LEVEL LOWERING FOR HALF-INTEGRAL WEIGHT MODULAR FORMS

JIM BROWN AND YINGKUN LI

Abstract. In this paper we provide a level lowering result for half-integral weight modular forms. The main ingredients are the Shimura map from half-integral weight modular forms to integral weight modular forms along with a level lowering result for integral weight modular forms due to Ribet. It is necessary to keep track of the parity of the weight as well as the character involved so that one can apply the Shintani lift to go back to a half-integral weight modular form and establish the result.

Let $k \geq 2$ and $N \geq 1$ be integers with $4|N$. Write $N = M\ell^m$ with $\ell \nmid M$ for an odd prime ℓ. Let χ be a Dirichlet character modulo N. Given an eigenform $F \in S_{k+1/2}(\Gamma_0(N), \chi)$, by "lowering the level" of F we mean finding an eigenform $G \in S_{k'+1/2}(\Gamma_1(M))$ for some integer $k' \geq 2$ so that

$$\lambda_F(p) \equiv \lambda_G(p) \pmod{\varpi}$$

for primes $p \nmid \ell M$ with ϖ a uniformizer of some finite extension $\mathcal{O}$ of $\mathbb{Z}_\ell$ where $\lambda_F(p), \lambda_G(p)$ denote the $T(p^2)$th eigenvalues of F and G respectively. The goal of this paper is to show one can lower the level of F to remove the powers of ℓ.

We will require the Shimura correspondence [3], which we now recall. There is a linear map, called the Shimura lifting map,

$$\mathrm{Sh} : S_{k+1/2}(\Gamma_0(N), \chi) \to S_{2k}(\Gamma_0(N), \chi^2)$$

defined as follows. Let $G(z) = \sum_{n=1}^{\infty} a_G(n)q^n \in S_{k+1/2}(\Gamma_0(N), \chi)$. The Shimura lift of G, $\mathrm{Sh}(G)(z)$, is defined by $\mathrm{Sh}(G)(z) := \sum_{n=1}^{\infty} a_{\mathrm{Sh}(G)}(n)q^n$ where the coefficients $a_{\mathrm{Sh}(G)}(n)$ are defined by

$$\sum_{n=1}^{\infty} a_{\mathrm{Sh}(G)}(n)n^{-s} = L(s-k+1, \chi\chi_{-1}^{k}) \cdot \sum_{n=1}^{\infty} a_G(tn^2)n^{-s}$$

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where $\chi_{-1} = \left(\frac{-1}{\bullet}\right)$. We also recall that the Shimura lifting is a Hecke-equivariant map, i.e., it commutes with the action of Hecke operators where the Hecke operator $T(p^2, k+1/2, \chi)$ acting on the space $S_{k+1/2}(\Gamma_0(N), \chi)$ corresponds to the Hecke operator $T(p, 2k, \chi^2)$ acting on the space $S_{2k}(\Gamma_0(N), \chi^2)$.

Let $f = \mathrm{Sh}(F) \in S_{2k}(\Gamma_0(N), \chi^2)$ be the Shimura lift of F. Without loss of generality we can assume that f is a normalized eigenform. Our goal now is to find a normalized eigenform g of level $\Gamma_0(M')$ for some $M' \mid M$ with $\lambda_g(p) \equiv \lambda_f(p) (\mathrm{mod}\, \varpi)$ for every prime $p \nmid \ell M$.

Let $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be the Galois representation attached to f as constructed by Deligne. In particular, we know that ρ_f is irreducible, unramified outside of $M\ell$, and for all primes $p \nmid M\ell$ we have

$$\mathrm{tr}(\rho_f(\mathrm{Frob}_p)) = a_f(p) \tag{1}$$

$$\det(\rho_f(\mathrm{Frob}_p)) = \chi^2(p) p^{2k-1}. \tag{2}$$

It is well known that it is possible to choose a basis so that the image of ρ_f is contained in $\mathrm{GL}_2(\mathcal{O})$ where $\mathcal{O}$ is a finite extension of $\mathbb{Z}_\ell$. One can then consider the residual representation that consists of the composition of ρ_f and the natural projection of $\mathrm{GL}_2(\mathcal{O})$ onto $\mathrm{GL}_2(\mathbb{F})$ where $\mathbb{F} = \mathcal{O}/\varpi$ with ϖ a uniformizer of $\mathcal{O}$. However, this depends on the choice of the basis so that the image of ρ_f lands in $\mathrm{GL}_2(\mathcal{O})$. The semi-simplification of this map does not depend on this choice. We write $\overline{\rho}_f$ to denote this semi-simplification. Note that if the composition of ρ_f with the natural projection is irreducible then the composition is equal to $\overline{\rho}_f$. Equations (1) and (2) show that finding g as above is equivalent to finding a g with level $\Gamma_0(M')$ with $\overline{\rho}_g \simeq \overline{\rho}_f$. To accomplish this we use the following level lowering result due to Ribet.

Theorem 1. *([2], Theorem 2.1) Assume $\ell \geq 3$. Suppose that $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$ arises from $\Gamma_1(N)$ where $N = M\ell^\alpha$ with $\gcd(M, \ell) = 1$. Then ρ arises from $\Gamma_1(M)$.*

Applying this theorem to our setting we see that there exists an integer k', a Dirichlet character $\varepsilon : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathcal{O}^\times$, and a normalized eigenform $g \in S_{k'}(\Gamma_0(M), \varepsilon)$ so that $\overline{\rho}_f \simeq \overline{\rho}_g$. It remains to show that k' is even and that there exists a character ψ so that $\varepsilon = \psi^2$.

Observe that we have

$$\begin{aligned} \varepsilon \omega_\ell^{k'-1} &\equiv \det \overline{\rho}_g \pmod{\varpi} \\ &\equiv \det \overline{\rho}_f \pmod{\varpi} \\ &\equiv \chi^2 \omega_\ell^{2k-1} \pmod{\varpi} \end{aligned}$$

where ω_ℓ denotes the reduction of the ℓ-adic cyclotomic character. Using the fact that $\ell \nmid M$ we see that ε is unramified at ℓ and so restricting to the inertia group I_ℓ we have

$$\omega_\ell^{k'-1}|_{I_\ell} \equiv \chi^2 \omega_\ell^{2k-1}|_{I_\ell} \pmod{\varpi}.$$

If we further restrict to $\mathrm{Gal}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q})$, we have that χ can be written as a power of ω_ℓ, say $\chi = \omega_\ell^i$. Thus, we have

$$\omega_\ell^{k'+1}|_{\mathrm{Gal}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q})} \equiv \omega_\ell^{2k+2i-1}|_{\mathrm{Gal}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q})} \pmod{\varpi}.$$

Thus, we see that $k' \equiv 2k + 2i \pmod{\ell - 1}$ and so k' must be even as desired.

Let $\varepsilon' = \varepsilon\chi^{-2}\omega_\ell^{k'-2k}$. Observe that since k' is even it is enough to show that $\varepsilon' = \psi^2$ for some character ψ. However, we know that $\varepsilon\chi^{-2}\omega_\ell^{k'-2k} \equiv 1 \pmod{\varpi}$ and so $\varepsilon' \equiv 1 \pmod{\varpi}$. Thus, we have that ε' must have order a power of ℓ. In particular, ε' has odd order and so there exists such a ψ. We have proven the following proposition.

Proposition 2. *Let $f \in S_{2k}(\Gamma_0(M\ell^m), \chi^2)$ be a normalized eigenform with $\gcd(\ell, M) = 1$. There exists an integer m, a character ψ, and a normalized eigenform $g \in S_{2m}(\Gamma_0(M), \psi^2)$ with $\lambda_f(p) \equiv \lambda_g(p) \pmod{\varpi}$ for all $p \nmid \ell M$.*

It is critical in the above proposition that the weight is even and the character is squared. We are now able to apply the Shintani map $\mathcal{SH}$, a Hecke-equivariant map from $S_{2m}(\Gamma_0(M), \psi^2)$ to $S_{m+1/2}(\Gamma_0(4M), \psi)$. In fact, the Shintani map is a Hecke-equivariant isomorphism between $S_{2m}(\Gamma_0(M), \psi^2)$ and Kohnen's $+$-space $S^+_{m+1/2}(\Gamma_0(4M), \psi)$ where

$$S^+_{m+1/2}(\Gamma_0(4M), \psi) = \{G \in S_{m+1/2}(\Gamma_0(4M), \psi) : a_G(n) = 0 \text{ if } (-1)^{m+1}n \equiv 2, 3 \pmod{4}\}.$$

One can see [1] for a proof of this fact. Denote the image of g under the Shintani map by $G := \mathcal{SH}(g)$. If we follow through our mappings, we have that for $p \nmid \ell M$,

$$\begin{aligned} \lambda_F(p) &= \lambda_f(p) \\ &\equiv \lambda_g(p) \pmod{\varpi} \\ &\equiv \lambda_G(p) \pmod{\varpi}. \end{aligned}$$

Thus, we have proven the following theorem

Theorem 3. *Let $F \in S_{k+1/2}(\Gamma_0(M\ell^m), \chi)$ be an eigenform with $\ell \nmid M$. There exists an integer m, a character ψ and an eigenform $G \in S_{m+1/2}(\Gamma_0(M), \psi)$ so that*

$$\lambda_F(p) \equiv \lambda_G(p) \pmod{\varpi}$$

where ϖ is the uniformizer of a finite extension $\mathcal{O}$ of $\mathbb{Z}_\ell$.

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Department of Mathematical Sciences, Clemson University, Clemson, SC 29634

E-mail address: jimlb@clemson.edu

Department of Mathematics, California Institute of Technology, Pasadena, CA 91125

E-mail address: yingkun@caltech.edu