ON THE ACTION OF THE U_p OPERATOR ON THE LOCAL (AT p) REPRESENTATION ATTACHED TO CONGRUENCE LEVEL SIEGEL MODULAR FORMS

JIM BROWN AND KRZYSZTOF KLOSIN

Abstract. In this article we study the action of the U_p Hecke operator on the normalized spherical vector ϕ in the representation of $\mathrm{GSp}_4(\mathbf{Q}_p)$ induced from a character on the Borel subgroup. We compute the Petersson norm of $U_p\phi$ in terms of certain local L-values associated with ϕ.

1. Introduction

In the theory of automorphic forms on algebraic groups the U_p Hecke operator arises in many applications. For instance, it plays a prominent role in the study of theta cycles of modular forms on GL_2 [2] and in applications to producing congruences between values of the partition function $p(n)$ [3, 12]. The U_p operator is used in the theory of harmonic weak Maass forms to "shed" the non-holomorphic part [15] and in studying class polynomials [4]. The study of the properties of this operator plays a crucial role both in Hida theory [13] (modular forms of slope zero), where it is used to define the ordinary projector, and in the theory developed in the GL_2 context by Coleman and Mazur [11] (modular forms of finite slope) and later generalized to other algebraic groups by several authors (see e.g., [8, 9, 10, 21]).

In this article we study the U_p operator acting on automorphic forms on the symplectic group GSp_4. Some of its properties in this setting were initially studied by Taylor in his thesis [20, Chapter 3] with the aim of formulating the theory of Λ-adic Siegel modular forms.

An important theme in the study of p-adic properties of automorphic forms on a reductive group G is the construction of congruences among them. A new method of exhibiting such congruences via computing Petersson norms of automorphic forms arising as lifts from proper subgroups of G was developed by several people including the authors (see e.g., [1, 5, 6, 14]).

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This method can be extended to the context of p-adic families resulting in the need for computing certain ratios of Petersson norms. One such ratio of interest is the focus of this paper, namely we compute $\frac{\langle U_p\phi, U_p\phi\rangle}{\langle\phi,\phi\rangle}$ for a spherical vector ϕ lying in the local at p component of an automorphic representation associated to a general Siegel Hecke eigenform on the group $\mathrm{GSp}_4(\mathbf{A_Q})$ of congruence level prime to p. In doing so we show that this ratio is very closely related to a ratio of certain local L-functions attached to ϕ (cf. Theorem 1.1). This is in line with a similar result for automorphic forms on the group GL_2, where one shows that the ratio of the norm of $U_p\phi$ and the norm of ϕ is related to a special value of the symmetric square L-function of ϕ (cf. [7]).

Let us now state the main result of the paper. Let p be a prime and N a positive integer with $p \nmid N$. Let F be a Siegel modular form of genus 2, congruence level $K_0(N)$ (see below), and trivial character. We assume that F is an eigenform for all Hecke operators. Write π_F for any (of the finitely many) irreducible cuspidal automorphic representations generated by the automorphic form associated to F. The isomorphism class of the local component $\pi_{F,p}$ (at p) of π_F depends only on F and not on the choice of π_F. Pitale and Schmidt (see e.g., [16, Proposition 1.1] or [17, Theorem 3.2]) give a complete description of the representations of $\mathrm{GSp}_4(\mathbf{Q}_p)$ which can occur as $\pi_{F,p}$ for some F as above. In particular they divide the possible local representations into three groups (the tempered case (T), the complimentary case (C) and the Saito-Kurokawa case (SK)). However, thanks to the result of Weissauer proving the Ramanujan conjecture for (non-CAP cuspidal) Siegel modular forms [22], we can dispose of the complimentary case as it would violate the conjecture, leaving us to consider only the tempered and the Saito-Kurokawa case.

Since $p \nmid N$ the form F gives rise to a spherical vector ϕ inside the space of $\pi_{F,p}$, which we fix by demanding that $\phi(k) = 1$ for all $k \in K_0(1) := \mathrm{GSp}_4(\mathbf{Z}_p)$. Let $T_p = K_0(1)\,\mathrm{diag}(p,p,1,1)K_0(1)$ be the standard Hecke operator at p. Since it is possible to treat F (and hence ϕ) simultaneously as a form of level N and as a form of level Np, we can study the action of a different Hecke operator, namely the operator $U_p' = K_0(p)\,\mathrm{diag}(p,p,1,1)K_0(p)$ which we normalize to $U_p := [K_0(1) : K_0(p)]U_p'$, where $K_0(N)$ denotes the subgroup of $\mathrm{GSp}_4(\mathbf{Z}_p)$ consisting of matrices all of whose entries in the lower-left 2×2 block are divisible by N. The goal of this article is to compute $\frac{\langle U_p\phi, U_p\phi\rangle}{\langle\phi,\phi\rangle}$ for the (unique up to scalar) inner product $\langle\cdot,\cdot\rangle$ on $\pi_{F,p}$.

For the convenience of the reader let us state here the main result of the paper - cf. also Theorem 9.1 and Corollary 9.3. For the definitions and notation we refer the reader to the main content of the article.

Theorem 1.1. *Suppose that* $\Pi := \pi_{F,p} = \chi_1 \times \chi_2 \rtimes \sigma$. *Then*

$$\frac{\langle U_p\phi, U_p\phi\rangle}{\langle\phi,\phi\rangle} = p^2|\sigma(p)|^2\left(1 + \frac{p^3(p-1)}{p^3+p^2+p+1}X\right),$$

where

$$X = \frac{L(0,\Pi,\mathrm{St})}{\zeta(0)L(0,\mathrm{BC}(\pi(\chi_1,\chi_2)))L(0,\mathrm{BC}(\pi(\chi_1^{-1},\chi_2^{-1})))}.$$

Here $L(s,\Pi,\mathrm{St})$, (resp. $L(s,\mathrm{BC}(\pi(\chi,\chi')))$) denotes the pth Euler factor of the standard L-function of Π (resp. of the base change L-function to the unique unramified quadratic extension of $\mathbf{Q}_p$ of the GL_2-representation $\pi(\chi,\chi')$ induced from the characters χ and χ'). Also $\zeta(s) := (1-p^{-s})^{-1}$ is the pth Euler factor of the Riemann zeta function and while it is undefined at 0, the ratio $L(0,\Pi,\mathrm{St})/\zeta(0)$ still makes sense.

The proof of Theorem 1.1 is elementary in nature. It consists of a careful study of the permutation action of an arbitrary element $g \in K_0(1)$ on the left cosets $bK_0(1)$ of the operator T_p (see section 2 for details).

2. The U_p operator

Let p be a prime. Set

$$H := \mathrm{GSp}_4 = \left\{g \in \mathrm{GL}_4 : gJ\,{}^tg = \mu(g)J, \mu(g) \in \mathbf{G}_m\right\}$$

where $J = \begin{bmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{bmatrix}$ and $\mathbf{G}_m$ is the multiplicative group. For a positive integer N set

$$K_0(N) := \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in H(\mathbf{Z}_p) \mid A,B,C,D \in \mathrm{Mat}_2(\mathbf{Z}_p), \quad C \equiv 0_2 \pmod{N} \right\}.$$

In particular, $K_0(N) = H(\mathbf{Z}_p)$ if $p \nmid N$. The group $H(\mathbf{Q}_p)$ is endowed with a unique Haar measure normalized so that $\mathrm{vol}(K_0(1)) = 1$. For a right $K_0(1)$-invariant continuous function ϕ and $g \in H(\mathbf{Q}_p)$ define the T_p Hecke operator as

$$(T_p\phi)(g) = \int_{K_0(1)\,\mathrm{diag}(p,p,1,1)K_0(1)} \phi(gh)dh.$$

Lemma 2.1. [20, p. 38] *One has*

$$K_0(1)\,\mathrm{diag}(p,p,1,1)K_0(1) = \bigsqcup_{b\in\mathcal{B}} bK_0(1),$$

where $\mathcal{B} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \mathcal{B}_3 \sqcup \mathcal{B}_4$ *and*

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1_2 & E \\ 0_2 & 1_2 \end{bmatrix} \mathrm{diag}(p,p,1,1) : E \in \mathrm{Mat}_2(\mathbf{F}_p),\ {}^tE = E \right\},$$
$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & -z \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathrm{diag}(1,p,p,1) : x, z \in \mathbf{F}_p \right\},$$
$$\mathcal{B}_3 = \left\{ \begin{bmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathrm{diag}(p,1,1,p) : x \in \mathbf{F}_p \right\},$$
$$\mathcal{B}_4 = \{\mathrm{diag}(1,1,p,p)\}.$$

Note we have $\#\mathcal{B}_1 = p^3$, $\#\mathcal{B}_2 = p^2$, $\#\mathcal{B}_3 = p$, and $\#\mathcal{B}_4 = 1$. We define the action of the U_p operator on continuous functions which are right-invariant under $K_0(1)$ as

$$(U_p\phi)(g) = \int_{\mathcal{B}_1 K_0(1)} \phi(gh)dh.$$

Remark 2.2. One can check that Lemma 2.1 holds also if one replaces $K_0(1)$ with $K_0(p)$ and the union over $\mathcal{B}$ with the union over $\mathcal{B}_1$. This implies that our definition of U_p agrees up to a scalar multiple (with the scalar equal to p^3+p^2+p+1 - cf. Lemma 4.2) with the definition (mentioned in section 1) in which one integrates over the double coset $K_0(p)\,\mathrm{diag}(p,p,1,1)K_0(p)$.

Lemma 2.3. *If* $g \in K_0(1)$ *there is an injection* $\sigma_g : \mathcal{B}_1 \hookrightarrow \mathcal{B}$ *such that for every* $\beta \in \mathcal{B}_1$ *there exists an element* $k(g,\beta) \in K_0(1)$ *with the property that* $g\beta = \sigma_g(\beta)k(g,\beta)$.

Proof. As $g\beta \in K_0(1)\,\mathrm{diag}(p,p,1,1)K_0(1)$, the existence of σ_g and $k(g,\beta)$ follows directly from Lemma 2.1. If $\sigma_g(\beta) = \sigma_g(\beta')$ then one gets $\beta k(g,\beta)^{-1} = \beta' k(g,\beta')^{-1}$ and hence $\beta = \beta'$ by the disjointness of the union in Lemma 2.1. □

3. The norm of $U_p\phi$

From now on we denote by $|\cdot|$ the complex modulus and by $|\cdot|_p$ the p-adic norm normalized so that $|p|_p = p^{-1}$. We fix an inner product on the space of continuous functions on $K_0(1)$ as follows. Given two such functions ϕ and ψ, we set

$$\langle \phi, \psi \rangle_{K_0(1)} = \int_{K_0(1)} \phi(g)\overline{\psi(g)}dg.$$

Following the conventions of [20] we fix a Borel subgroup $B \subset H$ defined by

$$B = \left\{ \begin{bmatrix} a & & * & * \\ * & b & * & * \\ & & ua^{-1} & * \\ & & & ub^{-1} \end{bmatrix} \mid a, b, u \in \mathbf{G}_m \right\}.$$

We caution the reader that the Borel B chosen here differs from the one used in [16], a source we will sometimes refer to. The relation between B and B_S (the Borel used in [16]) is the following

$$B = \operatorname{diag}(A, A) B_S{}^t \operatorname{diag}(A, A), \quad \text{where } A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In particular the representation of $H(\mathbf{Q}_p)$ which in [16] is denoted by $\chi_1 \times \chi_2 \rtimes \sigma$ in our setup is the representation induced from the character Ψ given by:

$$\Psi : \begin{bmatrix} a & & * & * \\ * & b & * & * \\ & & ua^{-1} & * \\ & & & ub^{-1} \end{bmatrix} \mapsto \chi_1(ub^{-1})\chi_2(ua^{-1})\sigma(u).$$

More precisely, $\chi_1 \times \chi_2 \rtimes \sigma$ is the representation whose space consists of smooth functions

$$f : H(\mathbf{Q}_p) \to \mathbf{C} \text{ such that } f(bg) = \delta(b)^{1/2}\Psi(b)f(g),\ b \in B(\mathbf{Q}_p), g \in H(\mathbf{Q}_p).$$

Here δ is the modulus character given by (cf. [20, p. 37])

$$\delta : \begin{bmatrix} a & & * & * \\ * & b & * & * \\ & & ua^{-1} & * \\ & & & ub^{-1} \end{bmatrix} \mapsto |a^2 b^4 u^{-3}|_p.$$

The normalized spherical vector ϕ is then defined as

$$\phi\left(\begin{bmatrix} a & & * & * \\ * & b & * & * \\ & & ua^{-1} & * \\ & & & ub^{-1} \end{bmatrix} \right) = |a^2 b^4 u^{-3}|_p^{1/2} \chi_1(ub^{-1})\chi_2(ua^{-1})\sigma(u).$$

In particular for $\beta \in \mathcal{B}_j$ we have $\phi(\beta) = \gamma_j \sigma(p)$ where

$$\gamma_j = \begin{cases} p^{-3/2} & j = 1 \\ p^{-1/2}\chi_2(p) & j = 2 \\ p^{1/2}\chi_1(p) & j = 3 \\ p^{3/2}\chi_1(p)\chi_2(p) & j = 4. \end{cases}$$

We will now express the value $(U_p\phi)(g)$ for an arbitrary $g \in K_0(1)$ in terms of the volumes of certain subsets K_s of $K_0(1)$. Given $g \in K_0(1)$, let

$n_j(g)$ be the number of elements of $\mathcal{B}_j$ that are in the image of $\sigma_g : \mathcal{B}_1 \to \mathcal{B}$ for $j = 1, 2, 3, 4$. We have

$$(U_p\phi)(g) = \sum_{\beta \in \mathcal{B}_1} \int_{\beta K_0(1)} \phi(gh)dh = \sum_{j=1}^{4} \sum_{\beta \in \mathcal{B}_j \cap \mathrm{Im}\sigma_g} \phi(\beta) = \sigma(p) \sum_{j=1}^{4} n_j(g)\gamma_j.$$

This gives

$$\begin{aligned} \langle U_p\phi, U_p\phi \rangle_{K_0(1)} &= \int_{K_0(1)} (U_p\phi)(g)\overline{(U_p\phi)(g)}dg \\ &= |\sigma(p)|^2 \sum_{i,j=1}^{4} \int_{K_0(1)} n_i(g)n_j(g)\gamma_i\overline{\gamma_j}dg. \end{aligned} \tag{3.1}$$

Remark 3.1. Any $H(\mathbf{Q}_p)$-invariant inner product $\langle \cdot, \cdot \rangle'$ is a scalar multiple of $\langle \cdot, \cdot \rangle$. Hence (3.1) may be rephrased independently of the choice of $\langle \cdot, \cdot \rangle'$ as

$$\frac{\langle U_p\phi, U_p\phi \rangle'_{K_0(1)}}{\langle \phi, \phi \rangle'_{K_0(1)}} = |\sigma(p)|^2 \sum_{i,j=1}^{4} \int_{K_0(1)} n_i(g)n_j(g)\gamma_i\overline{\gamma_j}dg.$$

This is so since $\langle \phi, \phi \rangle_{K_0(1)} = 1$ as $\phi(k) = 1$ for all $k \in K_0(1)$ by our choice of ϕ and $\mathrm{vol}(K_0(1)) = 1$.

For $s = (s_1, s_2, s_3, s_4) \in \mathbf{Z}^4$, define

$$K_s := \{g \in K_0(1) \mid n_j(g) = s_j \text{ for all } j = 1, 2, 3, 4\}.$$

Then we obtain

$$\begin{aligned} \langle U_p\phi, U_p\phi \rangle_{K_0(1)} &= |\sigma(p)|^2 \sum_{i,j=1}^{4} \sum_{s \in \mathbf{Z}^4} \int_{K_s} s_i s_j \gamma_i\overline{\gamma_j}dg \\ &= |\sigma(p)|^2 \sum_{s \in \mathbf{Z}^4} \mathrm{vol}(K_s) \sum_{i,j=1}^{4} s_i s_j \gamma_i\overline{\gamma_j}. \end{aligned} \tag{3.2}$$

Note that, of course, the sum over $\mathbf{Z}^4$ is in fact a finite sum. Our goal in the next few sections will be to calculate $\mathrm{vol}(K_s)$ for each 4-tuple s. To achieve this we proceed as follows. After some preliminaries (section 4) we will show in section 5 that for an arbitrary $g \in K_0(1)$, $n_4(g) = 1$ if the determinant of the lower-left 2×2-block of g is not zero mod p and $n_4(g) = 0$ otherwise. Then in section 6 we will compute the values $n_3(g)$ for an arbitrary element $g \in K_0(1)$ and in section 7 we will compute the values $n_2(g)$. From these three the value $n_1(g)$ is determined - see also section 5. Finally, in section 8 given all the possible combinations of s_1, s_2, s_3 and s_4 we will compute the corresponding volumes $\mathrm{vol}(K_s)$.

4. Some decompositions

We begin by recalling a couple of elementary lemmas. The first is the well-known formula giving the order of the symplectic group over a finite field. From now on we set $G = H(\mathbf{F}_p)$.

Lemma 4.1. *The order of G is given by*

$$\#G = p^4(p-1)^3(p+1)^2(p^2+1).$$

Proof. For the order of $\mathrm{Sp}_4(\mathbf{F}_p)$ see e.g. [19]. The lemma follows from this and the fact the similitude map $\mu : G \to \mathbf{F}_p^\times$ is a surjection. □

Lemma 4.2. *(*[18, Lemma 5.1.1]*) We have*

$$[K_0(1) : K_0(p)] = p^3 + p^2 + p + 1$$

In fact, we have the following coset representatives:

$$\mathfrak{s}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathfrak{s}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & z & 0 & 1 \end{bmatrix} w_2, \quad z \in \mathbf{F}_p$$

$$\mathfrak{s}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & y & 1 & 0 \\ y & z & 0 & 1 \end{bmatrix} w_2 w_1, \quad y, z \in \mathbf{F}_p$$

$$\mathfrak{s}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x & y & 1 & 0 \\ y & z & 0 & 1 \end{bmatrix} w_2 w_1 w_2, \quad x, y, z \in \mathbf{F}_p$$

where

$$w_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad w_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Let P denote the standard Siegel parabolic subgroup of G. Note here that P is just $K_0(p)$ modulo p and so $\#P = p^4(p-1)^3(p+1)$ by Lemmas 4.1 and 4.2.

Let

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{pmatrix} a'_{11} & a'_{12} & b'_{11} & b'_{12} \\ a'_{21} & a'_{22} & b'_{21} & b'_{22} \\ 0 & 0 & d'_{11} & d'_{12} \\ 0 & 0 & d'_{21} & d'_{22} \end{pmatrix} \in P.$$

Note that

$$D' = \alpha \cdot {}^t(A')^{-1} = \frac{\alpha}{\det A'} \begin{bmatrix} a'_{22} & -a'_{21} \\ -a'_{12} & a'_{11} \end{bmatrix} \quad \text{for some } \alpha \in \mathbf{F}_p^\times. \tag{4.1}$$

The group $K_0(1)$ can be partitioned into the following collections, the first one equal to $K_0(p)$, the second one containing p cosets of $K_0(p)$ in $K_0(1)$, the third one containing p^2 such cosets and the last one containing p^3. They are:

(4.2)

$$\text{collection } \mathcal{G}_1,\ \det C = 0, \quad g = \begin{bmatrix} A' & B' \\ 0_2 & D' \end{bmatrix} \mathfrak{s}_1 = \begin{pmatrix} a'_{11} & a'_{12} & b'_{11} & b'_{12} \\ a'_{21} & a'_{22} & b'_{21} & b'_{22} \\ 0 & 0 & d'_{11} & d'_{12} \\ 0 & 0 & d'_{21} & d'_{22} \end{pmatrix}$$

$$\text{collection } \mathcal{G}_2,\ \det C = 0, \quad g = \begin{bmatrix} A' & B' \\ 0_2 & D' \end{bmatrix} \mathfrak{s}_2 = \begin{pmatrix} a'_{11} & -b'_{12} & b'_{11} & b'_{12}z + a'_{12} \\ a'_{21} & -b'_{22} & b'_{21} & b'_{22}z + a'_{22} \\ 0 & -d'_{12} & d'_{11} & d'_{12}z \\ 0 & -d'_{22} & d'_{21} & d'_{22}z \end{pmatrix}$$

$$\text{collection } \mathcal{G}_3,\ \det C = 0, \quad g = \begin{bmatrix} A' & B' \\ 0_2 & D' \end{bmatrix} \mathfrak{s}_3 = \begin{pmatrix} -b'_{12} & b'_{12}y + a'_{11} & b'_{11}y + b'_{12}z + a'_{12} & b'_{11} \\ -b'_{22} & b'_{22}y + a'_{21} & b'_{21}y + b'_{22}z + a'_{22} & b'_{21} \\ -d'_{12} & d'_{12}y & d'_{11}y + d'_{12}z & d'_{11} \\ -d'_{22} & d'_{22}y & d'_{21}y + d'_{22}z & d'_{21} \end{pmatrix}$$

$$\text{collection } \mathcal{G}_4,\ \det C \neq 0, \quad g = \begin{bmatrix} A' & B' \\ 0_2 & D' \end{bmatrix} \mathfrak{s}_4$$

$$= \begin{pmatrix} -b'_{12} & -b'_{11} & b'_{11}y + b'_{12}z + a'_{12} & b'_{11}x + b'_{12}y + a'_{11} \\ -b'_{22} & -b'_{21} & b'_{21}y + b'_{22}z + a'_{22} & b'_{21}x + b'_{22}y + a'_{21} \\ -d'_{12} & -d'_{11} & d'_{11}y + d'_{12}z & d'_{11}x + d'_{12}y \\ -d'_{22} & -d'_{21} & d'_{21}y + d'_{22}z & d'_{21}x + d'_{22}y \end{pmatrix}.$$

Whenever we use primed variables in the following sections, they will always denote the variables above.

5. The $\mathcal{B}_1$ and $\mathcal{B}_4$ cases

In this and the following two sections we will compute the numbers $n_j(g)$ (cf. section 3 for definition) for $g \in K_0(1)$ and $j = 2, 3, 4$. However, let us record here an easy lemma pertaining to $n_1(g)$. This lemma will allow us to restrict to the case $C \not\equiv 0 \pmod p$ in all future considerations.

Lemma 5.1. *If $g \in K_0(p)$, then $\mathrm{Im}\sigma_g = \mathcal{B}_1$, i.e., $n_1(g) = p^3$ (and thus $n_2(g) = n_3(g) = n_4(g) = 0$).*

Proof. This is straightforward. □

We now deal with the case when $\beta = \mathrm{diag}(1,1,p,p)$, i.e., $\beta \in \mathcal{B}_4$, as this case is simplest.

Proposition 5.2. *Let* $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in K_0(1)$ *with* $A, B, C, D \in \mathrm{Mat}_2(\mathbf{Z}_p)$ *and let* $\sigma_g : \mathcal{B}_1 \to \mathcal{B}$ *be the induced injection. Then* $\mathrm{diag}(1,1,p,p)$ *is in the image of* σ_g *(i.e.,* $n_4(g) = 1$*) if and only if* $\det C \not\equiv 0 \pmod{p}$.

Proof. Let $\begin{bmatrix} p1_2 & E \\ & 1_2 \end{bmatrix} \in \mathcal{B}_1$. Here E is a symmetric matrix with entries in $\mathbf{Z}_p$ and to get all the elements of $\mathcal{B}_1$ we need to run over all E mod p. We would like to show that there exists such a symmetric matrix E and a matrix $\widetilde{g} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \in K_0(1)$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} p1_2 & E \\ & 1_2 \end{bmatrix} = \begin{bmatrix} 1_2 & \\ & p1_2 \end{bmatrix} \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \tag{5.1}$$

if and only if $\det C \not\equiv 0 \pmod{p}$. Reducing the matrices mod p we obtain the following set of equations

$$\begin{aligned} 0_2 &= \widetilde{A} \\ AE + B &= \widetilde{B} \\ CE + D &= 0_2, \end{aligned} \tag{5.2}$$

where all of the equalities (as well as the equalities below) are mod p. Note that if such an E exists, it is necessarily unique.

Suppose first that $\det C \neq 0$. Then $E = -C^{-1}D$ is a solution to (5.2). Here the only thing to check is that $-AC^{-1}D + B$ is invertible, but this follows from the fact that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} AC^{-1} & 1_2 \\ 1_2 & 0_2 \end{bmatrix} \begin{bmatrix} C & D \\ 0_2 & B - AC^{-1}D \end{bmatrix}.$$

Suppose now that $\det C = 0$ and that a unique E exists. Write g as $\alpha\pi$, where $\pi \in P$ and α is one of the coset representatives of G/P as in (4.2). Similarly write $\widetilde{g} = \widetilde{\alpha}\widetilde{\pi}$. Then (5.1) becomes

$$\alpha\pi \begin{bmatrix} 0_2 & E \\ & 1_2 \end{bmatrix} (\widetilde{\pi})^{-1} = \begin{bmatrix} 1_2 & \\ & 0_2 \end{bmatrix} \widetilde{\alpha}. \tag{5.3}$$

Computing the left hand side we get $\alpha \begin{bmatrix} 0_2 & \widetilde{E} \\ 0_2 & X \end{bmatrix}$, where $\widetilde{E}$ is symmetric (and uniquely determined by E) and $X \in \mathrm{GL}_2(\mathbf{F}_p)$. Now redefine $A, B, C, D, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ according to:

$$\alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad \widetilde{\alpha} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix}.$$

Then (5.3) translates to the following system of equations

$$\begin{aligned} 0_2 =& \widetilde{A} \\ A\widetilde{E} + BX =& \widetilde{B} \\ C\widetilde{E} + DX =& 0_2. \end{aligned} \tag{5.4}$$

The last equality implies that if $\det C = 0$, then $\det D = 0$.

Multiplying the matrices in Lemma 4.2 we see that both $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix}$ belong to the set

$$\left\{ 1_4, \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & z \end{bmatrix}, \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & y & 1 \\ -1 & y & z & \end{bmatrix}, \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & y & x \\ -1 & & z & y \end{bmatrix} \right\}.$$

Since $\det C = 0$ and $\det D = 0$, we see that the only possibilities for α are the second matrix (with $z = 0$) or the third one (with $z = 0$ and arbitrary y). However, in both cases it follows from the third equality in (5.4) that X cannot be invertible, which leads to a contradiction. $\square$

6. The $\mathcal{B}_3$ case

Proposition 6.1. *Let*

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{12} & d_{22} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{bmatrix} \in K_0(1) - K_0(p).$$

As above let $n_j = n_j(g)$ denote the number of elements of $\mathcal{B}_j$ in the image of σ_g. Let C_2 denote the second row of C. Below the conditions on the entries of C are to be taken modulo p.

1. *If $\det C \neq 0$, then $n_3 = p - 1$.*
2. *If $\det C = 0$ and $C_2 \neq [0, 0]$, then $n_3 = p$.*
3. *Otherwise $n_3 = 0$.*

Proof. We must calculate for how many values of $x \in \mathbf{F}_p$ there exists a symmetric $E \in \mathrm{Mat}_2(\mathbf{F}_p)$ and $\widetilde{g} \in K_0(1)$ such that

$$g \begin{bmatrix} p1_2 & E \\ & 1_2 \end{bmatrix} = \begin{bmatrix} p & & x & \\ & 1 & & \\ & & 1 & \\ & & & p \end{bmatrix} \widetilde{g}.$$

Reducing modulo p and eliminating the primed variables we see that such a pair $\widetilde{g}, E$ exists if and only if one has

$$\begin{aligned} \text{first row of } AE+B &= x\cdot(\text{first row of } CE+D);\\ \text{second row of } CE+D &= [0,0], \end{aligned} \tag{6.1}$$

(where this equality and all subsequent ones are modulo p) which in matrix form can be re-written as

$$\mathcal{Y}\mathcal{E} := \begin{bmatrix} a_{11}-xc_{11} & a_{12}-xc_{12} & 0 \\ 0 & a_{11}-xc_{11} & a_{12}-xc_{12} \\ c_{21} & c_{22} & 0 \\ 0 & c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \\ e_{22} \end{bmatrix} = \mathcal{D} := \begin{bmatrix} xd_{11}-b_{11} \\ xd_{12}-b_{12} \\ -d_{21} \\ -d_{22} \end{bmatrix}.$$

We now use Gaussian elimination to study the existence of solutions to the above equation. The matrix $\mathcal{Y}$ can be reduced to $\operatorname{diag}(1,1,\mathcal{Y}_1,0)$ with $\mathcal{Y}_1 = (\det C)x - \det F$ where $F = \begin{bmatrix} a_{11} & a_{12} \\ c_{21} & c_{22} \end{bmatrix}$. One can quickly see that no solution exists unless $\mathcal{Y}_1 \neq 0$, in which case $\mathcal{Y}$ and $[\mathcal{Y}|\mathcal{D}]$ both have rank 3 so there is a unique solution $\mathcal{E}$. If $\det C \neq 0$, this inequality is satisfied for exactly $p-1$ values of x. Let $\det C = 0$. Then we see $\mathcal{Y}_1 \neq 0$ for each possible value of x if and only if $\det F \neq 0$. Clearly we have if $C_2 = [0,0]$ then $\det F = 0$. Assume $c_{21} \neq 0$. This forces g to be in collection $\mathcal{G}_3$ and so $\det F = \alpha(d'_{22})^2/\det D' = \alpha c_{21}^2/\det D'$ for some $\alpha \neq 0$, and so $\det F \neq 0$. (Note that when primed variables are used these are as in the collections given in (4.2).) Thus, if $\det C = 0$ and $c_{21} \neq 0$ we have $n_3 = p$. Now suppose $c_{21} = 0$ and so $\det F = a_{11}c_{22}$. If $c_{22} \neq 0$ then g must be in collection $\mathcal{G}_2$. Using this we see that $a_{11} = -(\det A')c_{22}/\alpha$ for some nonzero α and so again we have $\det F \neq 0$. Thus, if $\det C = 0$ and $c_{22} \neq 0$ we have $n_3 = p$.

Finally, if $\det C = \det F = 0$ then $n_3 = 0$. □

7. The $\mathcal{B}_2$ case

Proposition 7.1. *Let*

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{12} & d_{22} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{bmatrix} \in K_0(1) - K_0(p).$$

As above let $n_j = n_j(g)$ denote the number of elements of $\mathcal{B}_j$ in the image of σ_g. Let C_2 denote the second row of C.

(i) *If $C_2 \neq [0,0]$, then $n_2 = p(p-1)$.*
(ii) *If $C_2 = 0$ but $C \neq 0$, then $n_2 = p^2$.*
(iii) *Otherwise $n_2 = 0$.*

Proof. Let $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in K_0(1)$. We need to calculate for how many values of $x, z \in \mathbf{F}_p$ there exists a symmetric $E \in \operatorname{Mat}_2(\mathbf{F}_p)$ and $\widetilde{g} \in K_0(1)$ such that

$$g\begin{bmatrix} p1_2 & E \\ & 1_2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ x & p & & -z \\ & & p & -x \\ & & & 1 \end{bmatrix} \widetilde{g}.$$

As in the previous section this leads to a matrix equation, which in this case is of the form:

$$\mathcal{Y}\mathcal{E} := \begin{bmatrix} c_{11}+xc_{21} & c_{12}+xc_{22} & 0 \\ 0 & c_{11}+xc_{21} & c_{12}+xc_{22} \\ a_{21}-xa_{11}+zc_{21} & a_{22}-xa_{12}+zc_{22} & 0 \\ 0 & a_{21}-xa_{11}+zc_{21} & a_{22}-xa_{12}+zc_{22} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \\ e_{22} \end{bmatrix}$$

$$= \mathcal{D} := \begin{bmatrix} -d_{11}-xd_{21} \\ -d_{12}-xd_{22} \\ -b_{21}+xb_{11}-zd_{21} \\ -b_{22}+xb_{12}-zd_{22} \end{bmatrix}.$$

Row-reducing the matrix $[\mathcal{Y}|\mathcal{D}]$ and assuming $c_{11}+xc_{21} \neq 0$ we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathcal{Y}_1 & \mathcal{Y}_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{7.1}$$

where

$$\mathcal{Y}_1 = -a_{12}x + c_{22}z + \frac{(c_{22}x+c_{12})(a_{11}x-c_{21}z-a_{21})}{c_{21}x+c_{11}} + a_{22} \tag{7.2}$$

and we note the bottom right entry is 0 due to g being symplectic. As before, there is a solution $\mathcal{E}$ if and only if either $\mathcal{Y}_1 \neq 0$ or $\mathcal{Y}_1 = \mathcal{Y}_2 = 0$, but as before we do not need to consider the latter. On the other hand if $c_{21}x + c_{11} = 0$, when we apply Gaussian elimination to $[\mathcal{Y}|\mathcal{D}]$ we obtain the same form as in (7.1) but with $\mathcal{Y}_1$ replaced by

$$\mathcal{Y}_3 := (c_{21}x+c_{11})\mathcal{Y}_1|_{c_{21}x+c_{11}=0} = (c_{22}x+c_{12})(a_{11}x-c_{21}z-a_{21}). \tag{7.3}$$

In this case again we only need to consider the case $\mathcal{Y}_3 \neq 0$. Hence, in both cases (which we will now treat simultaneously) we can consider the inequality

$$(c_{21}x+c_{11})\mathcal{Y}_1 \neq 0.$$

7.1. **Suppose first that $c_{21} \neq 0$.** We have

$$(c_{21}x+c_{11})\mathcal{Y}_1 = \alpha(x) + (\det C)z, \tag{7.4}$$

where

$$\alpha(x) = (\det F)x^2 + (\det F' - \det F'')x - \det F''' \tag{7.5}$$

and $F = \begin{bmatrix} a_{11} & a_{12} \\ c_{21} & c_{22} \end{bmatrix}$, $F' := \begin{bmatrix} a_{11} & a_{12} \\ c_{11} & c_{12} \end{bmatrix}$, $F'' := \begin{bmatrix} a_{21} & a_{22} \\ c_{21} & c_{22} \end{bmatrix}$, $F''' := \begin{bmatrix} a_{21} & a_{22} \\ c_{11} & c_{12} \end{bmatrix}$.
If $x = -\frac{c_{11}}{c_{21}}$, i.e., $c_{21}x + c_{11} = 0$, then

$$\begin{aligned} \alpha\left(-\frac{c_{11}}{c_{21}}\right) &= \frac{\det C}{c_{21}}\left(a_{11}\frac{c_{11}}{c_{21}} + a_{21}\right); \\ \mathcal{Y}_3 = (c_{21}x + c_{11})\mathcal{Y}_1|_{x=-\frac{c_{11}}{c_{21}}} &= \alpha\left(-\frac{c_{11}}{c_{21}}\right) + (\det C)z \end{aligned} \tag{7.6}$$

7.1.1. *Suppose that* $\det C \neq 0$. The inequality $c_{21}x + c_{11} \neq 0$ has exactly $p-1$ solutions in x. Fix such a solution. Then for such a fixed x, $\mathcal{Y}_1 \neq 0$ has exactly $p-1$ solutions in z by (7.4). Also, since $\det C \neq 0$, there are $p-1$ values of z that make $\mathcal{Y}_3 \neq 0$ for $x = -\frac{c_{11}}{c_{21}}$ by (7.6). Hence we conclude that if $c_{21} \neq 0$ and $\det C \neq 0$, then there are exactly $p-1$ values of x and for each such x there are exactly $p-1$ values of z such that both $c_{21}x + c_{11} \neq 0$ and $\mathcal{Y}_1 \neq 0$ hold. Also, there is exactly one value of x and $p-1$ values of z such that both $c_{21}x + c_{11} = 0$ and $\mathcal{Y}_3 \neq 0$ hold. Altogether this gives $(p-1)^2 + p - 1 = p(p-1)$ matrices, hence $n_2 = p(p-1)$ in this case.

7.1.2. *Suppose that* $\det C = 0$. If $\alpha(x) \neq 0$, then for such x the inequality $\mathcal{Y}_j \neq 0$ (for $j = 1, 3$) has p solutions in z (from (7.4) or (7.6)). If $\alpha(x) = 0$, then for such x we get that $\mathcal{Y}_j = 0$ ($j = 1, 3$) for all values of z and for every fixed value of z we would get p solutions E, so this is impossible. Let us now see for what values of x we have $\alpha(x) \neq 0$. Note that the conditions $c_{21} \neq 0$ and $\det C = 0$ force g to be in collection $\mathcal{G}_3$, and hence $c_{21} = -d'_{22}$. Thus $\det F = a'_{11}d'_{22} = \alpha(d'_{22})^2/\det D'$ for some $\alpha \neq 0$ and hence $\det F \neq 0$. We compute the discriminant Δ of $\alpha(x)$ and get $\Delta = (\det F' + \det F'')^2$. Again using the fact that g is in collection $\mathcal{G}_3$, a quick calculation shows that $\Delta = 0$ and thus $x = -\frac{c_{11}}{c_{21}}$ is the unique solution to $\alpha(x) = 0$, i.e., there are $p-1$ values of x for which $\alpha(x) \neq 0$. Hence we have shown that whenever $c_{21} \neq 0$, then $n_2 = p(p-1)$.

7.2. **Suppose that** $c_{21} = 0$ **and** $c_{11} \neq 0$**.** Then the inequality $c_{21}x + c_{11} \neq 0$ always holds, i.e., has p solutions in x. The analysis is exactly the same as before. We need to look for solutions in z to the inequality

$$c_{11}\mathcal{Y}_1 = \alpha(x)|_{c_{21}=0} + c_{11}c_{22}z \neq 0.$$

If $c_{22} \neq 0$, this has $p-1$ solutions in z. This proves (i) except in the case when $c_{21} = c_{11} = 0$ and $c_{22} \neq 0$. We will analyze this case below.

If $c_{22} = 0$ and $\alpha(x) \neq 0$, we get p solutions in z and if $c_{22} = \alpha(x) = 0$, we get no solutions in z. Let us see when $\alpha(x) \neq 0$ under the assumption that $c_{22} = 0$. From (7.5) we get

$$\alpha(x) = (\det F')x - \det F'''.$$

The conditions $\det C = 0$ and $c_{11} \neq 0$ alone force g to be in collection $\mathcal{G}_3$, so $\det F' = d'_{22}a'_{11} = -c_{21}a'_{11} = 0$. On the other hand for a matrix in collection $\mathcal{G}_3$, we have $\det F''' = d'_{12}a'_{21} = \alpha d'_{12} = -\alpha c_{11}$ for a non-zero α, hence $\det F''' \neq 0$. Thus we get that $\alpha(x) \neq 0$ has p solutions in x and for each such x we get p solutions in z.

7.3. **Suppose** $c_{21} = c_{11} = 0$**.** First suppose $C \neq 0$. For all x we have $c_{21}x + c_{11} = 0$. Hence we must consider $\mathcal{Y}_3 \neq 0$ and in fact we get

$$\mathcal{Y}_3 = c_{22}a_{11}x^2 + (-c_{22}a_{21} + c_{12}a_{11})x - a_{21}c_{12}. \tag{7.7}$$

Note that $\mathcal{Y}_3$ does not depend on z this time. We examine the solutions in x to $\mathcal{Y}_3 \neq 0$.

Suppose first that $c_{22} \neq 0$. Since $c_{21} = c_{11} = 0$ and $C \neq 0$ force g to be in collection $\mathcal{G}_2$, this implies that $a_{11} \neq 0$ (since $a_{11} = a'_{11} = \alpha d'_{22} = -\alpha c_{22}$ for nonzero α). Then (7.7) is quadratic with discriminant $(c_{22}a_{21} + c_{12}a_{11})^2$, which must be zero again by virtue of g being in collection $\mathcal{G}_2$. Since we want $\mathcal{Y}_3 \neq 0$, this gives us $p-1$ solutions in x. This finishes the proof of (i).

Suppose now that $c_{22} = 0$ but still $C \neq 0$. This implies $c_{12} \neq 0$. Then

$$\mathcal{Y}_3 = c_{12}(a_{11}x - a_{21}).$$

The conditions $c_{21} = c_{11} = 0$, but $C \neq 0$ imply that g is in collection $\mathcal{G}_2$. Since we also assume that $c_{22} = 0$, we must have $d'_{22} = 0$ and thus by (4.1) we must have $a'_{11} = 0$. But since in collection $\mathcal{G}_2$ one has $a'_{11} = a_{11}$, we must have $a'_{21} \neq 0$ since A' is invertible and $a_{21} = a'_{21}$. Hence the inequality $\mathcal{Y}_3 \neq 0$ has p solutions in x (and for each of them p solutions in z).

Finally if $C = 0$, then $n_2 = 0$ by Lemma 5.1. □

8. The volumes of K_s

Recall that for any $s = (s_1, s_2, s_3, s_4) \in \mathbf{Z}^4$ we defined $K_s = \{g \in K_0(1) \mid n_j(g) = s_j \text{ for all } j = 1, 2, 3, 4\}$. The following result gives volumes of K_s for all possible s.

Proposition 8.1. *Recall that* $\#\mathcal{B} = p^3 + p^2 + p + 1$. *The only values of* $s \in \mathbf{Z}^4$ *for which one has* $\operatorname{vol} K_s \neq 0$ *are listed in Table 1.*

Table 1. Values of s and volumes of K_s

Value of s	*Volume of* K_s
$(p^3 - p^2, p^2 - p, p - 1, 1)$	$p^3/\#\mathcal{B}$
$(p^3 - p^2, p^2 - p, p, 0)$	$p^2/\#\mathcal{B}$
$(p^3 - p^2, p^2, 0, 0)$	$p/\#\mathcal{B}$
$(p^3, 0, 0, 0)$	$1/\#\mathcal{B}$

Proof. Propositions 5.2, 6.1 and 7.1 give conditions (modulo p) that $g \in K_0(1)$ needs to satisfy so that one obtains particular values of $n_4(g)$, $n_3(g)$ and $n_2(g)$ respectively (note that one has $n_1(g) = p^3 - n_2(g) - n_3(g) - n_4(g)$). So, the proof is just an elementary count how many mod p residue classes of matrices in $K_0(1)$ satisfy these conditions. The results of that count are summarized in Table 2 (recall that C_2 denotes the second row of the matrix

C and $\#P = p^4(p-1)^3(p+1)$). The volumes in Table 1 follow immediately

TABLE 2. Cases, values of s and numbers of matrices

Cases	Value of s	Matrix count
$\det C \neq 0$	$(p^3-p^2, p^2-p, p-1, 1)$	$p^3\#P$
$\det C = 0, C_2 \neq 0$	$(p^3-p^2, p^2-p, p, 0)$	$p^2\#P$
$C_2 = 0, C \neq 0$	$(p^3-p^2, p^2, 0, 0)$	$p\#P$
$C = 0$	$(p^3, 0, 0, 0)$	$\#P$

from the matrix count in Table 2 and the fact that $\operatorname{vol} K_0(1) = 1$, so it is enough to show how we obtain the matrix count in Table 2. If we are in case (i) of Proposition 6.1, i.e., $\det C \neq 0$, then we clearly must be in case (i) of Proposition 7.1, i.e., $C_2 \neq 0$. These two conditions are equivalent to g being in collection $\mathcal{G}_3$, which has $p^3\#P$ elements when counted mod p. Hence (using also Proposition 5.2) we obtain the first line in Table 2. Next, if we are in case (ii) of Proposition 6.1, i.e., $\det C = 0$, but $C_2 \neq 0$, then we must be in case (ii) of Proposition 7.1. Then $g \in \mathcal{G}_2 \cup \mathcal{G}_3$ and g must satisfy that $d'_{22} \neq 0$, so using (4.1) we are counting the number of pairs (α, D'), where $\alpha \in \mathbf{F}_p^\times$ and $D' \in \mathrm{GL}_2(\mathbf{F}_p)$ is a matrix that lies outside the Borel of $\mathrm{GL}_2(\mathbf{F}_p)$ given by the condition $d'_{22} = 0$. There are $(p-1)(p(p-1)^2(p+1) - p(p-1)^2) = p^2(p-1)^3$ such matrices. For each such pair (α, D') the matrix A' is completely determined, but we get p^3 choices of B'. We then multiply the number $p^5(p-1)^3$ by the number of coset representatives of $K_0(p)$ which lie in collection $\mathcal{G}_2$ (this number equals p) or in $\mathcal{G}_3$ (this number equals p^2). Thus we get $p^6(p-1)^3(p+1) = p^2\#P$ matrices. This gives line 2 in Table 2. Now, if we are in case (iii) of Proposition 6.1, i.e., $C_2 = 0$, then we may be in case (ii) or case (iii) of Proposition 7.1. In the latter case, g must be in collection $\mathcal{G}_1$ and we obtain the last line in the table. So, suppose that $C_2 = 0$, but $C \neq 0$. Then $g \in \mathcal{G}_2 \cup \mathcal{G}_3$ and $d'_{22} = 0$. Thus, the analysis is exactly the same as for line 2 of the Table, but we need to replace the number of matrices in $\mathrm{GL}_2(\mathbf{F}_p)$ that lie outside a given Borel with the cardinality of a Borel in $\mathrm{GL}_2(\mathbf{F}_p)$. This amounts to dividing the count in the case of line 2 by p and thus we get line 3 in the Table. □

9. Calculation of the Petersson norm of $U_p\phi$

We now use the calculations from Table 1 to compute the Petersson norm of $U_p\phi$ using the formula (3.2) which we repeat here

$$\langle U_p\phi, U_p\phi\rangle_{K_0(1)} = |\sigma(p)|^2 \sum_{s\in\mathbf{Z}^4} \operatorname{vol}(K_s) \sum_{i,j=1}^{4} s_i s_j \gamma_i \overline{\gamma_j}, \tag{9.1}$$

where

$$\gamma_j = \begin{cases} p^{-3/2} & j = 1 \\ p^{-1/2}\chi_2(p) & j = 2 \\ p^{1/2}\chi_1(p) & j = 3 \\ p^{3/2}\chi_1(p)\chi_2(p) & j = 4. \end{cases}$$

Using (9.1) and Proposition 8.1 we get

$$\begin{aligned} \frac{\#\mathcal{B}\,\langle U_p\phi, U_p\phi\rangle_{K_0(1)}}{|\sigma(p)|^2} =& p^2(p^4 - p^3 + 1) \\ &+|\chi_2(p)|^2 p^3(p^3 - p^2 + 1) \\ &+|\chi_1(p)|^2 p^4(p^2 - p + 1) \\ &+|\chi_1(p)|^2|\chi_2(p)|^2 p^6 \\ &+(\chi_2(p) + \overline{\chi_2(p)})p^5(p-1) \\ &+(\chi_1(p) + \overline{\chi_1(p)})p^5(p-1) \\ &+(\chi_1(p)\chi_2(p) + \overline{\chi_1(p)\chi_2(p)})p^5(p-1) \\ &+(\chi_1(p)\overline{\chi_2(p)} + \overline{\chi_1(p)}\chi_2(p))p^5(p-1) \\ &+|\chi_2(p)|^2(\chi_1(p) + \overline{\chi_1(p)})p^5(p-1) \\ &+|\chi_1(p)|^2(\chi_2(p) + \overline{\chi_2(p)})p^5(p-1). \end{aligned} \tag{9.2}$$

Using the fact that $\overline{\chi}_j = |\chi_j|^2\chi_j^{-1}$ and that $|\chi_j(p)|^2 = p^{t_j}$ for some real number t_j we obtain the following result.

Theorem 9.1. *Let χ_1, χ_2, σ be unramified characters of $\mathbf{G}_m(\mathbf{Q}_p)$. Let ϕ be the normalized spherical vector in the representation $\chi_1 \times \chi_2 \rtimes \sigma$. Then one has*

(9.3)

$$\begin{aligned} \frac{\#\mathcal{B}\,\langle U_p\phi, U_p\phi\rangle_{K_0(1)}}{\mathrm{vol}(K_0(1))^2|\sigma(p)|^2} =& p^2 + p^{t_2+3} + p^{t_1+4} + p^{t_1+t_2+5} \\ &+p^5(p-1)(1+\chi_1(p))(1+p^{t_1}\chi_1(p)^{-1})(1+\chi_2(p))(1+p^{t_2}\chi_2(p)^{-1}). \end{aligned}$$

Note that in the case when $t_1 = t_2 = 0$ (case (T) below), and the case when $t_1 = -1$ and $t_2 = 1$ (the SK case below) the constant term just gives $p^2\#\mathcal{B}$.

Let us quote the following result from [16].

Proposition 9.2. *Let N and k be positive integers with $k > 2$. Let $F \in S_k(K_0(N))$ be a Siegel Hecke eigenform on $\mathrm{GSp}_4(\mathbf{A_Q})$. For $p \nmid N$ let $\pi_{F,p}$ be the corresponding local representation of $\mathrm{GSp}_4(\mathbf{Q}_p)$. Then $\pi_{F,p}$ can only be one of the following:*

(T) *$\chi_1 \times \chi_2 \rtimes \sigma$ irreducible with $|\chi_1| = |\chi_2| = |\sigma| = 1$ (the tempered case); or*

(C) *$\chi_1 \times \chi_2 \rtimes \sigma$ irreducible with $\chi_1 = |\cdot|_p^{\beta}\chi$, $\chi_2 = |\cdot|_p^{\beta}\chi^{-1}$, $|\chi| = 1$, $e(\sigma) = \beta$ with $0 < \beta < 1/2$ (the complementary series case); or*

(SK) *the spherical constituent of $|\cdot|_p^{1/2}\chi \times |\cdot|_p^{-1/2}\chi \rtimes \sigma$, with $|\chi| = 1$ (the Saito-Kurokawa case).*

The characters χ_1, χ_2, χ and σ above are unramified.

Now recall that for $\Pi = \chi_1 \times \chi_2 \rtimes \sigma$ we have the standard local L-function of Π is defined by

$$L(s, \Pi, \mathrm{St}) = (1 - p^{-s})^{-1} \prod_{i=1}^{2} (1 - \chi_i(p)p^{-s})^{-1}(1 - \chi_i^{-1}(p)p^{-s})^{-1}.$$

Let $E/\mathbf{Q}_p$ be the unique quadratic unramified extension. Recall the local L-function of the base change $\mathrm{BC}(\pi(\chi_1, \chi_2))$ is defined by

$$L(s, \mathrm{BC}(\pi(\chi_1, \chi_2))) = (1 - \chi_1(p)^2p^{-2s})^{-1}(1 - \chi_2(p)^2p^{-2s})^{-1}.$$

Given a character ψ, we write $L(s, \psi) = (1 - \psi(p)p^{-s})^{-1}$. Using Proposition 9.2 above we obtain the following corollary to Theorem 9.1.

Corollary 9.3. *With the notation from Proposition 9.2 and one has:*

$$\frac{\#\mathcal{B}\, \langle U_p\phi, U_p\phi\rangle_{K_0(1)}}{|\sigma(p)|^2} = p^2 + p^{t_2+3} + p^{t_1+4} + p^{t_1+t_2+5} + p^5(p-1)X,$$

where

$$X = \begin{cases} \frac{L(0,\Pi,\mathrm{St})}{\zeta(0)L(0,\mathrm{BC}(\pi(\chi_1,\chi_2)))L(0,\mathrm{BC}(\pi(\chi_1^{-1},\chi_2^{-1})))} & \text{(T) and (SK)} \\ \frac{L(0,\mathrm{BC}(\pi(\chi_1,\chi_2))L(-3\beta/2,\mathrm{BC}(\pi(\chi_1,\chi_2)))}{L(0,\chi_1)L(0,\chi_2)L(-3\beta/4,\chi_1)L(-3\beta/4,\chi_2)} & \text{(C)} \end{cases}$$

and

$$(t_1, t_2) = \begin{cases} (0,0) & \text{(T)} \\ (-2\beta, -2\beta) & \text{(C)} \\ (-1, 1) & \text{(SK).} \end{cases}$$

We note here that $\zeta(0)$ is the pth Euler factor of the Riemann zeta function at $s = 0$. While this term is undefined on its own, since we only consider the ratio $L(0, \Pi, \mathrm{St})/\zeta(0)$, this makes sense.

Proof. From Theorem 9.1 and Proposition 9.2 we immediately obtain

$$\frac{\#\mathcal{B}\, \langle U_p\phi, U_p\phi\rangle_{K_0(1)}}{|\sigma(p)|^2} = p^2 + p^{t_2+3} + p^{t_1+4} + p^{t_1+t_2+5} + p^5(p-1)X,$$

with
(9.4)

$$X = \begin{cases} (1+\chi_1(p))(1+\chi_1(p)^{-1})(1+\chi_2(p))(1+\chi_2(p)^{-1}) & \text{(T)} \\ (1+p^{-\beta}\chi(p))^2(1+p^{-\beta}\chi(p)^{-1})^2 & \text{(C)} \\ (1+p^{-1/2}\chi(p))(1+p^{-1/2}\chi(p)^{-1})(1+p^{1/2}\chi(p))(1+p^{1/2}\chi(p)^{-1}) & \text{(SK)} \end{cases}$$

with the values of t_i listed in the statement of the corollary. The value of X is now obtained by elementary calculations. □

Remark 9.4. By a result of Weissauer [22] which proves the Ramanujan Conjecture for (non-CAP cuspidal) Siegel modular forms we know that the complimentary case does not occur. Hence using that in the (T) and (SK) case we have $p^2 + p^{t_2+3} + p^{t_1+4} + p^{t_1+t_2+5} = p^2\#\mathcal{B}$ we obtain Theorem 1.1 in the Introduction.

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Department of Mathematical Sciences, Clemson University, Clemson, SC 29634
E-mail address: jimlb@g.clemson.edu

Department of Mathematics, Queens College, City University of New York, Flushing, NY 11367
E-mail address: klosin@qc.cuny.edu