AN INNER PRODUCT RELATION ON SAITO-KUROKAWA LIFTS

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ABSTRACT. Let f be a newform of weight $2k-2$ and level M with M an odd square-free integer. Via the Saito-Kurokawa correspondence there is associated to f a Siegel newform F_f of weight k and level M. In this paper we provide a formula relating the Petersson products $\langle F_f, F_f\rangle$ and $\langle f, f\rangle$. We use this result to give a new proof of a special case of a well-known result of Shimura on the algebraicity of a special value of a Rankin convolution L-function.

1. Introduction

Let k be a positive integer. Based on numerical evidence, H. Saito and N. Kurokawa conjectured that there exists a map from the space of classical cuspidal eigenforms of weight $2k-2$ and level 1 to the space of cuspidal Siegel eigenforms of even weight k and level 1. This conjecture was proven in a series of papers by Maass ([17]-[19]), Andrianov ([1]), and Zagier ([26]). This result was generalized to odd square free levels by M. Manickham, B. Ramakrishnan, and T. C. Vasudevan ([21]) and then to arbitrary level by M. Manickham and B. Ramakrishnan ([23]). This correspondence is known in the language of automorphic forms via the work of Piatetski-Shapiro ([24]).

We view the Saito-Kurokawa correspondence as a series of isomorphisms. The first of these isomorphisms relates classical newforms of weight $2k-2$ on $\Gamma_0(M)$ to newforms of weight $k-1/2$ in Kohnen's $+$-space on $\Gamma_0(4M)$. The second isomorphism relates the half-integer weight newforms to Jacobi newforms of weight k and index 1 on the space $\Gamma_0^{\mathrm{J}}(M)$. Finally, one has an isomorphism between the space of Jacobi newforms to Siegel newforms of weight k in the "Maass spezialschar" on the space $\Gamma_0^4(M)$. One should note here that when the term "newforms" is used in relation to Siegel eigenforms we mean newforms as defined in [21]. Using these isomorphisms we calculate a relation between the inner products of the related forms at each stage. Combining the formulas we obtain we have the following theorem.

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Theorem 1.1. *Let $M = p_1 \dots p_n$ with the p_i odd distinct primes, $f \in S_{2k-2}^{new}(\Gamma_0(M))$ a newform, and $F_f \in \mathcal{S}_k^{*,new}(\Gamma_0^4(M))$ the Siegel modular form associated to f via the Saito-Kurokawa correspondence. Let D be a fundamental discriminant with $(-1)^{k-1}D > 0$, $\gcd(M, D) = 1$, and $c_g(|D|) \neq 0$ where the c_g are the Fourier coefficients of the half-integral weight modular form associated to f via the Saito-Kurokawa correspondence. Then one has*

$$\langle F_f, F_f \rangle = \mathcal{B}_{k,M} \frac{|c_g(|D|)|^2 \, L(k, f)}{\pi \, |D|^{k-3/2} \, L(k-1, f, \chi_D)} \langle f, f \rangle \tag{1}$$

where

$$\mathcal{B}_{k,M} = \frac{M^k \, (k-1) \prod_{i=1}^n \left(p_i^{2m_i - 2}(p_i^4 + 1) \right)}{2^{\nu(M)+3} \, 3 \, [\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^4(M)] \, [\Gamma_0(M) : \Gamma_0(4M)]}.$$

The case $M = 1$ of Theorem 1.1 has essentially been shown in ([7], Theorem 1) and ([14], Corl. 2). In these papers the result is given in terms of algebraicity of the ratio of the inner products. All of the main ingredients of the formula for $M = 1$ are provided in [15] without gathering them together into a single formula.

Note that in light of [23] this theorem can be extended to arbitrary odd levels, but we do not require such a result for future applications so restrict ourselves here to the case of odd square-free level for ease of exposition.

We conclude the paper with a simple proof of the algebraicity of a Rankin convolution L-function. Let k be even, $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ a normalized eigenform, and $h \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ a normalized eigenform. Associated to f and h is a Rankin L-function $\mathcal{D}(s, f, h)$ (see Section 7 for the definition.) Associated to f and h are complex periods that allow one to normalize the L-functions associated to f and h so that the special values of these L-functions are algebraic. Shimura proved in [25] that one has

$$\frac{\mathcal{D}(m, f, h)}{\pi^{2m+2-k} \langle f, f \rangle} \in \overline{\mathbb{Q}}$$

for $k \leq m \leq 2k-3$. Using the formula in Theorem 1.1 and a result of Heim we are able to give a simple proof of a particular case of this result, namely, the fact that

$$\frac{\mathcal{D}(2k-3, f, h)}{\pi^{3k-4} \langle f, f \rangle} \in \overline{\mathbb{Q}}.$$

Though the formula given in the theorem is interesting in its own right and does yield a new proof of the algebraicity of $\mathcal{D}(2k-3, f, h)$, our main motivation for studying such a relation is the desire to produce congruences between the eigenvalues of Saito-Kurokawa lifts and the eigenvalues of Siegel eigenforms that do not arise as Saito-Kurokawa lifts. One can see how such a formula is used to accomplish this in the level 1 case as well as applications to the non-vanishing of Selmer groups in [3]. The formula established above will be used in a subsequent paper to produce a similar congruence in the case of square-free odd level. It is the author's hope to investigate similar relationships for the correspondence established by Ikeda ([10]) between elliptic cusp forms and Siegel cusp forms of genus n in future work. An algebraicity result in this direction has been shown in [4], though a specific formula has yet to be worked out.

The author would like to thank the referee for pointing out the references [4], [7], and [14] of which the author was previously unaware. The author would also like to thank Ameya Pitale for pointing out an error in the statement of Theorem 7.3 that appeared in the printed version of this article.

2. Notation and Definitions

In this section we fix notation and definitions that will be used throughout the paper.

For a ring R, we let $\mathrm{M}_n(R)$ denote the set of n by n matrices with entries in R. Given a matrix $x \in \mathrm{M}_{2n}(R)$, we write

$$x = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix}$$

where a_x, b_x, c_x and d_x are all in $\mathrm{M}_n(R)$; dropping the subscript x when it is clear from the context. The groups $\mathrm{GL}_n(R)$ and $\mathrm{SL}_n(R)$ have their standard definition here. For a positive integer M we recall that the Hecke congruence subgroup of level M is defined by

$$\Gamma_0(M) = \{x \in \mathrm{SL}_2(\mathbb{Z}) : c_x \equiv 0 (\operatorname{mod} M)\}.$$

We let $\Gamma_0^{\mathrm{J}}(M) = \Gamma_0(M) \ltimes \mathbb{Z}^2$ be the Hecke-Jacobi modular group of level M as defined in [6]. Define $\mathrm{GSp}_4(\mathbb{R})$ by

$$\mathrm{GSp}_4(\mathbb{R}) = \{\gamma \in \mathrm{GL}_4(\mathbb{R}) : {}^t\gamma \iota_2 \gamma = \mu(\gamma)\iota_2 \text{ with } \mu(\gamma) \in \mathbb{R}^*\}$$

where $\iota_2 = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}$. Recall that the symplectic group $\mathrm{Sp}_4(\mathbb{R})$ is defined to be the subgroup of $\mathrm{GSp}_4(\mathbb{R})$ obtained when one requires $\mu = 1$. The Siegel-Hecke congruence subgroup of level M is defined by

$$\Gamma_0^4(M) = \{\gamma \in \mathrm{Sp}_4(\mathbb{Z}) : c_\gamma \equiv 0 (\operatorname{mod} M)\}$$

where the congruence is a congruence on the entries of the matrix c_γ.

We write $\mathfrak{h}^1$ to denote the complex upper half-plane. The group $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathfrak{h}^1$ via linear fractional transformations. The Siegel upper half-space is defined by

$$\mathfrak{h}^2 = \{Z \in \mathrm{M}_2(\mathbb{C}) : {}^tZ = Z, \, \mathrm{Im}(Z) > 0\}.$$

Siegel upper half-space comes equipped with an action of $\mathrm{Sp}_4(\mathbb{R})$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ+B)(CZ+D)^{-1}.$$

For positive integers k and M we write $M_k(\Gamma_0(M))$ to denote the space of modular forms of weight k on the congruence subgroup $\Gamma_0(M)$. For $f \in M_k(\Gamma_0(M))$, we denote the n^{th} Fourier coefficient of f by $a_f(n)$. We let $S_k(\Gamma_0(M))$ denote the space of cusp forms and $S_k^{\mathrm{new}}(\Gamma_0(M))$ the space of newforms. For $f_1, f_2 \in M_k(\Gamma_0(M))$ with f_1 or f_2 a cusp form, the Petersson product of f_1 and f_2 is given by

$$\langle f_1, f_2 \rangle = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(M)]} \int_{\Gamma_0(M)\backslash \mathfrak{h}^1} f_1(z)\overline{f_2(z)} y^{k-2} dx\, dy.$$

For $f \in S_k(\Gamma_0(M))$, one has the associated L-function defined by

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}.$$

The only half-integral weight modular forms we are interested in are the ones in Kohnen's $+$-space defined by

$$S_{k-1/2}^+(\Gamma_0(4M)) = \{g \in S_{k-1/2}(\Gamma_0(4M)) : \, a_g(n) = 0 \text{ if } (-1)^{k-1}n \equiv 2,3 (\mathrm{mod}\, 4)\}.$$

The Petersson product on $S_{k-1/2}^+(\Gamma_0(4M))$ is given by

$$\langle g_1, g_2 \rangle = \frac{1}{[\Gamma_0(4) : \Gamma_0(4M)]} \int_{\Gamma_0(4M)\backslash \mathfrak{h}^1} g_1(z)\overline{g_2(z)} y^{k-5/2} dx\, dy.$$

We denote the space of Jacobi cusp forms on $\Gamma_0^{\mathrm{J}}(M)$ by $J_{k,1}^{\mathrm{cusp}}(\Gamma_0^{\mathrm{J}}(M))$. The Petersson product is given by

$$\langle \phi_1, \phi_2 \rangle = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(M)]} \int_{\Gamma_0^{\mathrm{J}}(M)\backslash \mathfrak{h}^1 \times \mathbb{C}} \phi_1(\tau,z)\overline{\phi_2(\tau,z)} v^{k-3} e^{-4\pi y^2/v} dx\, dy\, du\, dv$$

for $\phi_1, \phi_2 \in J_{k,1}^{\mathrm{cusp}}(\Gamma_0^{\mathrm{J}}(M))$ and $\tau = u+iv$, $z = x+iy$.

We denote the space of Siegel modular forms of weight k on $\Gamma_0^4(M)$ by $\mathcal{M}_k(\Gamma_0^4(M))$. The space of cusp forms is denoted by $\mathcal{S}_k(\Gamma_0^4(M))$. For $\gamma \in \mathrm{Sp}_4^+(\mathbb{R})$, the slash operator of γ on a Siegel modular form F of weight k is given by $(F|_k\gamma)(Z) = \det(C_\gamma Z + D_\gamma)^{-k} F(\gamma Z)$. For F and

G two Siegel modular forms on $\Gamma_0^4(M)$ of weight k with at least one of them a cusp form, we define the Petersson product of F and G by

$$\langle F, G\rangle = \frac{1}{[\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^4(M)]} \int_{\Gamma_0^4(M)\backslash \mathfrak{h}^2} F(Z)\overline{G(Z)} \det(Y)^k d\mu(Z).$$

Associated to a Siegel Hecke eigenform F are two L-functions: the standard and the Spinor L-functions. Here we are only interested in the Spinor L-function. It is defined by

$$L_{\mathrm{spin}}(s, F) = \zeta(2s - 2k + 4) \sum_{m=1}^{\infty} \lambda_F(m) m^{-s}$$

where the $\lambda_F(m)$ are the Hecke eigenvalues of F. The Euler product of $L_{\mathrm{spin}}(s, F)$ is given by

$$L_{\mathrm{spin}}(s, F) = \prod_p L_{\mathrm{spin},(p)}(s, F)$$

where

$$L_{\mathrm{spin},(p)}(s, F) = 1-\lambda_F(p)p^{-s}+(\lambda_F(p)^2-\lambda_F(p^2)-p^{2k-4})p^{-2s}-\lambda_F(p)p^{2k-3-3s}+p^{4k-6-4s}$$

for $p \nmid M$ and

$$L_{\mathrm{spin},(p)}(s, F) = 1 - \lambda_F(p)p^{-s}$$

for $p \mid M$ ([2]). Alternatively, one has a description of the Spinor L-function in terms of the Satake parameters $\alpha_0, \alpha_1, \alpha_2$ attached to F. One has

$$L_{\mathrm{spin}}(s, F) = \prod_p Q_p(p^{-s})^{-1}$$

where the $Q_p(X)$ are the Hecke polynomials given by

$$Q_p(p^{-s}) = (1 - \alpha_0 p^{-s})(1 - \alpha_0\alpha_1 p^{-s})(1 - \alpha_0\alpha_2 p^{-s})(1 - \alpha_0\alpha_1\alpha_2 p^{-s}).$$

If F has level M we define the modified Spinor L-function $L^*_{\mathrm{spin}}(s, F)$ by

$$L^*_{\mathrm{spin}}(s, F) = \left(\prod_{p\mid M} [(1 - p^{k-1-s})(1 - p^{k-2-s})]^{-1}\right) L_{\mathrm{spin}}(s, F).$$

The Maass spezialschar $\mathcal{M}^*_k(\Gamma_0^4(M)) \subset \mathcal{M}_k(\Gamma_0^4(M))$ play an important role in the Saito-Kurokawa correspondence. A Siegel modular form F is in the Maass spezialchar if the Fourier coefficients of F satisfy the relation

$$A_F(n, r, m) = \sum_{d\mid \gcd(n,r,m)} d^{k-1} A_F\left(\frac{nm}{d^2}, \frac{r}{d}, 1\right)$$

for every $m, n, r \in \mathbb{Z}$ with $m, n, 4mn - r^2 \geq 0$ ([26]).

3. The Saito-Kurokawa Correspondence

In this section we briefly outline the Saito-Kurokawa correspondence for level M odd and square-free as established in [21]. For arbitrary M the reader should consult [22] and [23].

As in the case of level 1, the first step in establishing the correspondence is to relate integer weight forms to half-integer weight forms. Let D be a fundamental discriminant with $(-1)^{k-1}D > 0$. There exists a Shimura lifting ζ_D that maps $S^+_{k-1/2}(\Gamma_0(4M))$ to $M_{2k-2}(\Gamma_0(M))$ and a Shintani lifting ζ^*_D mapping $S_{2k-2}(\Gamma_0(M))$ to $S^+_{k-1/2}(\Gamma_0(4M))$. These maps are adjoint on cusp forms with respect to the Petersson products. Explicitly, for

$$g(z) = \sum c_g(n)q^n \in S^+_{k-1/2}(\Gamma_0(M))$$

one has

$$\zeta_D g(z) = \sum_{n\geq 1}\left(\sum_{\substack{d \mid n \\ \gcd(d,M)=1}} \left(\frac{D}{d}\right) d^{k-2} c_g(|D|n^2/d^2)\right) q^n$$

and for $f \in S^{\text{new}}_{2k-2}(\Gamma_0(M))$ a newform one has

$$\zeta^*_D f(z) = (-1)^{[(k-1)/2]} 2^{k-1} \sum r_{k-1,M,D}(f;|D|n)q^n$$

where the sums defining g and $\zeta^*_D f$ are over all $n \geq 1$ so that $(-1)^{k-1}n \equiv 0, 1 \pmod 4$ and $r_{k-1,M,D}(f;|D|m)$ is a certain integral. For the definition of these integrals see ([13], Section 1).

Using these liftings, one has the following theorem.

Theorem 3.1. *([12], [13], [20]) For D a fundamental discriminant with $(-1)^{k-1}D > 0$ and $\gcd(D,M) = 1$, the Shimura and Shintani liftings give Hecke-equivariant isomorphisms between $S^{+,\text{new}}_{k-1/2}(\Gamma_0(4M))$ and $S^{\text{new}}_{2k-2}(\Gamma_0(M))$.*

The correspondence between half-integral weight modular forms and Jacobi forms is given by the following theorem.

Theorem 3.2. *([21], Theorem 4) The map defined by*

$$\sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2 \pmod 4}} c(D,r)e\left(\frac{r^2-D}{4}\tau + rz\right) \mapsto \sum_{\substack{0>D\in\mathbb{Z} \\ D\equiv 0,1 \pmod 4}} c(D)e(|D|\tau),$$

is a canonical isomorphism between $J_{k,1}^{\text{cusp,new}}(\Gamma_0^{\text{J}}(M))$ *and* $S_{k-1/2}^{+,\text{new}}(\Gamma_0(4M))$ *which commutes with the action of Hecke operators.*

And finally one relates Jacobi forms to Siegel forms. Let $F \in \mathcal{M}_k^*(\Gamma_0^4(M))$ have Fourier-Jacobi expansion

$$F(\tau, z, \tau') = \sum_{m \geq 0} \phi_m(\tau, z) e(m\tau') \tag{2}$$

where the ϕ_m are Jacobi forms of weight k, index m, and level M.

Theorem 3.3. *([21], Theorem 6) The association* $F \mapsto \phi_1$ *gives an isomorphism between* $\mathcal{S}_k^{*,\text{new}}(\Gamma_0^4(M))$ *and* $J_{k,1}^{\text{cusp,new}}(\Gamma_0^J(M))$*. This isomorphism commutes with the action of Hecke operators.*

The inverse map to the map $F \mapsto \phi_1$ is given as follows. Let $\phi(\tau, z) \in J_{k,1}(\Gamma_0^J(M))$. Define

$$F(\tau, z, \tau') = \sum_{m \geq 0} V_m \phi(\tau, z) e(m\tau')$$

where V_m is the linear operator defined in [6]. Then F is in the Maass spezialschar. For the details consult ([6], §6) and [21].

We have the following theorem giving the Saito-Kurokawa correspondence.

Theorem 3.4. *([21], Theorem 8) The space* $\mathcal{S}_k^{*,\text{new}}(\Gamma_0^4(M))$ *is isomorphic to* $S_{2k-2}^{\text{new}}(\Gamma_0(M))$ *for* M *odd and square-free. Given a newform* $f \in S_{2k-2}^{\text{new}}(\Gamma_0(M))$*, the corresponding* $F_f \in \mathcal{S}_k^{*,\text{new}}(\Gamma_0^4(M))$ *has Spinor* L*-function satisfying*

$$L_{\text{spin}}^*(s, F) = \zeta(s-k+1)\zeta(s-k+2)L(s, f). \tag{3}$$

4. Relating $\langle F_f, F_f \rangle$ to $\langle \phi_f, \phi_f \rangle$

In this section we seek to generalize the following result of Kohnen and Skoruppa from level $M = 1$ to M odd and square-free.

Theorem 4.1. *([15], Corollary to Theorem 2) Let* $f \in S_{2k-2}(\text{SL}_2(\mathbb{Z}))$ *be a normalized Hecke eigenform,* $F_f \in \mathcal{S}_k^*(\text{Sp}_4(\mathbb{Z}))$ *the Saito-Kurokawa lift of* f*, and* ϕ_f *the Jacobi form associated via the Saito-Kurokawa correspondence. Then the formula*

$$\langle F_f, F_f \rangle = \frac{\langle \phi_f, \phi_f \rangle}{\pi^k c_k} L(k, f)$$

holds, where $c_k = \dfrac{3 \cdot 2^{2k+1}}{(k-1)!}$.

We follow Kohnen and Skoruppa's arguments, generalizing results where needed. Let $F, G \in \mathcal{S}_k^*(\Gamma_0^4(M))$ be eigenforms with Fourier-Jacobi expansions given by

$$F(Z) = \sum_{N \geq 1} \phi_N(\tau, z) e(N\tau')$$

and

$$G(Z) = \sum_{N \geq 1} \psi_N(\tau, z) e(N\tau').$$

Define a Dirichlet series attached to F and G by

$$D_{F,G}(s) = \zeta(2s - 2k + 4) \sum_{N \geq 1} \langle \phi_N, \psi_N \rangle N^{-s}$$

and set

$$D_{F,G}^*(s) = (2\pi)^{-2s} \Gamma(s) \Gamma(s - k + 2) \prod_{p|M} (1 - p^{-2(s-k+2)}) D_{F,G}(s). \tag{4}$$

It is shown in [9] that $D_{F,G}^*(s)$ has meromorphic continuation to $\mathbb{C}$, is entire if $\langle F, G \rangle = 0$ and otherwise has a simple pole at $s = k$. Calculating the residue of $D_{F,G}$ at $s = k$ provides the desired generalization of Theorem 4.1.

Define an Eisenstein series

$$E_{s,M}(Z) = \sum_{\gamma \in C_{2,1}(M) \backslash \Gamma_0^4(M)} \left(\frac{\det(\mathrm{Im} \gamma Z)}{\mathrm{Im}(\gamma Z)_1} \right)^s$$

where $(\gamma Z)_1$ denotes the upper left entry of γZ and

$$C_{2,1}(M) = \left\{ \begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_0^4(M) \right\}, \ (\lambda', \mu') = (\lambda, \mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Set

$$E_{s,M}^*(Z) = \pi^{-s} \Gamma(s) \zeta(2s) \prod_{p|M} (1 - p^{-2s}) E_{s,M}(Z). \tag{5}$$

One has that $E_{s,M}^*(Z)$ has mermomorphic continuation to $\mathbb{C}$ with possible simple poles at $s = 0, 2$ ([9]). It is known that $\operatorname{res}_{s=2} E_{s,1}^*(Z) = 1$ ([15]). Note that this is independent of Z, so we have $\operatorname{res}_{s=2} E_{s,1}^*(NZ) = 1$ for all positive integers N. Equation 5 gives $\operatorname{res}_{s=2} E_{s,1}(Z) = \dfrac{90}{\pi^2}$. As above, this residue is independent of Z so we have $\operatorname*{res}_{s=2} E_{s,1}(NZ) = \dfrac{90}{\pi^2}$ for all positive integers N. The following formula is given in [9]:

$$E_{s,1}(MZ) = \frac{1}{M^s} \sum_{d|M} d^{2s} \prod_{p|d} (1 - p^{-2s}) E_{s,d}(Z).$$

This formula allows one to calculate the residue of $E_{s,M}(Z)$ inductively in terms of $E_{s,d}(Z)$ for $d \mid M$. In fact, for $M = p_1^{m_1} \dots p_n^{m_n}$, we have

$$\operatorname*{res}_{s=2} E_{s,M}(Z) = \left(\frac{90}{\pi^2}\right) h(p_1, \dots, p_n) \prod_{i=1}^{n} \left(\frac{1}{p_i^{2m_i - 2}(p_i^4 - 1)}\right) \tag{6}$$

where h is a polynomial with coefficients in $\mathbb{Z}$ uniquely determined by M. For example, if $M = p^n$ for a prime p, we have

$$h(p) = p^2 - 1$$

and if $M = p_1 \dots p_n$ is a product of distinct primes, we have

$$h(p_1, \dots, p_n) = \prod_{i=1}^{n} (p_i^2 - 1).$$

Returning to the case we are interested in, namely $M = p_1 \dots p_n$ odd and square-free, appealing to Equation 5 one obtains

$$\operatorname*{res}_{s=2} E^*_{s,M}(Z) = \prod_{i=1}^{n} \left(\frac{1 - p_i^{-4}}{p_i^{2m_i - 2}(p_i^2 + 1)}\right). \tag{7}$$

We now turn our attention back to calculating the residue of $D_{F,G}(s)$ at $s = k$. We have the following equation ([9]):

$$\pi^{-k+2}[\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^4(M)] \langle F E^*_{s-k+2,M}, G \rangle = M^s D^*_{F,G}(s).$$

Taking the residue of this equation at $s = k$ and solving for $\operatorname*{res}_{s=k} D^*_{F,G}(s)$ we obtain

$$\begin{aligned} \operatorname*{res}_{s=k} D^*_{F,G}(s) &= \frac{\pi^{2-k}\,[\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^4(M)]}{M^k} \operatorname*{res}_{s=2} E^*_{s,M}(Z) \langle F, G \rangle \\ &= \frac{\pi^{2-k}\,[\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^4(M)]}{M^k} \prod_{i=1}^{n} \left(\frac{1 - p_i^{-4}}{p^{2m_i - 2}(p_i^2 + 1)}\right) \langle F, G \rangle. \end{aligned}$$

On the other hand, taking the residue at $s = k$ of Equation 4 we have

$$\operatorname*{res}_{s=k} D^*_{F,G}(s) = (2\pi)^{-2k}(k-1)! \prod_{i=1}^{n} (1 - p_i^{-4}) \operatorname*{res}_{s=k} D_{F,G}(s).$$

Combining these two results and solving for $\operatorname*{res}_{s=k} D_{F,G}(s)$ we obtain

$$\operatorname*{res}_{s=k} D_{F,G}(s) = \frac{2^{2k}\pi^{k+2}\,[\mathrm{Sp}_4(\mathbb{Z}):\Gamma_0^4(M)]}{M^k(k-1)!\prod_{i=1}^{n}[p_i^{2m_i-2}(p_i^2+1)]}\langle F,G\rangle. \tag{8}$$

Lemma 4.2. *Let $f \in S_{2k-2}^{\mathrm{new}}(\Gamma_0(M))$ be a newform, F_f the Saito-Kurokawa lift of f, and ϕ_f the corresponding Jacobi form obtained in the Saito-Kurokawa correspondence. Then we have*

$$\langle F_f, F_f\rangle = \frac{\langle \phi_f,\phi_f\rangle}{\pi^k c_k} L(k,f)$$

where

$$c_k = \frac{3\cdot 2^{2k+1}[\mathrm{Sp}_4(\mathbb{Z}):\Gamma_0^4(M)]}{M^k(k-1)!\prod_{i=1}^{n}[p_i^{2m_i-2}(p_i^2+1)]}.$$

Proof. Dabrowski ([5], Theorem 4.2) gives the formula

$$D_{F_f,F_f}(s) = \langle \phi_f,\phi_f\rangle\, L_{\mathrm{spin}}^*(s,F_f) \tag{9}$$

for M odd and square-free. In fact, this result was originally proven by Kohnen and Skoruppa for level 1: see ([15], Theorem 2). Equation 3 gives us

$$L_{\mathrm{spin}}^*(s,F_f) = \zeta(s-k+1)\zeta(s-k+2)L(s,f).$$

Combining this with Equation 9 and taking residues at $s=k$ gives

$$\operatorname*{res}_{s=k} D_{F_f,F_f}(s) = \frac{\pi^2}{6} L(k,f)\langle \phi_f,\phi_f\rangle.$$

This along with Equation 8 gives the result. □

5. Relating $\langle \phi_f,\phi_f\rangle$ to $\langle g_f,g_f\rangle$

Let g_f denote the half-integral weight modular form associated to f via the Saito-Kurokawa correspondence. In this section we will calculate a relationship between $\langle \phi_f,\phi_f\rangle$ and $\langle g_f,g_f\rangle$. Combining this with Lemma 4.2 we will obtain a relationship between $\langle F_f,F_f\rangle$ and $\langle g_f,g_f\rangle$.

Let $g_f(z) = \sum_{n=1}^{\infty} c_g(n)q^n$ be the Fourier expansion of g_f. Consider the summation $\sum_{n=1}^{\infty} \frac{c_g(n)^2}{n^{s+k-3/2}}$. Applying the Rankin-Selberg method to this summation we have for sufficiently large s:

$$\begin{aligned}\frac{\Gamma(s+k-3/2)}{(4\pi)^{s+k-1/2}}\sum_{n=1}^{\infty}\frac{c_g(n)^2}{n^{s+k-3/2}} &= \int\limits_{\mathfrak{h}^1/\Gamma_\infty}|g_f(z)|^2 y^{s+k-5/2}dxdy\\ &= \int\limits_{\mathfrak{h}^1/\Gamma_0(4M)} y^{k-1/2}|g_f(z)|^2 E_s^{4M}(z)\frac{dxdy}{y^2}\end{aligned}$$

where $E_s^{4M}(z)=\sum\limits_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4M)}(\mathrm{Im}\,(\gamma z))^s$ and Γ_∞ the stabilizer of ∞. In other words,

(10)

$$\sum_{n=1}^{\infty}\frac{c_g(n)^2}{n^{s+k-3/2}}=\frac{(4\pi)^{s+k-1/2}}{\Gamma(s+k-3/2)}\int\limits_{\Gamma_0(4M)\backslash\mathfrak{h}^1}E_s^{4M}(z)g_f(z)\overline{g_f(z)}y^{k-1/2}\frac{dxdy}{y^2}.$$

Taking residues at $s=1$ we obtain

$$\begin{aligned}\operatorname*{res}_{s=1}\left(\sum_{n=1}^{\infty}\frac{c(n)^2}{n^{s+k-3/2}}\right) &= \frac{(4\pi)^{k-1/2}\,[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(4M)]}{\Gamma(k-1/2)}\langle g_f,g_f\rangle\operatorname*{res}_{s=1}E_s^{4M}(z)\\ &= \frac{3\cdot 2^{k-1}(4\pi)^{k-1/2}}{\pi^{3/2}(2k-3)!!}\langle g_f,g_f\rangle\end{aligned}$$

where

$$n!!=\begin{cases} n(n-2)\dots 5\cdot 3\cdot 1 & n>0,\text{ odd}\\ n(n-2)\dots 6\cdot 4\cdot 2 & n>0,\text{ even}\end{cases}$$

and we have used that

$$\begin{aligned}\operatorname*{res}_{s=1}E_s^{4M}(z) &= \frac{1}{[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(4M)]}\operatorname*{res}_{s=1}E_s(z)\\ &= \frac{1}{[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(4M)]}\left(\frac{3}{\pi}\right)\end{aligned}$$

where $E_s(z)$ is the Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$. Solving the above residue calculation for $\langle g_f,g_f\rangle$ we obtain

(11) $$\langle g_f,g_f\rangle=\frac{(2k-3)!!}{3\cdot 2^{3k-2}\pi^{k-2}}\operatorname*{res}_{s=1}\left(\sum_{n=1}^{\infty}\frac{c_g(n)^2}{n^{s+k-3/2}}\right).$$

We define two half-integral weight modular forms g_0 and g_1 by

$$g_j(z)=\sum_{n\equiv j\,(\mathrm{mod}\,4)}c_g(n)q^{n/4}$$

for $j = 0, 1$ as in ([6], Page 64-65). Using that g_f is in Kohnen's $+-$space, we see that $g_f(z) = g_0(4z) + g_1(4z)$. Applying the same process to g_0 and g_1 we obtain

$$\langle g_j, g_j \rangle = \frac{(2k-3)!!}{3 \cdot 2^{3k-2} \pi^{k-2}} \cdot 2^{2k-1} \operatorname*{res}_{s=1} \left(\sum_{n \equiv j} \frac{c_g(n)^2}{n^{s+k-3/2}} \right).$$

Thus we have

$$\langle g_0, g_0 \rangle + \langle g_1, g_1 \rangle = 2^{2k-1} \langle g_f, g_f \rangle. \tag{12}$$

We need a slight generalization of Theorem 5.3 in [6]. In [6], the formula given only deals with the case $M = 1$. However, the proof carries through verbatim to the general case.

Theorem 5.1. *([6], Theorem 5.3) For ϕ_f and g_j as defined above, one has*

$$\langle \phi_f, \phi_f \rangle = \frac{1}{2\left[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(M)\right]} \int_{\Gamma_0(M)\backslash\mathfrak{h}^1} \sum_{j=0}^{1} g_j(z)\overline{g_j(z)} v^{k-3/2} \frac{du dv}{v^2}. \tag{13}$$

Combining Equations 12 and 13 we have:

Lemma 5.2. *For ϕ_f and g_f defined as above we have*

$$\langle \phi_f, \phi_f \rangle = \frac{2^{2k-2}}{[\Gamma_0(M) : \Gamma_0(4M)]} \langle g_f, g_f \rangle.$$

6. Relating $\langle g_f, g_f \rangle$ to $\langle f, f \rangle$

The only remaining hurdle in establishing a relationship between $\langle F_f, F_f \rangle$ and $\langle f, f \rangle$ is to relate $\langle g_f, g_f \rangle$ to $\langle f, f \rangle$. Fortunately the work has already been done for us.

Let ℓ be a prime dividing M. Define the Atkin-Lehner involution on $S_{2k-2}^{new}(\Gamma_0(M))$ associated to ℓ by slashing f by the element

$$W_\ell = \frac{1}{\sqrt{l}} \begin{pmatrix} \ell & \alpha \\ M & \ell\beta \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{Z}$ and $\ell^2\beta - M\alpha = \ell$. We can define $w_\ell \in \{\pm 1\}$ for every $\ell \mid M$ by

$$f|_{W_\ell} = w_\ell f.$$

Lemma 6.1. *([11], Corollary 1) Let M be odd and let D be a fundamental discriminant with $(-1)^{k-1}D > 0$ and suppose that for all primes $\ell \mid M$ we have $\left(\frac{D}{\ell}\right) = w_\ell$. Then*

$$\frac{|c_g(|D|)|^2}{\langle g_f, g_f \rangle} = 2^{\nu(M)} \frac{(k-2)!}{\pi^{k-1}} |D|^{k-3/2} \frac{L(k-1, f, \chi_D)}{\langle f, f \rangle} \tag{14}$$

where $\nu(M)$ is the number of primes dividing M.

The condition on the discriminant in Lemma 6.1 is not a major restriction. If for some prime $\ell \mid M$ we have $w_\ell = -\left(\frac{D}{\ell}\right)$, then $c_g(|D|) = 0$ ([11], Page 243). So as long as we choose D so that $\gcd(M, D) = 1$ and $c_g(|D|) \neq 0$, then our condition will be satisfied.

We are now in a position to gather our results and state the relationship between $\langle f, f\rangle$ and $\langle F_f, F_f\rangle$.

Theorem 6.2. *Let $M = p_1 \dots p_n$ be odd and square-free, $f \in S_{2k-2}^{new}(\Gamma_0(M))$ a newform, and $F_f \in \mathcal{S}_k^{*,new}(\Gamma_0^4(M))$ the Siegel modular form associated to f via the Saito-Kurokawa correspondence. Let D be a fundamental discriminant with $(-1)^{k-1}D > 0$, $\gcd(M, D) = 1$, and $c_g(|D|) \neq 0$. Then one has*

$$\langle F_f, F_f\rangle = \mathcal{B}_{k,M} \, \frac{|c_g(|D|)|^2 \, L(k, f)}{\pi \, |D|^{k-3/2} \, L(k-1, f, \chi_D)} \langle f, f\rangle \tag{15}$$

with

$$\mathcal{B}_{k,M} = \frac{M^k \, (k-1) \, \prod_{i=1}^n \left(p_i^{2m_i-2}(p_i^2+1)\right)}{2^{\nu(M)+3} \, 3 \, [\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^4(M)] \, [\Gamma_0(M) : \Gamma_0(4M)]}.$$

In particular, applying this in the case of level 1 we are able to recover the following inner product relation stated in [3], but really an amalgamation of previous results. Note that in the level 1 case the fact that $c_g(|D|) \neq 0$ is automatic.

Corollary 6.3. *([15], [16]) Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized eigenform and $F_f \in \mathcal{S}_k^*(\mathrm{Sp}_4(\mathbb{Z}))$ the Siegel modular form associated to f via the Saito-Kurokawa correspondence. Let D be a discriminant with $(-1)^{k-1}D > 0$. Then one has*

$$\langle F_f, F_f\rangle = \mathcal{B}_k \frac{|c_g(|D|)|^2 L(k, f)}{\pi |D|^{k-3/2} L(k-1, f, \chi_D)} \langle f, f\rangle \tag{16}$$

where

$$\mathcal{B}_k = \frac{(k-1)}{2^4 \, 3^2}.$$

7. An algebraicity result on a Rankin L-functions

Let $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized eigenform with Fourier expansion given by

$$f(z) = \sum_{n=1}^{\infty} a_f(n) q^n.$$

Attached to f are complex periods $\Omega_f^{\pm}$ so that we have the following theorem.

Theorem 7.1. *([25], Theorem 1) For $1 \leq m < k$ one has*

$$\frac{L(m,f)}{\pi^m \Omega_f^{\pm}} \in \overline{\mathbb{Q}}$$

where we choose Ω_f^+ if m is even and Ω_f^- if m is odd.

The L-function of f can be factored as

$$L(s,f) = \prod_p [(1-\alpha_f(p)p^{-s})(1-\beta_f(p)p^{-s})]^{-1} \tag{17}$$

where $\alpha_f(p)+\beta_f(p) = a_f(p)$ and $\alpha_f(p)\beta_f(p) = p^{k-1}$. Let $h \in S_l(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized eigenform of weight l with $l < k$. Using the factorization of $L(s,f)$ and $L(s,h)$ we define the Rankin L-function associated to f and h by

$$\begin{aligned}\mathcal{D}(s,f,h) = \prod_p &[(1-\alpha_f(p)\alpha_h(p)p^{-s})(1-\alpha_f(p)\beta_h(p)p^{-s}) \\ &\cdot (1-\beta_f(p)\alpha_h(p)p^{-s})(1-\beta_f(p)\beta_h(p)p^{-s})]^{-1}.\end{aligned}$$

It is known that these Rankin L-functions can be normalized so they are algebraic at the special values. In particular, one has:

Theorem 7.2. *([25], Theorem 4) Let $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ and $h \in S_l(\mathrm{SL}_2(\mathbb{Z}))$ be normalized eigenforms with $l < k$. Then one has*

$$\frac{\mathcal{D}(m,f,h)}{\pi^{2m+2-l}\langle f,f\rangle} \in \overline{\mathbb{Q}}$$

for $l \leq m < k$.

We given an alternative proof of Shimura's result, though in a much more restrictive setting.

Theorem 7.3. *Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ and $h \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be normalized eigenforms. If k is even we have*

$$\frac{\mathcal{D}(2k-3,f,h)}{\pi^{3k-4}\langle f,f\rangle} \in \overline{\mathbb{Q}}.$$

The proof of this theorem is obtained by considering an L-function on $\mathrm{GSp}_4 \times \mathrm{GL}_2$. Let $F \in \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a Siegel eigenform and $h \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ a normalized eigenform with $\alpha_h(p)$ and $\beta_h(p)$ defined as above. We define the L-function $Z(s, F\otimes h)$ by

$$Z(s,F\otimes h) = \prod_p [Q_p(\alpha_h(p)p^{-s})Q_p(\beta_h(p)p^{-s})]^{-1}$$

where the Q_p are the Hecke polynomials defined in Section 2. We make use of the following result of Heim.

Theorem 7.4. *([8], Corl 5.4) Let $F \in \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a Siegel eigenform and $h \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ a normalized eigenform with k even. Then one has*

$$\frac{Z(2k-3, F \otimes h)}{\pi^{5k-8}\langle F, F\rangle\langle h, h\rangle} \in \overline{\mathbb{Q}}.$$

We begin by restricting to the case of $F = F_f$ a Saito-Kurokawa lift. In this case we are able to factor $Z(s, F_f \otimes h)$ into a product of familiar L-functions.

Proposition 7.5. *Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized eigenform, $h \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ a normalized eigenform, and F_f the Saito-Kurokawa lift of f. Then $Z(s, F_f \otimes h)$ has the following factorization:*

$$Z(s, F_f \otimes h) = L(s-k+1, h)L(s-k+2, h)\mathcal{D}(s, f, h). \tag{18}$$

Proof. Let α_0, α_1 and α_2 be the Satake parameters of F_f as defined in Section 2. We make use of the fact that we can write the Satake parameters of F_f in terms of α_f and β_f. In particular, we have $\alpha_0 = p^{k-1}$, $\alpha_1 = \beta_f p^{1-k}$ and $\alpha_2 = \alpha_f p^{1-k}$ ([3], Theorem 3.10). A short calculation now yields the desired result. □

It is now easy to combine our previous results to conclude Theorem 7.3. For a normalized eigenform f, one has that there exists an algebraic number ξ_f so that $\pi\langle f, f\rangle = \xi_f \Omega_f^+ \Omega_f^-$ ([25], Theorem 1). We write $\doteqdot$ to indicate equality up to an algebraic multiple. Then using Corollary 6.3 we see that

$$\begin{aligned}
\frac{Z(2k-3, F_f \otimes h)}{\pi^{5k-8}\langle F_f, F_f\rangle\langle h, h\rangle} &\doteqdot \frac{L(k-2, h)L(k-1, h)\mathcal{D}(2k-3, f, h)}{\pi^{5k-8}\langle F_f, F_f\rangle\langle h, h\rangle} \\
&\doteqdot \left(\frac{L(k-2, h), L(k-1, h)}{\pi^{2k-3}\Omega_h^+\Omega_h^-}\right) \cdot \left(\frac{\mathcal{D}(2k-3, f, h)}{\pi^{3k-4}\langle F_f, F_f\rangle}\right) \\
&\doteqdot \frac{\mathcal{D}(2k-3, f, h)}{\pi^{3k-4}\langle F_f, F_f\rangle} \\
&\doteqdot \frac{\mathcal{D}(2k-3, f, h)}{\pi^{3k-4}\langle f, f\rangle}.
\end{aligned}$$

Combining this with Theorem 7.4 finishes the proof of Theorem 7.3.

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