to the Petersson products. Explicitly, for

$$g(z) = \sum_{\substack{n\geq 1\\ n\equiv 0,3 (\mathrm{mod}\, 4)}} c_g(n)q^n \in S^+_{\kappa-1/2}(\Gamma_0(4M))$$

one has

$$\zeta_D g(z) = \sum_{n\geq 1}\left(\sum_{\substack{d\mid n\\ \gcd(d,M)=1}} \left(\frac{D}{d}\right) d^{\kappa-2} c_g(|D|n^2/d^2)\right) q^n$$

and for $f \in S^{\mathrm{new}}_{2\kappa-2}(\Gamma_0(M))$ a newform one has

$$\mathcal{SH}_D(f)(z) = (-1)^{[(\kappa-1)/2]} 2^{\kappa-1} \sum_{\substack{m \geq 1 \\ m \equiv 0,3 \pmod 4}} r_{\kappa-1,M,D}(f;|D|m) q^m$$

where the $r_{\kappa-1,M,D}(f;|D|m)$ are certain integrals. One can consult [11] for their precise definition; we will not need it here. Using the Shimura and Shintani liftings one has the following theorem.

Theorem 2.1. *([11],[12]) For $D < 0$ a fundamental discriminant with $\gcd(D, M) = 1$, the Shimura and Shintani liftings give a Hecke-equivariant isomorphism between $S^{+,\mathrm{new}}_{\kappa-1/2}(\Gamma_0(4M))$ and $S^{\mathrm{new}}_{2\kappa-2}(\Gamma_0(M))$.*

Let $\mathcal{O}$ be a ring so that an embedding of $\mathcal{O}$ into $\mathbb{C}$ exists. Choose such an embedding and identify $\mathcal{O}$ with its image in $\mathbb{C}$ via this embedding. Assume that $\mathcal{O}$ contains the Fourier coefficients of f. The Shintani lifting $g_f := \zeta_D^* f$ is determined up to normalization by a constant multiple. We have the following result of Stevens.

Theorem 2.2. *([18], Prop. 2.3.1) Let $f \in S^{\mathrm{new}}_{2\kappa-2}(\Gamma_0(M))$ be a newform. If the Fourier coefficients of f are in $\mathcal{O}$, then there exists a corresponding Shintani lifting $\mathcal{SH}_D(f)$ of f with Fourier coefficients in $\mathcal{O}$ as well.*

Remark 2.3. We fix a $\mathcal{SH}_D(f)$ as in the theorem throughout this paper. If $\mathcal{O}$ is a discrete valuation ring, we fix $\mathcal{SH}_D(f)$ to not only have its Fourier coefficients in $\mathcal{O}$, but to have one Fourier coefficient in $\mathcal{O}^\times$ as well.

Half-integer weight forms and Jacobi forms are related by the following theorem.

Theorem 2.4. *([12]) The linear map defined by*

$$\sum_{\substack{D<0, r \in \mathbb{Z} \\ D \equiv r^2 \pmod 4}} c(D,r) e\left(\frac{r^2 - D}{4}\tau + rz\right) \mapsto \sum_{\substack{D<0 \\ D \equiv 0,1 \pmod 4}} c(D) e(|D|r)$$

is a canonical isomorphism between $J^{\mathrm{cusp,new}}_{\kappa,1}(\Gamma_0^{\mathrm{J}}(M))$ and $S^{+,\mathrm{new}}_{\kappa-1/2}(\Gamma_0(4M))$ which commutes with the action of Hecke operators. It also preserves the Hilbert space structure.

The last step is to relate Jacobi forms to Siegel modular forms. First we define the space of Maass spezialschar. Recall that we can write the Fourier expansion of a Siegel modular form F as

$$F(\tau, z, \tau') = \sum_{\substack{m,n,r \in \mathbb{Z} \\ m,n,4mn - r^2 \geq 0}} A_F(n,r,m)\, e(n\tau + rz + m\tau') \tag{1}$$

where $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ with $\tau, \tau' \in \mathfrak{h}^1, z \in \mathbb{C}$, $\mathrm{Im}(z)^2 < \mathrm{Im}(\tau)\,\mathrm{Im}(\tau')$ and $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$. We say that F is in the space of Maass spezialschar if the Fourier coefficients of F satisfy the relation

$$A_F(n,r,m) = \sum_{d \mid \gcd(n,r,m)} d^{\kappa-1} A_F\left(\frac{nm}{d^2}, \frac{r}{d}, 1\right)$$

for every $m, n, r \in \mathbb{Z}$ with $m, n, 4mn - r^2 \geq 0$. We denote the space of Maass spezialschar by $S_\kappa^{\mathrm{M}}(\Gamma_0^{(2)}(M))$. Note that the space of Maass forms and Maass cusp forms are invariant under the action of Hecke operators. We now have the following theorem relating Jacobi forms to the Maass spezialschar.

Theorem 2.5. *([12]) Let $F \in S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))$ and write the Fourier-Jacobi expansion of F as*

$$F(\tau, z, \tau') = \sum_{m=1}^{\infty} \phi_m(\tau, z) e(m\tau').$$

Then one has

$$F(\tau, z, \tau') = \sum_{m=1}^{\infty} (V_m \phi_1)(\tau, z) e(m\tau')$$

and the association $F \mapsto \phi_1$ gives a Hecke equivariant isomorphism between the spaces $S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))$ and $J_{\kappa,1}^{\mathrm{new}}(\Gamma_0^{\mathrm{J}}(M))$. Here V_m is the index shifting operator defined by

$$V_m \phi = \sum_{\substack{D \leq 0, r \in \mathbb{Z} \\ D \equiv r^2 \,(\mathrm{mod}\, 4m)}} \left(\sum_{\substack{d \mid \gcd(n,r) \\ D \equiv r^2 \,(\mathrm{mod}\, 4md)}} d^{\kappa-1} c_\phi\left(\frac{D}{d^2}, \frac{r}{d}\right) \right) e\left(\frac{r^2 - D}{4m}\tau + rz\right)$$

for $\phi \in J_{\kappa,1}(\Gamma_0^{\mathrm{J}}(M))$.

Combining all of these results gives the Saito-Kurokawa correspondence for odd square-free levels.

Theorem 2.6. *([12]) There is a Hecke equivariant isomorphism between the spaces $S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$ and $S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))$ so that if $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$ maps to $F_f \in S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))$ one has*

$$L^*(s, F_f, \mathrm{Spin}) = \zeta(s - \kappa + 1)\zeta(s - \kappa + 2) L(s, f) \tag{2}$$

where we define the modified Spinor L-function by

$$L^*(s, F_f, \mathrm{Spin}) = \zeta_M(s - \kappa + 1)\zeta_M(s - \kappa + 2) L(s, F_f, \mathrm{Spin})$$

where

$$\zeta_M(s) = \prod_{p \mid M} (1 - p^{-s})^{-1}.$$

Corollary 2.7. *If $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M), \mathcal{O})$, then $F_f \in S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M), \mathcal{O})$. If $\mathcal{O}$ is a discrete valuation ring, then F_f has a Fourier coefficient in $\mathcal{O}^\times$.*

Proof. This follows from the formulas given above combined with the Maass relations and the formula for the index shifting operator V_m. □

Corollary 2.8. *Let $f \in S_k(\Gamma_0(M))$ be newform and F_f the Saito-Kurokawa lift of f. One has that $F_f^c = F_f$.*

Proof. This is straightforward by viewing the lift through the isomorphisms described above. For example, observe that $\mathcal{SH}_D(f)^c$ maps to f^c under the given isomorphism, as does $\mathcal{SH}_D(f^c)$. Thus, we must have $\mathcal{SH}_D(f^c) = \mathcal{SH}_D(f)^c$. The others are even more obvious. Thus, we have that $F_f^c = F_{f^c}$. However, since f is a newform we know that $f^c = f$. This follows immediately from the fact that the Hecke operators are self-adjoint with respect to the Petersson product. This gives the result. $\square$

The following theorem is an easy generalization of Theorem 3.10 of [4].

Theorem 2.9. *Let M and N be positive integers with M odd and square-free and $M \mid N$. Let χ be a Dirichlet character of conductor N. Let $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$ be a newform and F_f the Saito-Kurokawa lift of f. The standard zeta function of F_f factors as*

$$L^{\Sigma_N}(2s, F_f, \chi, \mathrm{St}) = L^{\Sigma_N}(2s-2,\chi)L^{\Sigma_N}(2s+\kappa-3, f, \chi)L^{\Sigma_N}(2s+\kappa-4, f, \chi)$$

where for an L-function $L(s) = \prod_p L_p(s)$, we write

$$L^{\Sigma_N}(s) = \prod_{p \notin \Sigma_N} L_p(s).$$

We will also make use of the following result relating $\langle F_f, F_f\rangle$ and $\langle f, f\rangle$.

Theorem 2.10. *([3], Theorem 1.1) Let $M = p_1 \dots p_n$, $f \in S_{2\kappa-2}^{new}(\Gamma_0(M))$ a newform, and F_f the Saito-Kurokawa lift of f. Let $D < 0$ be a fundamental discriminant with* $\gcd(M, D) = 1$, *χ_D the associated quadratic character, and $c_{\mathcal{SH}_D(f)}(|D|) \neq 0$. Then one has*

$$\langle F_f, F_f\rangle = \mathcal{B}_{\kappa,M}\, \frac{|c_{\mathcal{SH}_D(f)}(|D|)|^2\, L(\kappa, f)}{\pi\, |D|^{\kappa-3/2}\, L(\kappa-1, f, \chi_D)} \langle f, f\rangle \tag{3}$$

where

$$\mathcal{B}_{\kappa,M} = \frac{M^\kappa\,(\kappa-1)\prod_{i=1}^n \left(p_i^2+1\right)}{2^{n+4}\, 3^2\, [\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M)]}.$$

3. The conjecture

In this section we recall the conjecture of Katsurada for congruence primes of Saito-Kurokawa lifts of full level and then state a generalization of this conjecture to the case of odd square-free levels. We begin with some notation.

Definition 3.1. Let $f, g \in S_\kappa(\Gamma_0(M), \mathcal{O})$. Let $\mathfrak{p}$ be a prime of $\mathcal{O}$. We write

$$f \equiv g \pmod{\mathfrak{p}^m}$$

to indicate a congruence of Fourier coefficients, i.e., that $(a_f(n) - a_g(n)) \in \mathfrak{p}^m$ for all n. If f and g are Hecke eigenforms and we wish to denote a congruence of eigenvalues, we write

$$f \equiv_{\mathrm{ev}} g \pmod{\mathfrak{p}^m}.$$

Furthermore, if we wish to indicate a congruence of eigenvalues away from a finite set of places Σ we write

$$f \equiv_{\mathrm{ev},\Sigma} g \pmod{\mathfrak{p}^m}.$$

We use the same notation for Siegel modular forms.

Let $F \in S_\kappa(\Gamma_0(M))$ be a Hecke eigenform and let $\mathbb{Q}(F)$ denote the finite extension of $\mathbb{Q}$ generated by the eigenvalues of F. Let $V \subset S_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$ be a subspace that is stable under the Hecke algebra and assume that $V \subset (\mathbb{C}F)^\perp$ where $(\mathbb{C}F)^\perp$ is the orthogonal complement of $\mathbb{C}F$ in $S_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$ with respect to the Petersson product. Let $\mathcal{O}_{\mathbb{Q}(F)}$ be the ring of integers of $\mathbb{Q}(F)$.

Definition 3.2. A prime $\mathfrak{p}$ of $\mathcal{O}_{\mathbb{Q}(F)}$ is said to be a *congruence prime of F with respect to $V \subset (\mathbb{C}F)^\perp$* if there exists a Hecke eigenform $G \in V$ so that

$$F \equiv_{\mathrm{ev},\Sigma_M} G (\bmod \widetilde{\mathfrak{p}})$$

for $\widetilde{\mathfrak{p}}$ a prime of $\mathcal{O}_{\mathbb{Q}(F)\cdot\mathbb{Q}(G)}$ that divides $\mathfrak{p}$. We simply say $\mathfrak{p}$ is a *congruence prime of F* if $V = (\mathbb{C}F)^\perp$.

One has the following conjecture of Katsurada.

Conjecture 3.3. ([10]) Let $\kappa > 2$ be even and $f \in S_{2\kappa-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized eigenform. Let $\mathfrak{p} \subset \mathcal{O}_{\mathbb{Q}(f)}$ be a prime with $\mathfrak{p} \nmid (2\kappa-1)!$. Then $\mathfrak{p}$ is a congruence prime of F_f with respect to $S_\kappa^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))^\perp$ if and only if $\mathfrak{p} \mid L_{\mathrm{alg}}(\kappa, f)$.

In fact, Katsurada's conjecture is more general than stated here as it deals with Ikeda lifts, but we restrict to the case of interest. Katsurada provides evidence for this conjecture via the following theorem.

Theorem 3.4. *([10]) Let $\kappa > 2$ be even and $f \in S_{2\kappa-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized eigenform. Let $\mathfrak{p}$ be a prime of $\mathbb{Q}(f)$. Furthermore, assume that*

(1) $\mathfrak{p} \mid L_{\mathrm{alg}}(\kappa, f)$;
(2) $\mathfrak{p}$ *does not divide*

$$D \cdot (2\kappa-1)! \cdot \zeta_{\mathrm{alg}}(2m) \cdot L_{\mathrm{alg}}(\kappa-1, f, \chi_D) \cdot \prod_{i=1}^{2} L_{\mathrm{alg}}(2m+\kappa-i, f)$$

for some $2 \leq m \leq \kappa/2 - 1$ with $D < 0$ a fundamental discriminant.

Then $\mathfrak{p}$ is a congruence prime for F_f. Furthermore, if one assumes that

(1) f *is ordinary at* $\mathfrak{p}$;
(2) $\mathfrak{p}$ *does not divide*

$$\prod_{\ell \leq (\kappa-1)/6} (1+\ell) \cdot \frac{\langle f, f\rangle}{\Omega_f^+ \Omega_f^-}$$

then $\mathfrak{p}$ is a congruence prime of F_f with respect to $S_\kappa^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))^\perp$.

We have provided evidence in previous work on this conjecture as well. For example, we have the following theorem (rephrased here in terms of the conjecture):

Theorem 3.5. *([4], Theorem 6.5) Let $\kappa > 9$ be even and $f \in S_{2\kappa-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized eigenform. Let $\mathfrak{p}$ be a prime of $\mathbb{Q}(f)$ with residue characteristic p so that $p > 2\kappa - 2$. Furthermore, assume that $\overline{\rho}_{f,\mathfrak{p}}$ is irreducible, f is ordinary at $\mathfrak{p}$, and $\mathfrak{p} \mid L_{\mathrm{alg}}(\kappa, f)$. If there exists an integer $N > 1$, a fundamental discriminant $D < 0$ with $\mathfrak{p} \nmid ND$, and a Dirichlet character χ of conductor N so that*

$$\mathfrak{p} \nmid L^{\Sigma_N}(3-\kappa, \chi) L_{\mathrm{alg}}(\kappa-1, f, \chi_D) L_{\mathrm{alg}}(1, f, \chi) L_{\mathrm{alg}}(2, f, \chi),$$

then $\mathfrak{p}$ is a congruence prime of F_f with respect to $S_\kappa^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))^\perp$.

One should note the main difference in the two results is that while Katsurada's result allows one to vary the special value one wants to "miss", our result allows one to vary a character. One can find similar results in [5] where the result is in terms of twisting by a modular form instead of a character.

The point here is to extend this conjecture to the case where the Saito-Kurokawa lift has level $\Gamma_0^{(2)}(M)$. We propose the following conjecture.

Conjecture 3.6. Let $\kappa > 2$ be even, M a positive square-free integer, and $f \in S_{2\kappa-2}^{\text{new}}(\Gamma_0(M))$ a newform. Let $\mathfrak{p} \subset \mathcal{O}_{\mathbb{Q}(f)}$ be a prime with $\mathfrak{p} \nmid M(2\kappa-1)!$. Then $\mathfrak{p}$ is a congruence prime of F_f with respect to $S_\kappa^{\text{M,new}}(\Gamma_0^{(2)}(M))^\perp$ if and only if $\mathfrak{p} \mid L_{\text{alg}}(\kappa, f)$. Furthermore, if there is a basis of newforms for $S_{2\kappa-2}^{\text{new}}(\Gamma_0(M))$ or $\overline{\rho}_{f,\mathfrak{p}}$ has Serre level M, then $\mathfrak{p}$ is a congruence prime of F_f with respect to $S_\kappa^{\text{M}}(\Gamma_0^{(2)}(M))^\perp$ if and only if $\mathfrak{p} \mid L_{\text{alg}}(\kappa, f)$.

We will provide evidence for this conjecture in Theorem 5.9 and Corollary 5.10.

4. Eisenstein Series

In this section we define a general Siegel Eisenstein series as well as choose an appropriate section for our purposes. We then review relevant properties of this particular Siegel Eisenstein series. Many of the results are due to Shimura, and those not contained in the works of Shimura can be found in [4]. The interested reader is advised to consult there for proofs and a more complete exposition than is given here.

To ease notation we set $G_n = \text{Sp}_{2n}$ in this section. Let $\mathbb{S}_n$ be the set of n by n symmetric matrices. We let $P_n = U_n Q_n$ denote the Siegel parabolic subgroup with unipotent radical

$$U_n = \left\{ u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{S}_n \right\}$$

and Levi subgroup

$$Q_n = \left\{ Q(A) = \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} : A \in \text{GL}_n \right\}.$$

Let $N > 1$ be an integer. For a prime ℓ define

$$K_{0,\ell}(N) = \{ g \in G_n(\mathbb{Q}_\ell) : a_g, b_g, d_g \in \text{Mat}_n(\mathbb{Z}_\ell), c_g \in \text{Mat}_n(N\mathbb{Z}_\ell) \}$$

and set

$$K_{0,f}(N) = \prod_{\ell \nmid \infty} K_{0,\ell}(N).$$

Note that if $\ell \nmid N$ we have $K_{0,\ell}(N) = \text{Mat}_{2n}(\mathbb{Z}_\ell) \cap G_n(\mathbb{Q}_\ell)$. Set

$$K_\infty = \{ g \in G_n(\mathbb{R}) : g(i_{2n}) = i_{2n} \}$$

where $i_{2n} = i 1_{2n}$ and put

$$K_0(N) = K_\infty K_{0,f}(N).$$

Let $\chi = \otimes_v \chi_v$ be an idele class character. For $s \in \mathbb{C}$, consider the induced representation

$$I(\chi, s) = \bigotimes_v I_v(\chi_v, s) = \text{Ind}_{P_n(\mathbb{A})}^{G_n(\mathbb{A})}(\chi | \cdot |^{2s})$$

consisting of smooth functions $\mathfrak{f}$ on $G_n(\mathbb{A})$ satisfying

$$\mathfrak{f}(pg,s) = \chi(\det(A))|\det A|^{2s}\mathfrak{f}(g,s)$$

for $p = u(x)Q(A) \in P_n(\mathbb{A})$ and $g \in G_n(\mathbb{A})$. Given such a section, we define the associated Eisenstein series by

$$E_{\mathfrak{f}}(g,s) = \sum_{\gamma \in P_n(\mathbb{Q})\backslash G_n(\mathbb{Q})} \mathfrak{f}(\gamma g, s).$$

We now specialize to the case of interest for our results. Let N and κ be integers with $N > 1$ and $\kappa > n+1$. We require χ to satisfy

$$\chi_\infty(x) = \left(\frac{x}{|x|}\right)^\kappa,$$
$$\chi_v(x) = 1 \text{ if } v \nmid \infty,\ x \in \mathbb{Z}_v^\times, \text{ and } x \equiv 1 (\operatorname{mod} N).$$

We choose the section $\mathfrak{f} = \otimes_v \mathfrak{f}_v$ so that

(1) At the infinite place we set $\mathfrak{f}_\infty$ to be the unique vector in $I_\infty(\chi, s)$ so that

$$\mathfrak{f}_\infty(k,\kappa) = j(k,i)^{-\kappa}$$

for all $k \in K_\infty$.

(2) For all $v \nmid N$ we set $\mathfrak{f}_v$ to be the unique $K_{0,v}(N)$-fixed vector with

$$\mathfrak{f}_v(1) = 1.$$

(3) For all $v|N$ we set $\mathfrak{f}_v$ to be the vector given by

$$\mathfrak{f}_v(k) = \chi_v(\det(a_k))$$

for all $k \in K_{0,v}(N)$ with $k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$ and

$$\mathfrak{f}_v(g) = 0$$

if $g \notin P_n(\mathbb{Q}_v)K_{0,v}(N)$.

The Eisenstein series associated to this section is precisely the Eisenstein series used in [4] and [16].

As we will be interested in classical applications, we associate an Eisenstein series on $\mathfrak{h}^n \times \mathbb{C}$ to $E_{\mathfrak{f}}(g,s)$ by setting

$$E_{\mathfrak{f}}(Z,s) = j(g_\infty, i1_n)^\kappa E_{\mathfrak{f}}(g_\infty, s)$$

where $g_\infty \in K_\infty$ so that $g_\infty(i1_n) = Z$. The Eisenstein series $E_{\mathfrak{f}}(Z,s)$ converges locally uniformly on $\mathfrak{h}^n$ for $\operatorname{Re}(s) > (n+1)/2$.

Our next step is to study the Fourier coefficients of this Eisenstein series. It turns out that it is easier to study the Fourier coefficients of a simple translation of $E_{\mathfrak{f}}(g,s)$ given by

$$E_{\mathfrak{f}}^\sharp(g,s) = E_{\mathfrak{f}}(g\iota_n^{-1}, s)$$

where $\iota_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. We have the corresponding classical form $E_{\mathfrak{f}}^\sharp(Z,s)$.

Let $L = \mathbb{S}_n(\mathbb{Q}) \cap \mathrm{Mat}_n(\mathbb{Z})$, $L' = \{\mathfrak{s} \in \mathbb{S}_n(\mathbb{Q}) : \mathrm{Tr}(\mathfrak{s}L) \subseteq \mathbb{Z}\}$ and $\mathscr{N} = N^{-1}L'$. The Eisenstein series $E_{\mathfrak{f}}^{\sharp}(Z,s)$ has a Fourier expansion

$$E_{\mathfrak{f}}^{\sharp}(Z,s) = \sum_{h \in \mathscr{N}} a(h,Y,s) e(\mathrm{Tr}(hX))$$

for $Z = X+iY \in \mathfrak{h}^n$. Set $\Lambda^{\Sigma_N}(s,\chi) = L^{\Sigma_N}(2s,\chi) \prod_{j=1}^{[n/2]} L^{\Sigma_N}(4s-2j,\chi^2)$. We normalize $E_{\mathfrak{f}}^{\sharp}(Z,s)$ by setting at

$$D_{E_{\mathfrak{f}}^{\sharp}}(Z,s) = \pi^{-\frac{n(n+2)}{4}} \Lambda^{\Sigma_N}(s,\chi) E_{\mathfrak{f}}^{\sharp}(Z,s).$$

We then have the following theorem.

Theorem 4.1. *([4], Theorem 4.4) Let n, N, and κ be positive integers such that $N > 1$, $n \geq 2$, $\kappa > n+1$. Let χ be an idele class character as above. Let p be an odd prime such that $p > n$ and $p \nmid N$. Then $D_{E_{\mathfrak{f}}^{\sharp}}(Z,(n+1)/2-\kappa/2)$ is in $M_\kappa(\Gamma_0^{(n)}(N), \mathbb{Z}_p[\chi, i^{n\kappa}])$.*

We now restrict to the case of $n = 4$. Recall that one has maps

$$\begin{array}{ccc} \mathfrak{h}^2 \times \mathfrak{h}^2 & \hookrightarrow & \mathfrak{h}^4 \\ (Z,W) & \mapsto & \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix} = \mathrm{diag}[Z,W] \end{array}$$

and

$$\begin{array}{ccc} G_2 \times G_2 & \hookrightarrow & G_4 \\ (\alpha,\beta) & \mapsto & \alpha \times \beta = \begin{pmatrix} a_\alpha & 0 & b_\alpha & 0 \\ 0 & a_\beta & 0 & b_\beta \\ c_\alpha & 0 & d_\alpha & 0 \\ 0 & c_\beta & 0 & d_\beta \end{pmatrix}. \end{array}$$

One can use these maps to pull back the Eisenstein series $E_{\mathfrak{f}}(g,s)$ from an automorphic form on $G_4(\mathbb{A})$ to an automorphic form on $G_2(\mathbb{A}) \times G_2(\mathbb{A})$. In particular, one can pull back $E_{\mathfrak{f}}(\mathfrak{z},(5-\kappa)/2)$ to a modular form of weight κ and level $\Gamma_0^{(2)}(N)$ in each of the variables Z and W separately. There is extensive literature on such pullbacks ([1], [8], [9], [16], [17]).

Define $\sigma = \in G_4(\mathbb{A})$ by

$$\sigma_\upsilon = \begin{cases} I_8 & \text{if } \upsilon \nmid N \text{ or } \upsilon = \infty \\ \begin{pmatrix} I_4 & 0_4 \\ \begin{pmatrix} 0_2 & I_2 \\ I_2 & 0_2 \end{pmatrix} & I_4 \end{pmatrix} & \text{if } \upsilon \mid N. \end{cases}$$

Strong approximation provides an element $\rho \in \mathrm{Sp}_8(\mathbb{Z}) \cap K_0(N)\sigma$ with the property that $N_\upsilon | a(\sigma\rho^{-1})_\upsilon - I_4$ for every $\upsilon \mid N$. In particular, $E_{\mathfrak{f}} |_\rho$ corresponds to $E_{\mathfrak{f}}(g\sigma^{-1})$. Set

$$\mathscr{E}(Z,W) := D_{E_{\mathfrak{f}}|_{\rho(1\times\iota_2^{-1})}}(\mathrm{diag}[Z,W],(5-\kappa)/2). \tag{4}$$

We then have the following theorem.

Theorem 4.2. *([4], Theorem 4.5) Let $N > 1$ and $\kappa > 9$ be integers and Σ_N the set of primes dividing N. For $F \in S_\kappa(\Gamma_0^{(2)}(N), \mathbb{R})$ a Hecke eigenform, p a prime with $p > 2$ and $p \nmid N$ we have*

$$\langle \mathscr{E}(Z,W), F(W)\rangle = \pi^{-3}\, \mathcal{A}_{\kappa,N} L^{\Sigma_N}(5-\kappa, F, \chi, \mathrm{St}) F(Z) \tag{5}$$

with $\mathcal{E}(Z,W)$ having Fourier coefficients in $\mathbb{Z}_p[\chi]$ where

$$\mathcal{A}_{\kappa,N} = \frac{(-1)^\kappa\, 2^{2\kappa-3} v_N}{3\,[\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(N)]}$$

with $v_N = \pm 1$.

We note that $\mathscr{E}(Z,W)$ is a holomorphic cuspform in each of the variables Z and W. One can see [6] for a proof of this fact in a more general setting.

5. A Congruence

Let k, M, and N be positive integers with $\kappa > 9$ even, $N > 1$, and $M|N$ with M odd and square-free. Let $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$ be a newform. Let $\mathfrak{p}$ be a prime of $\mathcal{O}_{\mathbb{Q}(f)}$ of residue characteristic p so that $p > 2\kappa - 2$, $p \nmid N$, $\overline{\rho}_{f,\mathfrak{p}}$ is irreducible, and $\mathfrak{p} \mid L_{\mathrm{alg}}(\kappa, f)$. We fix once and for all compatible embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$. Let $\mathcal{O}$ be the $\mathfrak{p}$-adic completion of $\mathcal{O}_{\mathbb{Q}(f)}$ at $\mathfrak{p}$. Write ϖ for a uniformizer of $\mathcal{O}$. In this section we demonstrate how to use the Eisenstein series studied in the previous section to obtain a congruence between the Saito-Kurokawa lift $F_f \in S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))$ and a cuspidal Siegel eigenform in $S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))^\perp$.

Our first step is to replace $\mathcal{E}(Z,W)$ by a form of level $\Gamma_0^{(2)}(M)$ in each variable. This is accomplished by taking the trace of $\mathcal{E}(Z,W)$:

$$\mathcal{E}_M(Z,W) = \sum_{\gamma\times\delta \in (\Gamma_0^{(2)}(M)\times\Gamma_0^{(2)}(M))/(\Gamma_0^{(2)}(N)\times\Gamma_0^{(2)}(N))} \mathcal{E}(Z,W)|_{\gamma\times\delta}.$$

It is easy to check that $\mathcal{E}_M(Z,W)$ is a Siegel modular of weight κ and level $\Gamma_0^{(2)}(M)$ in each variable separately. The q-expansion principle for Siegel modular forms ([7], Prop. 1.5) gives that the Fourier coefficients of $\mathcal{E}_M(Z,W)$ lie in $\mathbb{Z}_p[\chi]$ in light of Theorem 4.1. Observe that for $F \in S_\kappa(\Gamma_0^{(2)}(M))$ one has

$$\langle \mathcal{E}(Z,W), F(W)\rangle_{\Gamma_0^{(2)}(N)} = \langle \mathcal{E}_M(Z,W), F(W)\rangle_{\Gamma_0^{(2)}(M)}.$$

Let $f_0 = f, \ldots, f_m$ be an orthogonal basis of newforms for $S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M), \mathcal{O})$. Let $F_0 = F_f, F_1, \ldots, F_m$ be the Saito-Kurokawa lifts of $f_0, \ldots, f_m$. Let $F_0, \ldots, F_m$, $F_{m+1}, \ldots, F_{m+r}$ be a basis of eigenforms of $S_\kappa(\Gamma_0^{(2)}(M))$ that are orthogonal with respect to the Petersson product. Enlarge $\mathcal{O}$ if necessary so that the f_i and the F_i are all defined over $\mathcal{O}$. Using that $\mathcal{E}_M(Z,W)$ is cuspidal in each variable, we can write

$$\mathcal{E}_M(Z,W) = \sum_{i,j} c_{i,j} F_i(Z) F_j(W).$$

Combining this expansion with equation (5) we obtain the following lemma.

Lemma 5.1. *One has*

$$\mathcal{E}_M(Z,W) = \sum_{i=0}^{m+r} c_{i,i} F_i(Z) F_i(W)$$

where

$$c_{i,i} = \mathcal{A}_{\kappa,N} \frac{L^{\Sigma_N}(5-\kappa, F_i, \chi, \mathrm{St})}{\pi^3 \langle F_i, F_i \rangle}.$$

It is now enough to study the $\mathfrak{p}$-divisibility of $c_{0,0}$. To see why this is the case, suppose we can write $c_{0,0} = \frac{u}{\varpi^b}$ with $b > 0$ for some ϖ-unit u. The following lemma will show that the congruence we obtain is a nontrivial one.

Lemma 5.2. *There is at least one $c_{i,i} \neq 0$ with $1 \leq i \leq m+r$ in the expansion*

$$\mathcal{E}_M(Z,W) = c_{0,0} F_f(Z) F_f(W) + \sum_{i=1}^{m+r} c_{i,i} F_i(Z) F_i(W)$$

if we can write $c_{0,0} = \frac{u}{\varpi^b}$ with $b > 0$ and u a ϖ-unit.

Proof. Suppose that $c_{i,i} = 0$ for all $1 \leq i \leq m+r$. We have

$$\varpi^b \mathcal{E}_M(Z,W) = u F_f(Z) F_f(W). \tag{6}$$

We know from Corollary 2.7 that F_f has Fourier coefficients in $\mathcal{O}$ and there exists T_0 so that $\varpi \nmid A_{F_f}(T_0)$. However, this gives a contradiction as we know the Fourier coefficients of $\mathcal{E}_M(Z,W)$ are ϖ-integral, and so in terms of Fourier coefficients $\varpi^b \mathcal{E}_M(Z,W)$ reduces to 0 modulo ϖ. However, in terms of Fourier coefficients we have that $u F_f(Z) F_f(W)$ does not reduce to 0. □

Applying the same type of argument used in the proof of Lemma 5.2 we have that

$$u A_{F_f}(T_0) F_f(W) \equiv -\varpi^b \sum_{i=1}^{m+r} c_{i,i} A_{F_i}(T_0) F_i(W) (\operatorname{mod} \varpi^b). \tag{7}$$

Thus, again using that u and $A_{F_f}(T_0)$ are ϖ-units we are able to conclude the following theorem.

Theorem 5.3. *Let M be odd and square free, $N > 1$ an integer so that $M|N$ and $\kappa > 9$ an even integer. Suppose that there exists $b \geq 1$ and a ϖ-unit u so that*

$$\mathcal{A}_{\kappa,N} \frac{L^{\Sigma_N}(5-\kappa, F_f, \chi, \mathrm{St})}{\pi^3 \langle F_f, F_f \rangle} = \frac{u}{\varpi^b}.$$

Then there exists a non-trivial $G \in S_\kappa(\Gamma_0^{(2)}(M), \mathcal{O})$ distinct from F_f so that

$$F_f \equiv G (\operatorname{mod} \varpi^b),$$

namely,

$$G(W) = \frac{-\varpi^b}{u A_{F_f}(T_0)} \sum_{i=1}^{m+r} c_{i,i} A_{F_i}(T_0) F_i(W),$$

i.e., $\mathfrak{p}$ is a congruence prime of F_f.

Before we study $c_{0,0}$ we deal with the issue that the G constructed in Theorem 5.3 could be in $S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))$. We will require a particular Hecke operator to remove other Saito-Kurokawa lifts in $S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))$ that are possibly congruent to F_f. Before we state that result, we need the following result on the existence of canonical periods associated to a newform.

Theorem 5.4. *([15], Theorem 1) Let $g \in S_\kappa(\Gamma_0(M), \mathcal{O})$ be a newform. There exists complex periods $\Omega_g^\pm$ such that for each integer j with $0 < j < \kappa$ and every Dirichlet character χ one has*

$$\frac{L(j,g,\chi)}{\tau(\chi)(2\pi i)^j} \in \begin{cases} \Omega_g^- \mathcal{O}_\chi & \text{if } \chi(-1) = (-1)^j, \\ \Omega_g^+ \mathcal{O}_\chi & \text{if } \chi(-1) = (-1)^{j-1}, \end{cases}$$

where $\tau(\chi)$ is the Gauss sum of χ and $\mathcal{O}_\chi$ is the extension of $\mathcal{O}$ generated by the values of χ. We write

$$L_{\mathrm{alg}}(j,g,\chi) = \frac{L(j,g,\chi)}{\tau(\chi)(2\pi i)^j \Omega_g^\pm}$$

where the appropriate $\Omega_g^\pm$ is chosen.

Theorem 5.5. *([2]) Let $f = f_0, \dots, f_m \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M), \mathcal{O})$ be a basis of newforms. Suppose that $\overline{\rho}_{f,\mathfrak{p}}$ is irreducible. If $M > 4$ there exists a Hecke operator $t_f \in \mathbb{T}^{(1)}(2\kappa - 2, \mathcal{O})$ so that*

$$t_f f_i = \begin{cases} \alpha_f f & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha_f = \frac{u_f \langle f, f \rangle}{\Omega_f^+ \Omega_f^-}$ and $u_f \in \mathcal{O}^\times$. We have the exact same result in the case $M \in \{1, 3\}$, we just require the additional assumption that f is ordinary at p.

We use the fact that the Saito-Kurokawa correspondence is Hecke-equivariant to conclude that there exists $t_f^{(2)} \in \mathbb{T}^{(2)}(\kappa, \mathcal{O})$ so that

$$t_f^{(2)} F_i = \begin{cases} \alpha_f F_f & \text{if } i = 0 \\ 0 & \text{if } 1 \leq i \leq m. \end{cases}$$

We now return to the congruence

$$F_f(W) \equiv \frac{-\varpi^b}{u A_{F_f}(T_0)} \sum_{i=1}^{m+r} c_{i,i} A_{F_i}(T_0) F_i(W) \pmod{\varpi^b}.$$

As the Hecke operator $t_f^{(2)}$ is defined over $\mathcal{O}$, we can apply it to the congruence to obtain

$$\alpha_f F_f(W) \equiv \frac{-\varpi^b}{u A_{F_f}(T_0)} \sum_{i=1}^{m+r} c_{i,i} A_{F_i}(T_0) t_f^{(2)} F_i(W) \pmod{\varpi^b}.$$

The operator $t_f^{(2)}$ kills any Saito-Kurokawa lifts in $S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))$ in the sum defining G. Thus, we have

$$F_f(W) \equiv \alpha_f^{-1} \sum_{i=m+1}^{m+r} c_i t_f^{(2)} F_i(W) \pmod{\varpi^b}$$

where the F_i are in $S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))^\perp$. Rename G as

$$G(W) = \alpha_f^{-1} \sum_{i=m+1}^{m+r} c_i \lambda_{F_i}(t_f^{(2)}) F_i(W).$$

We would like to have a congruence to an eigenform. We can achieve this, assuming we can write

$$\mathcal{A}_{\kappa,N} \frac{L^{\Sigma_N}(5-\kappa, F_f, \chi, \mathrm{St})}{\pi^3 \langle F_f, F_f \rangle} = \frac{u}{\varpi^b}$$

for some $b \geq 1$ as follows. Let $F \in S_\kappa(\Gamma_0^{(2)}(M), \mathcal{O})$ be an eigenform for $\mathbb{T}^{(2),\Sigma_M}(\kappa, \mathcal{O})$ where $\mathbb{T}^{(2),\Sigma_M}(\kappa, \mathcal{O})$ denotes the Hecke algebra generated over $\mathcal{O}$ by the Hecke operators $T(n)$ with $\gcd(n, M) = 1$. Recall we can associate to F a maximal ideal $\mathfrak{m}_F$ of $\mathbb{T}^{(2),\Sigma_M}(\kappa, \mathcal{O})$ by setting $\mathfrak{m}_F = \ker(\varsigma_F)$ where $\varsigma_F : \mathbb{T}^{(2),\Sigma_M}(\kappa, \mathcal{O}) \to \mathcal{O} \to \mathcal{O}/\varpi$ is defined by sending t to $\lambda_F(t) (\mathrm{mod}\, \varpi)$. We will make use of the following well known result.

Proposition 5.6. *The Hecke algebra $\mathbb{T}^{(2),\Sigma_M}(\kappa, \mathcal{O})$ factors as*

$$\mathbb{T}^{(2),\Sigma_M}(\kappa, \mathcal{O}) = \prod_{\mathfrak{m}} \mathbb{T}^{(2),\Sigma_M}(\kappa, \mathcal{O})_{\mathfrak{m}}$$

where the product runs over all maximal ideals of $\mathbb{T}^{(2),\Sigma_M}(\kappa, \mathcal{O})$ and $\mathbb{T}^{(2),\Sigma_M}(\kappa, \mathcal{O})_{\mathfrak{m}}$ denotes the localization of $\mathbb{T}^{(2),\Sigma_M}(\kappa, \mathcal{O})$ at $\mathfrak{m}$.

We have the following immediate corollary.

Corollary 5.7. *Let $F, G \in S_\kappa(\Gamma_0^{(2)}(M), \mathcal{O})$ be eigenforms for $\mathbb{T}^{(2),\Sigma_M}(\kappa, \mathcal{O})$. There exists a Hecke operator $t_G \in \mathbb{T}^{(2),\Sigma_M}(\kappa, \mathcal{O})$ so that $t_G G = G$ and $t_G F = 0$ if $F \not\equiv_{\mathrm{ev},\Sigma_M} G \pmod{\varpi}$. If $F \equiv_{\mathrm{ev},\Sigma_M} G \pmod{\varpi}$, then $t_G F = F$.*

Lemma 5.8. *Let $G \in \mathcal{S}_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))^\perp$ be as above so that we have the congruence $G \equiv F_f (\mathrm{mod}\, \varpi^b)$. Then there exists an eigenform F so that $F \in S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))^\perp$ so that*

$$F_f \equiv_{\mathrm{ev},\Sigma_M} F (\mathrm{mod}\, \varpi).$$

Proof. Let t_{F_f} be as above. If $F_i \not\equiv_{\mathrm{ev},\Sigma_M} F_f (\mathrm{mod}\, \varpi)$ for every i then applying t_{F_f} to the congruence $G \equiv F_f (\mathrm{mod}\, \varpi)$ would then yield $F_f \equiv 0 (\mathrm{mod}\, \varpi)$, a contradiction to the fact that $A_{F_f}(T_0) \in \mathcal{O}^\times$. Thus there must be an i so that $F_f \equiv_{\mathrm{ev},\Sigma_M} F_i (\mathrm{mod}\, \varpi)$. □

After acting by the Hecke operators, we have reduced the problem to studying the conditions under which we can write $\alpha_f c_{0,0} = \frac{\alpha}{\varpi^b}$ for some $b \geq 1$. We wish to give these conditions in terms of special values of the L-function associated to f as well as twists of this L-function by Dirichlet characters.

We can apply Theorems 2.9 and 2.10 to write

$$\alpha_f c_{0,0} = \mathcal{C}_{\kappa,M,N} \frac{L^{\Sigma_N}(3-\kappa, \chi) L(\kappa-1, f, \chi_D) L^{\Sigma_N}(1, f, \chi) L^{\Sigma_N}(2, f, \chi)}{\pi^2 L(\kappa, f) \Omega_f^+ \Omega_f^-}$$

where

$$\mathcal{C}_{\kappa,M,N} = \frac{\nu_N D^\kappa 2^{2\kappa+n+1} 3 u_f}{M^\kappa (\kappa-1) D^{3/2} |c_{\mathcal{SH}_D(f)}(|D|)|^2 [\Gamma_0^2(M) : \Gamma_0^2(N)] \prod_{i=1}^n (p_i^2+1)}$$

where $M = p_1 \cdots p_n$.

We begin by investigating $\mathcal{C}_{\kappa,M,N}$. Note that everything in the numerator (respectively the denominator) of $\mathcal{C}_{k,M,N}$ is integral. Note that as long as we choose D so that $p \nmid D$, we will not have any ϖ's in the numerator as we assumed $p > 3$ so ϖ cannot divide 2 or 3. Thus, we have that

$$\operatorname{ord}_{\varpi}(\mathcal{C}_{\kappa,M,N}) \leq 0.$$

We can write

$$\frac{L^{\Sigma_N}(1,f,\chi)L^{\Sigma_N}(2,f,\chi)}{\Omega_f^+\Omega_f^-} = \frac{\tau(\chi)^2(2\pi i)^3 L_{\mathrm{alg}}(1,f,\chi)L_{\mathrm{alg}}(2,f,\chi)}{L_{\Sigma_N}(1,f,\chi)L_{\Sigma_N}(2,f,\chi)}.$$

We also know that if $p \nmid N$ then $L^{\Sigma_N}(3-\kappa,\chi) \in \mathbb{Z}_p[\chi]$. If we require that $\chi_D(-1) = -1$, then we have

$$\frac{L(\kappa-1,f,\chi_D)}{L_{\mathrm{alg}}(\kappa,f)} = \frac{\tau(\chi_D)L_{\mathrm{alg}}(\kappa-1,f,\chi_D)}{(2\pi i)L_{\mathrm{alg}}(\kappa,f)}.$$

Since $p \nmid DN$ we have that $\tau(\chi)$ and $\tau(\chi_D)$ are ϖ units. Thus, we have that if

$$\operatorname{ord}_{\varpi}\left(\frac{L^{\Sigma_N}(3-\kappa,\chi)L_{\mathrm{alg}}(\kappa-1,f,\chi_D)L_{\mathrm{alg}}(1,f,\chi)L_{\mathrm{alg}}(2,f,\chi)}{L_{\mathrm{alg}}(\kappa,f)}\right) < 0$$

then we will have $\operatorname{ord}_{\varpi}(\alpha_f c_{0,0}) < 0$ as desired. This certainly is the case if we can choose D and χ so that $\mathfrak{p} \nmid L^{\Sigma_N}(3-\kappa,\chi)L_{\mathrm{alg}}(\kappa-1,f,\chi_D)L_{\mathrm{alg}}(1,f,\chi)L_{\mathrm{alg}}(2,f,\chi)$ since by assumption we have $\mathfrak{p} \mid L_{\mathrm{alg}}(\kappa,f)$.

We summarize with the following theorem.

Theorem 5.9. *Let M be a positive odd square-free integer and $\kappa > 9$ an even integer. Let $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$ be a newform, $\mathfrak{p} \subset \mathcal{O}_{\mathbb{Q}(f)}$ a prime with residue characteristic $p > 2\kappa - 2$ so that $p \nmid M$, $\overline{\rho}_{f,\mathfrak{p}}$ is irreducible, and $\mathfrak{p} \mid L_{\mathrm{alg}}(\kappa,f)$. (If $M = 1,3$ we also require f to be ordinary at $\mathfrak{p}$.) Let F_f be the Saito-Kurokawa lift of f. If there exists a fundamental discriminant D so that $\gcd(pM,D) = 1$, $D < 0$, $\chi_D(-1) = -1$ and $c_{\mathcal{SH}(f)}(|D|) \neq 0$, an integer $N > 1$ with $M|N$ and $\mathfrak{p} \nmid N$ and a Dirichlet character χ of conductor N such that*

$$\mathfrak{p} \nmid L^{\Sigma_N}(3-\kappa,\chi)L_{\mathrm{alg}}(\kappa-1,f,\chi_D)L_{\mathrm{alg}}(1,f,\chi)L_{\mathrm{alg}}(2,f,\chi),$$

then there exists a nonzero eigenform $G \in S_k^{\mathrm{M,new}}(\Gamma_0^2(M))^{\perp}$ so that

$$F_f \equiv_{\mathrm{ev},\Sigma_M} G (\operatorname{mod} \mathfrak{p}),$$

i.e. $\mathfrak{p}$ is a congruence prime of F_f with respect to $S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M)))^{\perp}$.

Note that if we add the condition that $S_{2\kappa-2}(\Gamma_0(M))$ has a basis of newforms, then we have $S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M)) = S_\kappa^{\mathrm{M}}(\Gamma_0^{(2)}(M))$. Alternatively, if we require that $\overline{\rho}_{f,\mathfrak{p}}$ has Serre level M then we have that f cannot be congruent modulo $\mathfrak{p}$ to a form of smaller level. In particular, this gives that F_f cannot be congruent to any $F_g \in S_\kappa^{\mathrm{M}}(\Gamma_0^{(2)}(M)) - S_\kappa^{\mathrm{M,new}}(\Gamma_0^{(2)}(M))$. Thus, we obtain the following corollary.

Corollary 5.10. *Let M be a positive odd square-free integer and $\kappa > 9$ an even integer. Let $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$ be a newform, $\mathfrak{p} \subset \mathcal{O}_{\mathbb{Q}(f)}$ a prime with residue characteristic $p > 2\kappa - 2$ so that $p \nmid M$, $\overline{\rho}_{f,\mathfrak{p}}$ is irreducible, and $\mathfrak{p} \mid L_{\mathrm{alg}}(\kappa,f)$. (If $M = 1,3$ we also require f to be ordinary at $\mathfrak{p}$.) Furthermore, assume that either*

$S_{2\kappa-2}(\Gamma_0(M))$ has a basis of newforms or $\overline{\rho}_{f,\mathfrak{p}}$ has Serre level M. Let F_f be the Saito-Kurokawa lift of f. If there exists a fundamental discriminant D so that $\gcd(pM, D) = 1$, $D < 0$, $\chi_D(-1) = -1$ and $c_{\mathcal{SH}(f)}(|D|) \neq 0$, an integer $N > 1$ with $M|N$ and $\mathfrak{p} \nmid N$ and a Dirichlet character χ of conductor N such that

$$\mathfrak{p} \nmid L^{\Sigma_N}(3-\kappa, \chi) L_{\mathrm{alg}}(\kappa - 1, f, \chi_D) L_{\mathrm{alg}}(1, f, \chi) L_{\mathrm{alg}}(2, f, \chi),$$

then there exists a nonzero eigenform $G \in S_k^{\mathrm{M}}(\Gamma_0^2(M))^{\perp}$ so that

$$F_f \equiv_{\mathrm{ev}, \Sigma_M} G (\operatorname{mod} \mathfrak{p})$$

i.e. $\mathfrak{p}$ is a congruence prime of F_f with respect to $S_\kappa^{\mathrm{M}}(\Gamma_0^{(2)}(M)))^{\perp}$.

References

[1] S. Böcherer, *Über die Funktionalgleichung automorpher L-Funktionen zur Siegelscher Modulgruppe*, J. reine angew. Math. **362** (1985) 146–168.

[2] J. Brown, *Pullbacks of Eisenstein series on $U(3,3)$ and nonvanishing of Selmer groups* 1–46. Preprint.

[3] ———, *An inner product relation on Saito-Kurokawa lifts*, The Ramanujan Journal **14** (2007), no. 1, 89–105.

[4] ———, *Saito-Kurokawa lifts and applications to the Bloch-Kato conjecture*, Compositio Math. **143** (2007), no. 2, 290–322.

[5] ———, *L-functions on* $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ *and the Bloch-Kato conjecture*, Int. J. Num. Th. (2008) 1–26. To appear.

[6] ———, *On the cuspidality of pullbacks of Siegel Eisenstein series to* $\mathrm{Sp}(2m) \times \mathrm{Sp}(2n)$, J. Number Theory **131** (2011) 106–119.

[7] C. Chai and G. Faltings, Degeneration of Abelian Varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3.Folge Band 22, A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin-Heidelberg-New York (1980).

[8] P. Garrett, *Pullbacks of Eisenstein series; applications*, in Automorphic forms of several variables (Katata, 1983), Vol. 46 of *Prog. in Math.*, 114–137, Birkhauser, Boston, MA (1984).

[9] ———, *On the arithmetic of Siegel-Hilbert cuspforms: Petersson inner products and Fourier coefficients*, Invent. Math. **107** (1992) 453–481.

[10] H. Katsurada, *Congruence of Siegel modular forms and their zeta functions*, in T. Yoshida, editor, Proceedings of the 8th Autumn Workshop on Number Theory, 47–59 (2006).

[11] W. Kohnen, *A remark on the Shimura correspondence*, Glasgow Math. J. **30** (1988) 285–291.

[12] M. Manickam, B. Ramakrishnan, and T. Vasudevan, *On Saito-Kurokawa descent for congruence subgroups*, Manuscripta Math. **81** (1993) 161–182.

[13] I. Piatetski-Shapiro, *On the Saito-Kurokawa Lifting*, Invent. Math. **71** (1983) 309–338.

[14] A. Pitale and R. Schmidt, *Sign changes of Hecke eigenvalues of Siegel cusp forms of degree 2*, Proc. Amer. Math. Soc. **136** (2008) 3831–3838.

[15] G. Shimura, *On the periods of modular forms*, Math. Ann. **229** (1977) 211–221.

[16] ———, *Eisenstein series and zeta functions on symplectic groups*, Invent. Math. **119** (1995) 539–584.

[17] ———, Euler products and Eisenstein series, Vol. 93 of *CBMS, Regional Conference Series in Mathematics*, AMS, Providence (1997).

[18] G. Stevens, *Λ-adic modular forms of half-integral weight and a Λ-adic Shintani lifting*, Contemp. Math. **174** (1994) 129–151.

[19] A. Wiles, *The Iwasawa conjecture for totally real fields*, Annals of Math. **131** (1990), no. 3, 493–540.

[20] D. Zagier, *Sur la conjecture de Saito-Kurokawa*, in Sé Delange-Pisot-Poitou 1979/80, Vol. 12 of *Progress in Math.*, 371–394, Boston-Basel-Stuttgart (1980).

Department of Mathematical Sciences, Clemson University, Clemson, SC 29631
E-mail address: jimlb@clemson.edu