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On the Bloch–Kato conjecture for elliptic modular forms of square-free level

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Abstract Let $\kappa \geq 6$ be an even integer, M an odd square-free integer, and $f \in S_{2\kappa-2}(\Gamma_0(M))$ a newform. We prove that under some reasonable assumptions that half of the λ-part of the Bloch–Kato conjecture for the near central critical value $L(\kappa, f)$ is true. We do this by bounding the ℓ-valuation of the order of the appropriate Bloch–Kato Selmer group below by the ℓ-valuation of algebraic part of $L(\kappa, f)$. We prove this by constructing a congruence between the Saito–Kurokawa lift of f and a cuspidal Siegel modular form.

Keywords Bloch–Kato conjecture · Congruences among automorphic forms · Galois representations · Saito–Kurokawa correspondence · Siegel modular forms · Special values of L-functions

Mathematics Subject Classification (1991) Primary 11F33 · 11F67; Secondary 11F46 · 11F80

1 Introduction

It is well known that L-functions encode deep and often subtle arithmetic information. One such conjectural relationship is the Bloch–Kato conjecture. This conjecture roughly states that given a motive $\mathcal{M}$, the order of the Bloch–Kato Selmer group should be controlled by a

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certain special value of the L-function attached to the motive $\mathcal{M}$ normalized by a canonical period. This paper deals with the case that the motive $\mathcal{M}$ arises from the twist of the motive associated to a newform.

Let $\kappa \geq 6$ be an even integer and M an odd square-free integer. Let $f \in S_{2\kappa-2}(\Gamma_0(M))$ be a classical newform. Let $\ell > \kappa, \ell \nmid M$ be an odd prime and E_0 be a sufficiently large number field. In particular, let E_0 contain all the eigenvalues of f. Let λ be a prime over ℓ in E_0. Set $E = E_{0,\lambda}$, $\mathcal{O}$ the ring of integers of E. We are interested in the Galois representation $(\rho_{f,\lambda}, V_\lambda)$ associated to f, $T_\lambda(\kappa-2)$ a Galois stable $\mathcal{O}$-lattice in $V_\lambda(\kappa-2)$, and set $W_\lambda(\kappa - 2) = V_\lambda(\kappa-2)/T_\lambda(\kappa-2)$. The Bloch–Kato Selmer group of $V_\lambda(\kappa-2)$ is denoted by $\mathrm{Sel}_\Sigma(\Sigma', W_\lambda(\kappa-2))$ where $\Sigma' = \{p \mid M\}$ and $\Sigma = \Sigma' \cup \{\ell\}$. One can see Sect. 8.1 for the precise definition of this Selmer group. Let $X_\Sigma(\Sigma', W_\lambda(\kappa-2))$ denote the Pontryagin dual of the Selmer group. Let $L_{\mathrm{alg}}(\kappa, f)$ denote the algebraic part of the L-function of f evaluated at $s = \kappa$. Under some reasonable assumptions we prove one half of the Bloch–Kato conjecture in this setting, namely, that we have

$$\mathrm{ord}_\ell(\# X_\Sigma(\Sigma', W_\lambda(\kappa-2))) \geq \mathrm{ord}_\ell(\#\mathcal{O}/L_{\mathrm{alg}}(\kappa, f)).$$

The proof of this result follows along the same lines as the main result of [10]. The main difference is that allowing for non-trivial level here adds several obstacles that must be overcome. The argument used here originates in the work of Ribet in his proof of the converse of Herbrand's theorem [29], but has now appeared in many forms. One can see [4,6–8,22,48] for some other examples. We now give a brief outline of the argument focusing on points of divergence from the arguments in the level one case.

The argument is given in essentially two main parts. The first part consists of constructing a congruence between the Saito–Kurokawa lift F_f of f and a cuspidal Siegel modular form and translating this congruence into a result about the CAP ideal associated to F_f. The construction of the Saito–Kurokawa lifting for congruence level $\Gamma_0^{(2)}(M)$ with M odd square-free is provided in [3]. This result is claimed in [25,26]. However, the generalized Maass lifting there is not defined correctly so Agarwal and Brown [3] combines known results from other sources to give this lifting. To construct the congruence, we study the pullback of a certain normalized Siegel Eisenstein series from $\mathrm{Sp}(8)$ to $\mathrm{Sp}(4) \times \mathrm{Sp}(4)$. It is here that the restriction $\kappa \geq 6$ (see Theorem 6.2) is needed. We use the pullback along with some norm computations to obtain a congruence of the form F_f with another modular form G modulo λ under certain conditions (see condition 3.8) on L-values. We write $G = \sum_i F_i$ as a sum of eigenforms orthogonal to F_f. We construct a Hecke operator that kills the Saito–Kurokawa lifts in the sum that arise as theta lifts from irreducible cuspidal automorphic representations of $\widetilde{\mathrm{SL}}_2$. A similar Hecke operator was constructed in [7], but the construction here is more involved and requires more care to deal with the possibility of Saito–Kurokawa lifts coming from oldforms. In addition to the Saito–Kurokawa lifts, we must also deal with the possibility that F_f could be congruent to a weak endoscopic lift or a form of Saito–Kurokawa type that arises as a non-trivial quadratic twist of a theta lift. The case of a weak endoscopic lift was ruled out easily in the case of full level and it was already known there are no forms of Saito–Kurokawa type that arise as twists of theta lifts in the case of full level. However, the results for non-full level require more work. We use a multiplicity one result from [33] for Saito–Kurokawa lifts in the case that M is odd and square-free. We show that such twists of theta lifts do not give rise to classical Siegel eigenforms due to the fact that we are assuming our level to be square-free. We also show that it is not possible to have a Saito–Kurokawa lift congruent to a weak endoscopic lift if $\overline{\rho}_{f,\lambda}$ is irreducible along with some other minor technical hypotheses. Once we know the congruence constructed is not to a Siegel eigenform which is CAP, we can use the congruence to give a lower bound on the size of the CAP ideal associated to F_f in terms of $L_{\mathrm{alg}}(\kappa, f)$.

The second main part of this argument involves turning our result on the size of the CAP ideal into a result bounding the size of the Selmer group from below. The work for this is primarily contained in [10]. The main difference here is in the relaxation of the conditions at the primes dividing the level of f. We cite the relevant results from [10] and give their adapted statements in our setting. Finally, we give an exposition of the Bloch–Kato conjecture in our setting and relate our results to the conjecture. Such a discussion was absent in [10], so this can serve as the relation of the results given there to the Bloch–Kato conjecture as well.

Let us now give a brief overview of the organization of the paper. Section 2 sets the notations for the paper. The first couple of subsections of Sect. 3 recall standard definitions of classical modular forms, Galois representations, etc. Later we define an integral period and construct a certain Hecke operator. Finally, we state the Bloch–Kato conjecture in our setting and relate it to the main result of the paper and work out a particular example. In Sect. 4 we recall the necessary facts about Siegel modular forms. We recall some results about Saito–Kurokawa lifting for congruence level $\Gamma_0^{(2)}(M)$ with M odd, square free in Sect. 5. The relevant congruence is constructed in Sect. 6. In the section following, we prove the results mentioned above in regards to CAP forms and weakly endoscopic lifts which ensure the constructed congruence has the desired property. In Sect. 8 we define the relevant Selmer group as well as give the main result of the paper along with its proof.

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2 Notation and basic setup

2.1 Number fields and Hecke characters

We write i to denote $\sqrt{-1}$. Throughout the paper ℓ denotes an odd prime.

We fix once and for all an algebraic closure $\overline{\mathbb{Q}}$ of the rationals and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ for each finite prime p. Also fix compatible embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ for all finite primes p. We write val_p for the p-adic valuation on $\mathbb{Q}_p$. The p-adic norm on $\mathbb{Q}_p$ is defined by $|x|_p = p^{-\mathrm{val}_p(x)}$ for $x \in \mathbb{Q}_p$. The archimedean valuation on $\mathbb{Q}_\infty = \mathbb{R}$ is the usual absolute value. We extend val_p to a function from $\overline{\mathbb{Q}}_p$ to $\mathbb{Q}$ and for convenience set $\mathrm{val}_p(\infty) = \infty$.

For a number field L, write $\mathcal{O}_L$ for its ring of integers. Given a place υ of L, denote the completion of L at υ by L_υ and $\mathcal{O}_{L,\upsilon}$ for the valuation ring of L_υ. We write G_L for $\mathrm{Gal}(\overline{L}/L)$. Given a prime $\mathfrak{p}$ of L we write $D_\mathfrak{p} \subset G_L$ for the decomposition group of $\mathfrak{p}$ and $I_\mathfrak{p} \subset D_\mathfrak{p}$ for the inertia group of $\mathfrak{p}$. One can identify $D_\mathfrak{p}$ with $\mathrm{Gal}(\overline{L}_\mathfrak{p}/L_\mathfrak{p})$ via the embeddings fixed above.

We write $\mathbb{A}_L$ for the adeles of L and $\mathbb{A}$ for the adeles of $\mathbb{Q}$. The infinite part of the adeles is denoted $\mathbb{A}_{L,\infty}$ and the finite part $\mathbb{A}_{L,f}$. Given $x = (x_p) \in \mathbb{A}$, we set $|x|_\mathbb{A} = |x|_\infty \prod_p |x|_p$. We say χ is a Hecke character of L if χ is a continuous homomorphism $\chi : L^\times \backslash \mathbb{A}_L^\times \to \mathbb{C}^\times$ with image contained in $\{z \in \mathbb{C} : |z| = 1\}$. The character χ factors into a product of local characters $\chi = \prod_\upsilon \chi_\upsilon$ where υ runs over all places of L. We say $\mathfrak{n} \subset \mathcal{O}_L$ is the conductor of χ if

(1) $\chi_\upsilon(x_\upsilon) = 1$ if υ is a finite place of L, $x_\upsilon \in \mathcal{O}_{L,\upsilon}^\times$ and $x - 1 \in \mathfrak{n}\mathcal{O}_{L,\upsilon}$.
(2) no ideal $\mathfrak{m}$ strictly containing $\mathfrak{n}$ has the above property.

Finally, for $z \in \mathbb{C}$ we will sometimes write q or $e(z)$ for $e^{2\pi i z}$.

2.2 Matrices over a ring

Given a ring R with identity, we write $\mathrm{Mat}_n(R)$ for the ring of n by n matrices with entries in R. We write 0_n for the zero matrix in $\mathrm{Mat}_n(R)$ and 1_n for the identity matrix in $\mathrm{GL}_n(R)$. The matrix $i1_n$ is denoted i_n.

3 Modular forms and the Bloch–Kato conjecture

3.1 Modular forms

We denote the upper half space by

$$\mathfrak{h}^1 = \{z \in \mathbb{C} : {}^t z = z, \mathrm{Im}(z) > 0\}.$$

We have an action of $\mathrm{GL}_2^+(\mathbb{R}) = \{\gamma \in \mathrm{GL}_2(\mathbb{R}) : \det(\gamma) > 0\}$ on $\mathfrak{h}^1$ given by

$$\gamma z = (a_\gamma z + b_\gamma)(c_\gamma z + d_\gamma)^{-1}$$

for $\gamma = \left(\begin{smallmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ and $z \in \mathfrak{h}^1$.

Given an integer $M \geq 1$, we will be interested in the congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ given by

$$\Gamma_0^{(1)}(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{M} \right\}.$$

For $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ and $z \in \mathfrak{h}^1$, we set

$$j(\gamma, z) = c_\gamma z + d_\gamma.$$

Let κ be a positive integer. Given a function $f : \mathfrak{h}^1 \to \mathbb{C}$, we set

$$(f|_\kappa \gamma)(z) = \det(\gamma)^\kappa j(\gamma, z)^{-\kappa} f(\gamma z).$$

We say such an f is a modular form of weight κ and level $\Gamma_0^{(1)}(M)$ if f is holomorphic and satisfies

$$(f|_\kappa \gamma)(z) = f(z)$$

for all $\gamma \in \Gamma_0^{(1)}(M)$ and is holomorphic at the cusps. We denote the space of modular forms of weight κ and level $\Gamma_0^{(1)}(M)$ by $M_\kappa(\Gamma_0^{(1)}(M))$. We denote the space of cusp forms of weight κ and level $\Gamma_0^{(1)}(M)$ by $S_\kappa(\Gamma_0^{(1)}(M))$. To ease notation we will omit the superscript 1 when it is clear from the context.

3.2 Hecke algebras

Given $g \in \mathrm{GL}_2^+(\mathbb{Q})$, we denote $\Gamma_0(M) g \Gamma_0(M)$ by $T(g)$. We define the usual action of $T(g)$ on classical modular forms by setting

$$T(g) f = \sum_i f|_\kappa g_i$$

where $\Gamma_0(M) g \Gamma_0(M) = \coprod_i \Gamma_0(M) g_i$ and $f \in M_\kappa(\Gamma_0(M))$. Let p be prime and define

$$T(p) = T(\mathrm{diag}(1, p)).$$

In the case that $p \mid M$ we will write $U(p)$ to denote the operator $T(p)$. Let Σ be a finite set of places. Set $\mathbb{T}_{\mathbb{Z}}^{\Sigma}$ to be the $\mathbb{Z}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(S_\kappa(\Gamma_0(M)))$ generated by $\{T(p) : p \nmid M, p \notin \Sigma\} \cup \{U(p) : p \mid M, p \notin \Sigma\}$. Given a $\mathbb{Z}$-algebra A, we set $\mathbb{T}_A^{\Sigma} = \mathbb{T}_{\mathbb{Z}}^{\Sigma} \otimes_{\mathbb{Z}} A$. In the case that $\Sigma = \emptyset$, we write $\mathbb{T}_{\mathbb{Z}}$ for $\mathbb{T}_{\mathbb{Z}}^{\emptyset}$.

Let E be a finite extension of $\mathbb{Q}_\ell$ and $\mathcal{O}_E$ the ring of integers of E. Then we have $\mathbb{T}_{\mathcal{O}_E}^{\Sigma}$ is a semi-local complete finite $\mathcal{O}_E$-algebra. One has

$$\mathbb{T}_{\mathcal{O}_E}^{\Sigma} = \prod_{\mathfrak{m}} \mathbb{T}_{\mathfrak{m}}^{\Sigma}$$

where the product runs over all maximal ideals of $\mathbb{T}_{\mathcal{O}_E}^{\Sigma}$ and $\mathbb{T}_{\mathfrak{m}}^{\Sigma}$ denotes the localization of $\mathbb{T}_{\mathcal{O}_E}^{\Sigma}$ at $\mathfrak{m}$.

3.3 Congruences

Let $f, g \in S_\kappa(\Gamma_0(M))$, K a field containing all the Fourier coefficients and eigenvalues of f and g, $\mathcal{O}$ be the ring of integers of K, and λ a prime of $\mathcal{O}$. We write

$$f \equiv g \pmod{\lambda^b}$$

to indicate that

$$\mathrm{ord}_\lambda(a_f(n) - a_g(n)) \geq b$$

for all $n \geq 1$.

Let Σ be a finite set of places. If f and g are eigenforms for all $t \in \mathbb{T}_{\mathcal{O}}^{\Sigma}$, we write

$$f \equiv_{\mathrm{ev},\Sigma} g \pmod{\lambda^b}$$

to indicate that

$$\mathrm{ord}_\lambda(\lambda_f(t) - \lambda_g(t)) \geq b$$

for all $t \in \mathbb{T}_{\mathcal{O}}^{\Sigma}$.

3.4 L-functions

Let $f \in S_\kappa(\Gamma_0(M))$ be an eigenform. The L-function associated to f is given by

$$L(s, f) = \prod_{p \nmid M} (1 - \lambda_f(p)p^{-s} + p^{\kappa-1-2s})^{-1} \prod_{p \mid M} (1 - \lambda_f(p)p^{-s})^{-1}$$

where $\lambda_f(p)$ is the eigenvalue of $T(p)$ corresponding to f. Given an L-function $L(s)$ with Euler product $L(s) = \prod_p L_p(s)$, we write

$$L^M(s) = \prod_{p \nmid M} L_p(s)$$

and

$$L_M(s) = \prod_{p \mid M} L_p(s).$$

3.5 Galois representations

The following well-known result of Deligne, et al. associates a λ-adic Galois representation to $f \in S_\kappa(\Gamma_0(M))$.

Theorem 3.1 *Let $\kappa \geq 2$, $M \geq 1$, and $f \in S_\kappa(\Gamma_0(M))$ a normalized eigenform. Let $\mathbb{Q}(f)$ be the number field generated by the eigenvalues of f, λ a prime of $\mathbb{Q}(f)$ over ℓ, and E the completion of $\mathbb{Q}(f)$ at λ. Then there exists a continuous, irreducible representation $(\rho_{f,\lambda}, V_{f,\lambda})$ of $G_\mathbb{Q}$ where $V_{f,\lambda}$ is a 2-dimensional E-vector space such that $(\rho_{f,\lambda}, V_{f,\lambda})$ is unramified at all primes $p \nmid \ell M$ and*

$$\det(1_2 - \rho_{f,\lambda}(\mathrm{Frob}_p)p^{-s}) = L_p(s, f)$$

for all $p \nmid \ell M$.

Note that we take geometric conventions throughout the paper, i.e., in the above theorem Frob_p is the geometric and not arithmetic Frobenius element.

3.6 Integral periods

We now define certain canonical integral periods that will be needed later. We assume for convenience that $M \geq 4$. This assumption is made so that we may follow Vatsal's construction [45] which uses that $\Gamma_1(M)$ is torsion-free. With more work one can also construct these canonical periods for $M < 4$ as is noted in [20, §3].

Let $\kappa \geq 2$ be an integer, $\ell > \kappa$ a prime with $\ell \nmid M$, and K be a suitably large finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}$. Let $f \in S_\kappa(\Gamma_1(M), \mathcal{O})$ be a newform. Furthermore, assume that $\overline{\rho}_{f,\lambda}$ is irreducible. The newform f defines a maximal ideal $\mathfrak{m}_f$ of $\mathbb{T}_\mathcal{O}$ given as the kernel of the composition of $\mathbb{T}_\mathcal{O} \to \mathcal{O} \to \mathcal{O}/\lambda$ where the first map is given by $t \mapsto \lambda_f(t)$.

Let $L_n(R)$ denote the set of homogeneous polynomials of degree n in two variables with coefficients in R. One has an action of $\mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{Mat}_2(\mathbb{Z})$ on $L_n(R)$ given by

$$\gamma \cdot P(x, y) = P((x, y)\det(\gamma)\gamma^{-1}).$$

Let $L = \mathrm{H}^1(\Gamma_1(M), L_{\kappa-2}(\mathcal{O}))_{\mathfrak{m}_f}$ and $L^\pm = \mathrm{H}^1(\Gamma_1(M), L_{\kappa-2}(\mathcal{O}))^\pm_{\mathfrak{m}_f}$ where we have used that $\mathrm{H}^1(\Gamma_1(M), L_{\kappa-2}(\mathcal{O}))$ decomposes as

$$\mathrm{H}^1(\Gamma_1(M), L_{\kappa-2}(\mathcal{O})) = \mathrm{H}^1(\Gamma_1(M), L_{\kappa-2}(\mathcal{O}))^+ \oplus \mathrm{H}^1(\Gamma_1(M), L_{\kappa-2}(\mathcal{O}))^-$$

with respect to the Atkin–Lehner involution. Since we are under the assumption that $\ell > \kappa$ and $\ell \nmid M$, we can use [45, Theorem 1.13] to conclude that the $\mathbb{T}_{\mathfrak{m}_f}$-modules $L^\pm$ are free of rank 1.

We define a differential form with values in $L_{\kappa-2}(\mathbb{C})$ by

$$\omega_f = f(z)(zx + y)^{\kappa-2}dz.$$

Then

$$\gamma \mapsto \int_{z_0}^{\gamma z_0} \omega_f$$

defines a 1-cocycle on $\Gamma_1(M)$, where z_0 is any basepoint in the upper half-plane. So f gives rise to a vector-valued differential form on the upper half plane and cohomology class ω_f in

$H^1(\Gamma_1(M), L_{\kappa-2}(\mathbb{C}))$. The cohomology class ω_f is an eigenvector for the Hecke operators, with the same eigenvalues as f.

Since we are assuming that $\ell > \kappa$, $\overline{\rho}_{f,\lambda}$ is irreducible and $M \geq 4$ is prime to ℓ, [45, Theorem 1.13] gives isomorphisms

$$\Theta^{\pm} : L^{\pm} \cong S_\kappa(\Gamma_1(M), \mathcal{O})_{\mathfrak{m}_f}.$$

Let $\delta_f^{\pm}$ be the cohomology classes in $L^{\pm}$ given by

$$\Theta^{\pm}(\delta_f^{\pm}) = f.$$

Let $\omega_f^{\pm}$ denote the projection of ω_f to the $\pm$-part in $H^1(\Gamma_1(M), L_{\kappa-2}(\mathbb{C}))$, f being a newform then implies that there exist complex numbers $\Omega_f^{\pm}$ such that

$$\omega_f^{\pm} = \Omega_f^{\pm}\delta_f^{\pm}.$$

The numbers $\Omega_f^{\pm}$ are canonical (determined up to ℓ-adic units), where the dependence comes from the choice of isomorphism

$$\Theta^{\pm} : L^{\pm} \cong S_\kappa(\Gamma_1(M), \mathcal{O})_{\mathfrak{m}_f}.$$

Theorem 3.2 [20,45] *Let $f \in S_\kappa(\Gamma_1(M), \mathcal{O})$ be a newform. Let $\ell > \kappa$ be a prime number with $\ell \nmid M$. For each integer j with $0 < j < \kappa$ and every Dirichlet character χ one has*

$$\frac{L(j, f, \chi)}{\tau(\chi)(2\pi i)^j} \in \begin{cases} \Omega_f^{-}\mathcal{O}_\chi & \text{if } \chi(-1) = (-1)^j, \\ \Omega_f^{+}\mathcal{O}_\chi & \text{if } \chi(-1) = (-1)^{j-1}, \end{cases}$$

where $\tau(\chi)$ is the Gauss sum of χ and $\mathcal{O}_\chi$ is the extension of $\mathcal{O}$ generated by the values of χ. We write

$$L_{\mathrm{alg}}(j, f, \chi) = \frac{L(j, f, \chi)}{\tau(\chi)(2\pi i)^j\Omega_f^{\pm}}$$

where the appropriate $\Omega_f^{\pm}$ is chosen as described above.

3.7 Construction of a certain Hecke operator

With the notation as in Sect. 3.6 we now outline the construction of a special Hecke operator that we will need later in this work.

Let M be a square-free positive integer, $\kappa \geq 2$ an integer, $\ell > \kappa$ be a prime with $\ell \nmid M$, and let K be a suitably large finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}$. Let $f \in S_\kappa(\Gamma_1(M), \mathcal{O})$ be a newform such that $\overline{\rho}_{f,\lambda}$ is irreducible. Since f is a newform we can write $S_{2\kappa-2}(\Gamma_1(M)) = \mathbb{C}f \oplus \mathfrak{S}$ with $\mathfrak{S}$ stable under the Hecke algebra. We denote the map from $\mathbb{T}_{\mathfrak{m}_f} \to \mathcal{O}$ given by $t \mapsto \lambda_f(t)$ by π_f. The fact that f is a newform allows us to write

$$\mathbb{T}_{\mathfrak{m}_f} \otimes_{\mathcal{O}} K = K \oplus D$$

with a K-algebra D so that π_f induces the projection of $\mathbb{T}_{\mathfrak{m}_f}$ onto K. In this direct sum, K corresponds to the Hecke algebra acting on the eigenspace generated by f and D corresponds to the Hecke algebra acting on the space $\mathfrak{S}$. Let ϱ be the projection map of $\mathbb{T}_{\mathfrak{m}_f}$ to D. Set I_f to be the kernel of ϱ. We can use that $\mathbb{T}_{\mathfrak{m}_f}$ is reduced to conclude that

$$I_f = \mathrm{Ann}(\wp_f)$$

where $\mathrm{Ann}(\wp_f)$ denotes the annihilator of the ideal $\wp_f$ and $\wp_f \cap I_f = 0$. Therefore we have that

$$\mathbb{T}_{\mathfrak{m}_f}/(\wp_f \oplus I_f) = \mathbb{T}_{\mathfrak{m}_f}/(\wp_f, I_f) \xrightarrow{\simeq} \mathcal{O}/\pi_f(I_f)$$

where we have used that

$$\pi_f : \mathbb{T}_{\mathfrak{m}_f}/\wp_f \xrightarrow{\simeq} \mathcal{O}.$$

The fact that $\mathcal{O}$ is a principal ideal domain implies that there exists $\alpha_f \in \mathcal{O}$ so that $\pi_f(I_f) = \alpha_f \mathcal{O}$. Thus, we have

$$\mathcal{O}/\alpha_f \mathcal{O} \cong \mathbb{T}_{\mathfrak{m}_f}/(\wp_f, I_f).$$

Since $\mathbb{T}_{\mathfrak{m}_f}/\wp_f \cong \mathcal{O}$, there exists a $t_f \in I_f$ that maps to α_f under the above isomorphism. Thus we have that $t_f f = \alpha_f f$ and $t_f g = 0$ for all $g \in \mathfrak{S}$. Recalling that $\mathbb{T}_{\mathcal{O}} = \prod_{\mathfrak{m}} \mathbb{T}_{\mathfrak{m}}$, we can view t_f as an element of $\mathbb{T}_{\mathcal{O}}$ and this is the Hecke operator we seek. We now calculate the eigenvalue α_f.

For $L = \mathrm{H}^1(\Gamma_1(M), L_{\kappa-2}(\mathcal{O}))_{\mathfrak{m}_f}$ there is a skew-symmetric perfect pairing

$$L^{\pm} \times L^{\mp} \to \mathcal{O}$$

adjoint with respect to the Hecke operators [20, Equation 3.3]. We write this pairing as $(x, y) \mapsto A(x, y)$. Note that to give such a perfect pairing is equivalent to giving an isomorphism

$$L^{\pm} \to \mathrm{Hom}_{\mathcal{O}}(L^{\mp}, \mathcal{O})$$

(the equivalence is given by $x \mapsto A(x, \cdot)$.)

Lemma 3.3 *The ideal $\alpha_f \mathcal{O}$ is generated by $A(\delta_f^+, \delta_f^-)$ where δ_f^+, δ_f^- are as defined in Sect. 3.6.*

Proof (See [13, Lemma 4.17]) □

We now calculate $A(\delta_f^+, \delta_f^-)$ in terms of $\Omega_f^{\pm}$ and $\langle f, f \rangle$. Applying [13, Theorem 4.20] we obtain (up to ℓ-adic unit in $\mathcal{O}$)

$$A(\delta_f^+, \delta_f^-) = \frac{(\kappa-1)! M \varphi(M)}{(-2\pi i)^{\kappa+1} \Omega_f^+ \Omega_f^-} L(\kappa, \mathrm{Sym}^2 f).$$

We note that $\det(A)$ in Theorem 4.20, [13] is the same as $\Omega_f^+ \Omega_f^-$ up to ℓ-units.

We combine this with the following equation of Shimura [35, Equation 2.5]

$$L(\kappa, \mathrm{Sym}^2 f) = \frac{2^{2\kappa} \pi^{\kappa+1}}{3(\kappa-1)!} \langle f, f \rangle$$

to obtain

$$A(\delta_f^+, \delta_f^-) = \frac{(-1)^{(\kappa+1)/2} M \varphi(M) 2^{\kappa-1}}{3} \frac{\langle f, f \rangle}{\Omega_f^+ \Omega_f^-}.$$

One should note that in the case that $M = 1, 3$ one calculates $\alpha_f \mathcal{O}$ via results of Hida if one assumes that f is ordinary at λ. For this calculation one can see [7, §5.2]. Summarizing, we have the following result.

Theorem 3.4 *Let M be a square-free positive integer, $\kappa \geq 2$ an integer, $\ell > \kappa$ be a prime with $\ell \nmid M$, and let K be a suitably large finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}$. Let $f \in S_\kappa(\Gamma_1(M), \mathcal{O})$ be a newform such that $\overline{\rho}_{f,\lambda}$ is irreducible. If $M = 1, 3$ we further assume that f is ordinary at ℓ. Then there exists $t_f \in \mathbb{T}_{\mathcal{O}}$ so that $t_f f = \alpha_f f$ with $\alpha_f = u\frac{\langle f,f\rangle}{\Omega_f^+\Omega_f^-}$ for $u \in \mathcal{O}^\times$ and $t_f g = 0$ if $g \in (\mathbb{C}f)^\perp$.*

3.8 The Bloch–Kato conjecture and the main theorem

In this section we recall the Bloch–Kato conjecture for our case of interest and relate it with the results in this work. We will follow the expositions found in [14] and [22, §9.3], which is also where one can consult for further details.

Let E_0 be a number field and $\mathcal{V}$ be a premotivic structure over $\mathbb{Q}$ with coefficients in E_0 as in [14], i.e.,

$$\mathcal{V} = \{\mathcal{V}_{\mathrm{B}}, \mathcal{V}_{\mathrm{dR}}, \{\mathcal{V}_\upsilon\}_\upsilon, I^\infty, \{I_{\mathrm{B}}^\upsilon\}_\upsilon, \{I^\upsilon\}_\upsilon, \{W^i\}_i\}$$

where

- υ runs over the finite places of E_0;
- $\mathcal{V}_{\mathrm{B}}$ is a finite dimensional E_0 vector space with an action of $G_{\mathbb{R}}$;
- $\mathcal{V}_{\mathrm{dR}}$ is a finite dimensional E_0 vector space with a finite decreasing filtration Fil^i;
- for each υ, $\mathcal{V}_\upsilon$ is a finite dimensional $E_{0,\upsilon}$ vector space with a continuous pseudo-geometric action of $G_{\mathbb{Q}}$;
- I^∞ is a $\mathbb{C} \otimes E_0$-linear isomorphism

$$I^\infty : \mathbb{C} \otimes \mathcal{V}_{\mathrm{dR}} \to \mathbb{C} \otimes \mathcal{V}_{\mathrm{B}}$$

  that respects the action of $G_{\mathbb{R}}$ where $G_{\mathbb{R}}$ acts on $\mathbb{C} \otimes \mathcal{V}_{\mathrm{dR}}$ via the first factor and acts on $\mathcal{V}_{\mathrm{B}}$ diagonally;
- for each υ, I_{B}^υ is a $E_{0,\upsilon}$-linear isomorphism

$$I_{\mathrm{B}}^\upsilon : E_{0,\upsilon} \otimes_{E_0} \mathcal{V}_{\mathrm{B}} \to \mathcal{V}_\upsilon$$

  that respects the action of $G_{\mathbb{R}}$ where $G_{\mathbb{R}}$ acts on $\mathcal{V}_\upsilon$ via the restriction $G_{\mathbb{R}} \to G_{\mathbb{Q}}$ determined by the choice of embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$;
- for each υ, I^υ is a $B_{\mathrm{dR},p} \otimes_{\mathbb{Q}_p} E_{0,\upsilon}$-linear isomorphism

$$B_{\mathrm{dR},p} \otimes_{\mathbb{Q}_p} E_{0,\upsilon} \otimes_{E_0} \mathcal{V}_{\mathrm{dR}} \to B_{\mathrm{dR},p} \otimes_{\mathbb{Q}_p} \mathcal{V}_\upsilon$$

  that respects filtrations and the action of $G_{\mathbb{Q}_p}$ where p is the prime υ divides, $E_{0,\upsilon}$ and $\mathcal{V}_\upsilon$ are given the degree-0 filtration, $E_{0,\upsilon}$ and V_{dR} are given the trivial $G_{\mathbb{Q}_p}$-action, the action on M_υ is determined by the choice of embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and $B_{\mathrm{dR},p}$ is defined as in [18];
- W^i are increasing weight filtrations on $\mathcal{V}_{\mathrm{B}}$, $\mathcal{V}_{\mathrm{dR}}$, and each $\mathcal{V}_\upsilon$ that respects all the above data and such that $\mathbb{R} \otimes \mathcal{V}_{\mathrm{B}}$ with its Galois action and weight filtration along with the Hodge filtration on $\mathbb{C} \otimes \mathcal{V}_{\mathrm{B}}$ defines a mixed Hodge structure over $\mathbb{R}$.

The fundamental lines for $\mathcal{V}$ are the E_0-lines defined by

$$\Delta_f^\pm(\mathcal{V}) = \mathrm{Hom}_{E_0}(\det_{E_0} \mathcal{V}_{\mathrm{B}}^\pm, \det_{E_0} t_{\mathcal{V}})$$

where the $\pm$ sign indicates $\pm$-isotypic subspace under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, $t_{\mathcal{V}} = \mathcal{V}_{\mathrm{dR}}/\operatorname{Fil}^0 \mathcal{V}_{\mathrm{dR}}$, and det denotes the highest exterior power. From the map I^{∞} we obtain $\mathbb{R} \otimes E_0$-linear isomorphisms

$$I^{\pm} : \mathbb{R} \otimes \mathcal{V}_{\mathrm{B}}^{\pm} \to (\mathbb{C} \otimes \mathcal{V}_{\mathrm{B}}) \xrightarrow{(I^{\infty})^{-1}} \mathbb{C} \otimes \mathcal{V}_{\mathrm{dR}} \to (\mathbb{C} \otimes \mathcal{V}_{\mathrm{dR}})^{\pm}.$$

The determinant of $I^{\pm}$ over $\mathbb{R} \otimes E_0$ defines a basis $c^{\pm}(\mathcal{V})$ for $\mathbb{R} \otimes \Delta_f^{\pm}(\mathcal{V})$. The bases $c^{\pm}(\mathcal{V})$ are referred to as the Deligne periods. The periods $c^{\pm}(\mathcal{V})$ are canonically defined.

We now specialize to the case of interest for this paper. Let κ and M be positive integers with κ even and M odd and square-free. Let $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$ be a newform. Let E_0 be a number field that is sufficiently large. In particular, we will always assume E_0 contains the eigenvalues of f. Let ℓ be an odd prime with $\ell > \kappa$, $\ell \nmid M$, and λ the prime over ℓ fixed by our choice of embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_{\ell}$. We assume that $\overline{\rho}_{f,\lambda}$ irreducible. Set $E = E_{0,\lambda}$ and let $\mathcal{O}$ be the ring of integers of E. Let $\widetilde{\mathcal{V}}$ be the premotivic structure defined over $\mathbb{Q}$ with coefficients in E_0 attached to our newform $f \in S_{2\kappa-2}(\Gamma_0(M))$ as in [34] and let $\mathcal{V} = \widetilde{\mathcal{V}}(\kappa)$. Let $\Sigma' = \{p|M\}$ and Σ a finite set of places containing $\Sigma' \cup \{\ell\}$. Note that for an integer j we have $L(s, \widetilde{\mathcal{V}}(j)) = L(s + j, f)$, so, in particular, $L(s, \mathcal{V}) = L(s + \kappa, f)$. From this point on write $\pm$ to denote the parity of κ. In this case, one knows Deligne's conjecture that there is a basis $b^{\pm}(\mathcal{V})$ of $\Delta_f^{\pm}(\mathcal{V})$ so that

$$L(0, \mathcal{V})(1 \otimes b^{\pm}(\mathcal{V})) = c^{\pm}(\mathcal{V}).$$

Choose a Galois stable $\mathcal{O}$-lattice $\mathcal{T}_{\lambda} \subset \mathcal{V}_{\lambda}$ and a free rank one $\mathcal{O}$-module $\omega \subset E \otimes \det_{E_0} t_{\mathcal{V}}$. Let $\theta^{\pm}(\mathcal{V}_{\lambda}) = \det_{\mathcal{O}} \mathcal{T}_{\lambda}^{\pm}$ regarded as a lattice in $E \otimes_{E_0} \det_{E_0} \mathcal{V}_{\mathrm{B}}^{\pm}$ via the comparison isomorphism I_{B}^{λ}. Set $\mathcal{T}_{\lambda}^{D} = \mathrm{Hom}_{\mathcal{O}}(\mathcal{T}_{\lambda}, O(1)) \subset \mathcal{V}_{\lambda}^{D} = \mathrm{Hom}_{E}(\mathcal{V}_{\lambda}, E(1))$. Then the Shafarevich–Tate group of $\mathcal{T}_{\lambda}$ is given by

$$Ш(\mathcal{T}_{\lambda}) = \frac{\mathrm{Sel}_{\Sigma}(\mathcal{V}_{\lambda}/\mathcal{T}_{\lambda})}{\mathrm{Sel}_{\Sigma}(\mathcal{V}_{\lambda}) \otimes E/\mathcal{O}}.$$

The precise definition of the various Selmer groups mentioned here can be found in Sect. 8.1. In particular, one knows this is always finite.

The Tamagawa ideal of $\mathcal{T}_{\lambda}$ relative to ω is given by

$$\mathrm{Tam}_{\omega}^{0}(\mathcal{T}_{\lambda}) = \mathrm{Tam}_{\ell,\omega}^{0}(\mathcal{T}_{\lambda})\, \mathrm{Tam}_{\infty}^{0}(\mathcal{T}_{\lambda}) \prod_{p \neq \ell} \mathrm{Tam}_{p}^{0}(\mathcal{T}_{\lambda})$$

where one can see [14] for the definition of the individual factors. In particular, for $p \nmid \ell M$ we have $\mathrm{Tam}_{p}^{0}(\mathcal{T}_{\lambda}) = \mathcal{O}$ and since ℓ is odd $\mathrm{Tam}_{\infty}^{0}(\mathcal{T}_{\lambda}) = \mathcal{O}$.

View $\mathrm{Hom}_{\mathcal{O}}(\theta^{\pm}(\mathcal{T}_{\lambda}), \omega)$ as a lattice in $E \otimes_{E_0} \Delta_f^{\pm}(\mathcal{V})$. Define

$$\delta_{f,\lambda}^{\pm}(\mathcal{V}) = \frac{\mathrm{Fitt}_{\mathcal{O}}\, H^0(\mathbb{Q}, \mathcal{V}_{\lambda}/\mathcal{T}_{\lambda}) \cdot \mathrm{Fitt}_{\mathcal{O}}\, H^0(\mathbb{Q}, \mathcal{V}_{\lambda}^{D}/\mathcal{T}_{\lambda}^{D})}{\mathrm{Fitt}_{\mathcal{O}}\, Ш(\mathcal{T}_{\lambda}) \cdot \mathrm{Tam}_{\omega}^{0}(\mathcal{T}_{\lambda})} \mathrm{Hom}_{\mathcal{O}}(\theta^{\pm}(\mathcal{T}_{\lambda}), \omega).$$

The λ-part of the Bloch–Kato conjecture can now be stated as follows.

Conjecture 3.5 (*λ-part of Bloch–Kato*) We have

$$\delta_{f,\lambda}^{\pm}(\mathcal{V}) = (1 \otimes b^{\pm}(\mathcal{V}))\mathcal{O}.$$

We will reframe this conjecture in terms of $\mathrm{Sel}_{\Sigma}(\Sigma', \mathcal{V}_{\lambda}/\mathcal{T}_{\lambda})$ to make it compatible with the terminology and results in this article (Sect. 8.2). Note that the main result in [21] gives that $\mathrm{Sel}_{\Sigma}(\mathcal{V}_{\lambda})$ is trivial and so

$$Ш(\mathcal{T}_\lambda) = \mathrm{Sel}_\Sigma(\mathcal{V}_\lambda/\mathcal{T}_\lambda).$$

Furthermore, the main result of [17] gives that

$$Ш(\mathcal{T}_\lambda)^\vee \cong Ш(\mathcal{T}_\lambda^D)$$

where for a locally compact abelian group G we write $G^\vee$ for the Pontryagin dual

$$G^\vee = \mathrm{Hom}_{\mathbb{Z}_\ell}(G, \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Set $b^{\pm,\Sigma'}(\mathcal{V}) = \prod_{p\in\Sigma'} L_p(0,\mathcal{V}) b^\pm(\mathcal{V})$. One can use the proof of I.4.2.2 in [19] to see that for $p \in \Sigma'$,

$$\mathrm{Fitt}_\mathcal{O} H^1_f(\mathbb{Q}_p, \mathcal{T}_\lambda) = L_p(0,\mathcal{V})^{-1}\, \mathrm{Tam}^0_p(\mathcal{T}_\lambda).$$

One has an exact sequence

$$0 \to \mathrm{Sel}_\Sigma(\mathcal{V}_\lambda^D/\mathcal{T}_\lambda^D) \to \mathrm{Sel}_\Sigma(\Sigma', \mathcal{V}_\lambda^D/\mathcal{T}_\lambda^D) \to \bigoplus_{p\in\Sigma'\cup\{\ell\}} H^1_f(\mathbb{Q}_p, \mathcal{T}_\lambda^D)^\vee$$
$$\to H^0(\mathbb{Q}, \mathcal{V}_\lambda^D/\mathcal{T}_\lambda^D)^\vee \to 0$$

via [14, Lemma 2.1]. Using this, the λ-part of the Bloch–Kato conjecture can be rewritten as follows.

Conjecture 3.6 (λ-*part of Bloch–Kato*)

$$\frac{\mathrm{Fitt}_\mathcal{O} H^0(\mathbb{Q}, \mathcal{V}_\lambda^D/\mathcal{T}_\lambda^D)}{\mathrm{Fitt}_\mathcal{O} \mathrm{Sel}_\Sigma(\Sigma', \mathcal{V}_\lambda^D/\mathcal{T}_\lambda^D)\, \mathrm{Tam}^0_{\ell,\omega}(\mathcal{T}_\lambda)} \mathrm{Hom}_\mathcal{O}(\theta^\pm(\mathcal{T}_\lambda), \omega) = (1 \otimes b^{\pm,\Sigma'}(\mathcal{V}))\mathcal{O}.$$

Since $\overline{\rho}_{f,\lambda}$ is irreducible, necessarily $H^0(\mathbb{Q}, \mathcal{V}_\lambda^D/\mathcal{T}_\lambda^D)$ is trivial.

The integral structures $\mathcal{T}_\lambda$ and ω give an identification of $E \otimes \Delta_f^\pm(M)$ with E. The quotient $\frac{\mathrm{Hom}_\mathcal{O}(\theta^\pm(\mathcal{T}_\lambda),\omega)}{\mathrm{Tam}^0_{\ell,\omega}(\mathcal{T}_\lambda)}$ is identified with a fractional ideal of E. We denote the inverse of this by $\mathrm{Tam}_\omega(\mathcal{T}_\lambda)$. Applying [15, Lemma 4.6] one concludes that the λ-part of $\mathrm{Tam}_\omega(\mathcal{T}_\lambda)$ is trivial, i.e., $\mathrm{Tam}_\omega(\mathcal{T}_\lambda) = \mathcal{O}$.

We also identify $(1 \otimes b^{\pm,\Sigma'}(\mathcal{V}))\mathcal{O}$ with a fractional ideal $\left(\frac{\Omega^\pm_\omega(\mathcal{T}_\lambda)}{L^{\Sigma'}(0,\mathcal{V})}\right)\mathcal{O}$ of E for some $\Omega^\pm_\omega(\mathcal{T}_\lambda) \in E^\times/\mathcal{O}^\times$. If we choose $\mathcal{T}_\lambda$ as in [15, §5] for an appropriate choice of ω we have that $\Omega^\pm_\omega(\mathcal{T}_\lambda) = u(2\pi i)^\kappa \Omega^\pm_f$ via [15, Lemma 4.1] for u a λ-unit. Since the order of the Selmer group is independent of the choice of lattice we use this lattice from now on.

Note that

$$\begin{aligned}\mathcal{V}_\lambda^D/\mathcal{T}_\lambda^D &\cong \mathcal{V}_\lambda(-2)/\mathcal{T}_\lambda(-2)\\ &\cong \widetilde{\mathcal{V}}_\lambda(\kappa-2)/\widetilde{\mathcal{T}}_\lambda(\kappa-2).\end{aligned}$$

Using this, we can rewrite the λ-part of the Bloch–Kato conjecture as follows.

Conjecture 3.7 (λ-*part of Bloch–Kato*) Let $f \in S_{2\kappa-2}(\Gamma_0(M))$ be a newform with M odd and square-free. Let E_0 be a number field containing $\mathbb{Q}(f)$. Let ℓ be an odd prime, λ a prime dividing ℓ, and $E = E_{0,\lambda}$ and $\mathcal{O}$ the ring of integers of E. Let $(\rho_{f,\lambda}, V_{f,\lambda})$ be the Galois representation of f. Choose a lattice as above and assume $\overline{\rho}_{f,\lambda}$ is irreducible. Then

$$\#X_\Sigma(\Sigma', W_{f,\lambda}(\kappa-2))\mathcal{O} = L^\Sigma_{\mathrm{alg}}(\kappa, f)\mathcal{O}$$

as fractional ideals of E.

We now state the main result of this article whose proof we defer until Sect. 8.2.

Theorem 3.8 *Let κ and M be positive integers with $\kappa \geq 6$ even and M odd and squarefree. Let $f \in S_{2\kappa-2}^{\text{new}}(\Gamma_0(M))$ be a newform. Let ℓ be an odd prime with $\ell > 2\kappa - 2$, $\ell \nmid M$, $\ell \nmid (p^2-1)$ for all $p \mid M$, $\mathcal{O}$ a sufficiently large extension of $\mathbb{Z}_\ell$, λ the prime of $\mathcal{O}$, $\overline{\rho}_{f,\lambda}$ irreducible, and $\lambda \mid L_{\text{alg}}(\kappa, f)$. Let $\Sigma' = \{p \mid M\}$ and $\Sigma = \Sigma' \cup \{\ell\}$. If there exists a fundamental discriminant $D < 0$ so that $\gcd(\ell M, D) = 1$, and $\chi_D(-1) = -1$, and an integer $N > 1$ with $M \mid N$, $\ell \nmid N$, and an even Dirichlet character χ of conductor N so that*

$$\text{ord}_\lambda\left(\frac{L^N(3-\kappa,\chi)L_{\text{alg}}(\kappa-1,f,\chi_D)L_{\text{alg}}(1,f,\chi)L_{\text{alg}}(2,f,\chi)}{L_{\text{alg}}(\kappa,f)}\right) = -b < 0,$$

then

$$\text{ord}_\ell(\#X_\Sigma(\Sigma', W_{f,\lambda}(\kappa-2))) \geq b.$$

In particular, if N, χ, and D can be chosen so that

$$\text{ord}_\lambda(L^N(3-\kappa,\chi)L_{\text{alg}}(\kappa-1,f,\chi_D)L_{\text{alg}}(1,f,\chi)L_{\text{alg}}(2,f,\chi)) = 0, \tag{1}$$

then we have

$$\text{ord}_\ell(\#X_\Sigma(\Sigma', W_{f,\lambda}(\kappa-2))) \geq \text{ord}_\ell(\#\mathcal{O}/L_{\text{alg}}(\kappa,f)).$$

The fact that $\ell \nmid (p^2-1)$ for all $p \mid M$ gives that $L_\Sigma(\kappa, f)$ is a λ-unit, so $L_{\text{alg}}^\Sigma(\kappa, f)\mathcal{O} = L_{\text{alg}}(\kappa, f)\mathcal{O}$. As such, we can rewrite the previous theorem to be in terms of $L_{\text{alg}}^\Sigma(\kappa, f)$. Again noting that the order of the Selmer group does not depend upon the lattice chosen, we have under the conditions given in the theorem that $L_{\text{alg}}^\Sigma(\kappa, f)\mathcal{O} \subset \#X_\Sigma(\Sigma', W_{f,\lambda}(\kappa-2))\mathcal{O}$, i.e., under these conditions we obtain half of the conjecture.

Remark 3.9 One should note that we could rephrase our results in terms of $X_\Sigma(W_{f,\lambda}(\kappa-2))$ and $L_{\text{alg}}(\kappa, f)$ by adding in the condition that for no prime $p \mid M$ is f congruent modulo λ to a newform of weight $2\kappa-2$, trivial character, and level dividing M/p. For the arguments needed to change our results to align with this, one can see [6,15]. We chose to phrase our results as above to avoid this extra congruence condition.

3.9 Example

We work out a complete example of our main theorem. Let $\kappa = 22$, $M = 1$, and $\ell = 1423$. We consider the space of newforms $S_{42}^{\text{new}}(\text{SL}_2(\mathbb{Z}))$. There are three Galois conjugate newforms in this space which we label as f_1, f_2, f_3. We have

$$\begin{aligned} f_1 = q + \alpha_1 q^2 &+ \left(-\frac{1}{576}\alpha_1^2 - \frac{783}{4}\alpha_1 + 3817732324\right)q^3 \\ &+ \left(\alpha_1^2 - 2199023255552\right)q^4 \\ &+ \left(-\frac{615}{16}\alpha_1^2 - 83724625\alpha_1 + 84494100859110\right)q^5 + \cdots \end{aligned}$$

where α_1 is a root of

$$g_1(x) = x^3 + 344688x^2 - 6374982426624x - 520435526440845312.$$

Let α_2 and α_3 be the other roots of $g(x)$ so that f_i is defined over $K_i = \mathbb{Q}(\alpha_i)$. Let $\mathcal{O}_{K_i}$ be the ring of integers of K_i and note that $L_{\text{alg}}(22, f_i) \in \mathcal{O}_{K_i}$.

We use MAGMA to determine that

$$1423 \mid \prod_{i=1}^{3} L_{\mathrm{alg}}(22, f_i).$$

We have that for each i there exists a prime λ_i of $\mathcal{O}_{K_i}$ over 1423 so that $\lambda_i \mid L_{\mathrm{alg}}(22, f_i)$. Note these primes are all Galois conjugate.

Using SAGE we check that

$$1423 \nmid L(-19, \chi_5)$$

and again using MAGMA we check that

$$1423 \nmid \prod_{i=1}^{3} L_{\mathrm{alg}}(j, f_i, \chi_5)$$

for $j = 1, 2$ and also check that

$$1423 \nmid \prod_{i=1}^{3} L_{\mathrm{alg}}(21, f_i, \chi_{-3}).$$

One uses SAGE to check that f_i is ordinary at λ_i. Thus, it only remains to show that the residual Galois representation associated to f_i is irreducible. Let $F_i = K_{i,\lambda_i}$ with ring of integers $\mathcal{O}_i$ and uniformizer ϖ_i. Set $\mathbb{F}_i = \mathcal{O}_i/\varpi_i = \mathbb{F}_\ell[\overline{\alpha}_i]$ where $\overline{\alpha}_i = \alpha_i \pmod{\lambda_i}$. Let $\rho_i : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}_i)$ be the Galois representation associated to f_i. We know ρ_i is unramified away from ℓ and crystalline at ℓ. Write $\overline{\rho}_i : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_i)$ for the residual representation. Suppose that $\overline{\rho}_i$ is reducible. One knows that $\overline{\rho}_i$ is nonsplit and we can write

$$\overline{\rho}_i = \begin{pmatrix} \phi & * \\ 0 & \psi \end{pmatrix}$$

with $* \neq 0$. Write $\omega : \mathrm{G}_{\mathbb{Q}} \to \mathbb{F}_\ell^\times$ for the mod ℓ cyclotomic character normalized geometrically. We have $\phi\psi = \omega^{41}$ and since ϕ and ψ are unramified away from ℓ and of order prime to ℓ we can write $\phi = \omega^a$ and $\psi = \omega^b$ with $0 \le a < b < \ell - 1$, and $a + b = 41$ or $a + b = \ell - 1 + 41$. One argues as in [29] to conclude that $*$ gives a non-zero cocycle class in $H^1(\mathbb{Q}, \mathbb{F}_i(a - b))$ since $a - b < 0$. This gives that $\ell \mid B_{b-a+1}$ where B_n denotes the nth Bernoulli number. However, one checks that for such a and b there are no Bernoulli numbers B_{b-a+1} divisible by 1423. Thus, it must be that the residual representation is irreducible and so we have our example. In particular, we have

$$1423 \mid \#X_\ell(W_{f_i,\lambda_i}(20)).$$

More computations can be found in [2].

4 Siegel modular forms

4.1 Siegel modular forms: definitions

Let $\iota_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$ and set

$$G_n = \mathrm{GSp}_{2n} = \{g \in \mathrm{GL}_{2n} : {}^t g \iota_n g = \mu_n(g)\iota_n, \mu_n(g) \in \mathrm{GL}_1\}.$$

Set $\mathrm{Sp}_{2n} = \ker(\mu_n)$. Given an element $g \in G_n$, we will often write $g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$ with $a_g, b_g, c_g, d_g \in \mathrm{Mat}_n$.

We denote the Siegel upper half space by

$$\mathfrak{h}^n = \{z \in \mathrm{Mat}_n(\mathbb{C}) : {}^t z = z, \mathrm{Im}(z) > 0\}.$$

We have an action of $G_n^+(\mathbb{R}) = \{\gamma \in G_n(\mathbb{R}) : \mu_n(\gamma) > 0\}$ on $\mathfrak{h}^n$ given by

$$\gamma z = (a_\gamma z + b_\gamma)(c_\gamma z + d_\gamma)^{-1}$$

for $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in G_n^+(\mathbb{R})$ and $z \in \mathfrak{h}^n$.

Given an integer $M \geq 1$, we will be interested in the congruence subgroups of $\mathrm{Sp}_{2n}(\mathbb{Z})$ given by

$$\Gamma_0^{(n)}(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Z}) : c \equiv 0 \pmod{M} \right\}.$$

For $\gamma \in G_n^+(\mathbb{R})$ and $z \in \mathfrak{h}^n$, we set

$$j(\gamma, z) = \det(c_\gamma z + d_\gamma).$$

Let κ be a positive integer. Given a function $F : \mathfrak{h}^n \to \mathbb{C}$, we set

$$(F|_\kappa \gamma)(z) = \mu_n(\gamma)^{n\kappa/2} j(\gamma, z)^{-\kappa} F(\gamma z).$$

We say such an F is a (genus n) Siegel modular form of weight κ and level $\Gamma_0^{(n)}(M)$ if F is holomorphic and satisfies

$$(F|_\kappa \gamma)(z) = F(z)$$

for all $\gamma \in \Gamma_0^{(n)}(M)$. If $n = 1$ we also require that F be holomorphic at the cusps and in this way recover the theory of elliptic modular forms. We denote the space of Siegel modular forms of weight κ and level $\Gamma_0^{(n)}(M)$ by $M_\kappa(\Gamma_0^{(n)}(M))$.

The Siegel operator is a linear operator from $M_\kappa(\Gamma_0^n(M))$ to $M_\kappa(\Gamma_0^{n-1}(M))$ defined by

$$(\Phi F)(z) = \lim_{\lambda \to \infty} F\left(\begin{pmatrix} z & 0 \\ 0 & i\lambda \end{pmatrix}\right).$$

We say $F \in M_\kappa(\Gamma_0^{(n)}(M))$ is a cusp form if $\Phi(F|_\gamma) = 0$ for every $\gamma \in \mathrm{Sp}_{2n}(\mathbb{Z})$. The space of weight κ level $\Gamma_0^{(n)}(M)$ cusp forms is denoted $S_\kappa(\Gamma_0^{(n)}(M))$.

If F is a Siegel modular form, it has a Fourier expansion of the form

$$F(z) = \sum_{T \in \mathbb{S}_n^{\geq 0}(\mathbb{Z})} a_F(T) e(\mathrm{Tr}(Tz))$$

where $\mathbb{S}_n^{\geq 0}(\mathbb{Z})$ is the semi-group of n by n positive semi-definite semi-integral matrices. We have $F \in S_\kappa(\Gamma_0^{(n)}(M))$ if and only if $a_{F|\gamma}(T) = 0$ for all $\gamma \in \mathrm{Sp}_{2n}(\mathbb{Z})$ when $\det T = 0$. Given a subalgebra A of $\mathbb{C}$, we write $M_\kappa(\Gamma_0^{(n)}(M), A)$ for the space of Siegel modular forms with Fourier coefficients in A and $S_\kappa(\Gamma_0^{(n)}(M), A)$ for the space of cusp forms with Fourier coefficients in A. Let F^c denote the Siegel modular form

$$F^c(z) = \sum_{T \in \mathbb{S}_n^{\geq 0}(\mathbb{Z})} \overline{a_F(T)} e(\mathrm{Tr}(Tz)).$$

Let $F, G \in M_\kappa(\Gamma_0^{(n)}(M))$ with at least one of them a cusp form. The Petersson product of F and G is given by

$$\langle F, G\rangle_{\Gamma_0^{(n)}(M)} = \int\limits_{\Gamma_0^{(n)}(M)\backslash\mathfrak{h}^n} F(z)\overline{G(z)}(\det y)^\kappa d\mu z$$

where $z = x + iy$ with $x = (x_{\alpha,\beta})$, $y = (y_{\alpha,\beta}) \in \mathrm{Mat}_n(\mathbb{R})$,

$$d\mu z = (\det y)^{-(n+1)} \prod_{\alpha\le\beta} dx_{\alpha,\beta} \prod_{\alpha\le\beta} dy_{\alpha,\beta}$$

with $dx_{\alpha,\beta}$ and $dy_{\alpha,\beta}$ the usual Lebesgue measure on $\mathbb{R}$. For $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{Z})$, we set

$$\langle F, G\rangle = \frac{1}{[\overline{\mathrm{Sp}_{2n}(\mathbb{Z})} : \overline{\Gamma}]}\langle F, G\rangle_\Gamma.$$

where $\overline{\mathrm{Sp}_{2n}(\mathbb{Z})} = \mathrm{Sp}_{2n}(\mathbb{Z})/\{\pm 1_{2n}\}$ and $\overline{\Gamma}$ is the image of Γ in $\overline{\mathrm{Sp}_{2n}(\mathbb{Z})}$. Note that $\langle F, G\rangle$ is independent of Γ.

4.2 Hecke algebras

Given $g \in G_n^+(\mathbb{Q})$, we write $T(g)$ to denote

$$\Gamma_0^{(n)}(M)g\Gamma_0^{(n)}(M).$$

We define the usual action of $T(g)$ on Siegel modular forms by setting

$$T(g)F = \sum_i F|_\kappa g_i$$

where $\Gamma_0^{(n)}(M)g\Gamma_0^{(n)}(M) = \coprod_i \Gamma_0^{(n)}(M)g_i$ and $F \in M_\kappa(\Gamma_0^{(n)}(M))$. Let p be prime and define

$$T^{(n)}(p) = T(\mathrm{diag}(1_n, p1_n))$$

and for $i = 1, \dots, n$ set

$$T_i^{(n)}(p^2) = T(\mathrm{diag}(1_{n-i}, p1_i, p^2 1_{n-i}, p1_i)).$$

In this paper we will be primarily interested in the cases $n = 1$ and $n = 2$. The $n = 1$ case has already been discussed in Sect. 3.2.

For the case $n = 2$ we have $T^{(2)}(p)$ is the usual pth Siegel Hecke operator, $T_1^{(2)}(p^2) = T(\mathrm{diag}(1, p, p^2, p))$, which is typically denoted by $T(p^2)$ in the literature, and $T_2^{(2)}(p^2) = p^2 1_4$. To distinguish with the elliptic modular case and to ease notation, we write $T_S(p) = T^{(2)}(p)$, $T_S(p^2) = T_1^{(2)}(p^2)$, and $U_S(p)$ for $T_S(p)$ when $p \mid M$. We follow the same conventions for the Hecke algebra as in Sect. 3.2, in this case writing $\mathbb{T}_{\mathbb{Z}}^{S,\Sigma}$.

Let E be a finite extension of $\mathbb{Q}_\ell$ and $\mathcal{O}_E$ the ring of integers of E. Then we have $\mathbb{T}_{\mathcal{O}_E}^{S,\Sigma}$ is a semi-local complete finite $\mathcal{O}_E$-algebra. One has

$$\mathbb{T}_{\mathcal{O}_E}^{S,\Sigma} = \prod_{\mathfrak{m}} \mathbb{T}_{\mathfrak{m}}^{S,\Sigma}$$

where the product runs over all maximal ideals of $\mathbb{T}^{S,\Sigma}_{\mathcal{O}_E}$ and $\mathbb{T}^{S,\Sigma}_{\mathfrak{m}}$ denotes the localization of $\mathbb{T}^{S,\Sigma}_{\mathcal{O}_E}$ at $\mathfrak{m}$.

One can define congruences of Siegel modular forms, analogous to those defined for classical modular forms in Sect. 3.3.

4.3 L-functions

Let $F \in S_\kappa(\Gamma_0^{(2)}(N))$ be an eigenform. The spinor L-function associated to F is given by

$$L(s, F, \mathrm{spin}) = \prod_p L_p(s, F, \mathrm{spin})$$

where

$$L_p(s, F, \mathrm{spin}) = (1 - \lambda_F(p)p^{-s} + (\lambda_F(p)^2 - \lambda_F(p^2) - p^{2\kappa-4})p^{-2s} \\ - \lambda_F(p)p^{2\kappa-3-3s} + p^{4\kappa-6-4s})^{-1}$$

for $p \nmid N$ and

$$L_p(s, F, \mathrm{spin}) = (1 - \lambda_F(p)p^{-s})^{-1}$$

for $p \mid N$ and we write $\lambda_F(p)$ (resp. $\lambda_F(p^2)$) for the $T_S(p)$ (resp. $T_S(p^2)$) eigenvalue of F. Note that we follow Andrianov's convention here for the Euler factors at primes dividing the level. The standard L-function associated to F is given by

$$L(s, F, \mathrm{st}) = \prod_p L_p(s, F, \mathrm{st}) = \prod_p W_p(p^{-s})$$

where

$$W_p(p^{-s}) = \begin{cases} \left((1 - p^{2-s}) \prod_{i=1}^2 (1 - \alpha_{p,i}p^{2-s})(1 - \alpha_{p,i}^{-1}p^{2-s})\right)^{-1} & p \nmid N \\ \left((1 - \alpha_{p,1}p^{2-s})(1 - \alpha_{p,2}p^{2-s})\right)^{-1} & p \mid N \end{cases}$$

where the α_i are the Satake parameters of F. If χ is a Dirichlet character, we have

$$L(s, F, \chi, \mathrm{st}) = \prod_p W_p(\chi(p)p^{-s}).$$

4.4 Galois representations

We have the following theorem giving the existence of Galois representations associated to cuspidal Siegel eigenforms.

Theorem 4.1 [47] *Let $F \in S_\kappa(\Gamma_0^{(2)}(M))$ be an eigenform, $\mathbb{Q}(F)$ the number field generated by the Hecke eigenvalues of F, and λ a prime of $\mathbb{Q}(F)$ over ℓ. There exists a finite extension E of the completion of $\mathbb{Q}(F)_\lambda$ of $\mathbb{Q}(F)$ at λ and a continuous semi-simple Galois representation*

$$\rho_{F,\lambda} : G_{\mathbb{Q}} \to \mathrm{GL}_4(E)$$

unramified away from ℓM so that for all $p \nmid \ell M$ we have

$$\det(1_4 - \rho_{F,\lambda}(\mathrm{Frob}_p)p^{-s}) = L_{(p)}(s, F, \mathrm{spin}).$$

4.5 Automorphic forms

For each finite prime p, we set

$$K_{0,p}^{(n)}(M) = \left\{g \in G_n(\mathbb{Q}_p) : a_g, b_g, d_g \in \mathrm{Mat}_n(\mathbb{Z}_p), c_g \in \mathrm{Mat}_n(M\mathbb{Z}_p)\right\}.$$

From this definition it is immediate that if $p \nmid M$ we have

$$K_{0,p}^{(n)}(M) = G_n(\mathbb{Q}_p) \cap \mathrm{Mat}_{2n}(\mathbb{Z}_p).$$

At the infinite place we put

$$K_\infty^{(n)} = \left\{g \in \mathrm{Sp}_{2n}(\mathbb{R}) : g(i_n) = i_n\right\}.$$

Set

$$K_{0,f}^{(n)}(M) = \prod_p K_{0,p}^{(n)}(M)$$

and

$$K_0^{(n)}(M) = K_\infty^{(n)} \prod_p K_{0,p}^{(n)}(M).$$

Let $F \in S_\kappa(\Gamma_0^{(2)}(M))$ be a Siegel eigenform. Define a cuspidal automorphic form $F_\mathbb{A}$ on $G_2(\mathbb{A})$ associated to F by

$$F_\mathbb{A}(\gamma g_\infty k) = \mu(g_\infty)^\kappa j(g_\infty, i_2)^{-\kappa} F(g_\infty i_2)$$

for $\gamma \in G_2(\mathbb{Q})$, $g_\infty \in G_2^+(\mathbb{R})$, and $k \in K_{0,f}^{(2)}(M)$. Let V_F be the automorphic representation generated by $F_\mathbb{A}$. This representation breaks into a finite sum of irreducible cuspidal automorphic representations of $G_2(\mathbb{A})$, all of which are isomorphic. We let Π_F be one of these irreducible components. We will always mean such a representation when we take an automorphic representation associated to $F_\mathbb{A}$. We have $\Pi_F = \otimes_\upsilon \Pi_{F,\upsilon}$ where

$$\Pi_{F,\upsilon} = \begin{cases} \text{holomorphic discrete series} & \text{if } \upsilon = \infty \\ \text{unramified spherical principal series} & \text{if } \upsilon \text{ is finite, } \upsilon \nmid M. \end{cases}$$

At the places $\upsilon \mid M$ the possibilities for $\Pi_{F,\upsilon}$ are given in Table 3 of [31]. In particular, if F is a newform of level $\Gamma_0(M)$ as defined in [31], then one knows $\Pi_{F,\upsilon}$ must be of the form III(a) of Table 3. We will only need the precise local representations for F of Saito–Kurokawa type, and we give these precisely when they arise.

5 Saito–Kurokawa lifts

While the Saito–Kurokawa lifting for square-free levels is well-understood via representation theory due to the works of Piatetski-Shapiro [27] and Schmidt [33], for our work an explicit classical construction of the lifting is required. In this section we summarize the results from [3], where we construct this classical Saito–Kurokawa lifting from $S_{2\kappa-2}(\Gamma_0(M))$ to $S_\kappa(\Gamma_0^{(2)}(M))$ for M odd, square-free and under some other minor hypotheses. Note that the square-free result is also stated in [26] and the more general result in [25], which relies on [24]. In [3], we provide a complete construction of the Saito–Kurokawa lifting in the square free case as many proofs are left to the reader in [24] and there are errors in [26]; for instance, the definition of the index-shifting operator on Jacobi forms stated there is incorrect.

Theorem 5.1 [3, Theorem 3.5] *Let $\kappa \geq 2$ be an even integer and $M \geq 1$ an odd square-free integer. Let $f \in S_{2\kappa-2}^{\text{new}}(\Gamma_0(M))$ be a newform. Then there exists a nonzero cuspidal Siegel eigenform $F_f \in S_\kappa(\Gamma_0^{(2)}(M))$ satisfying*

$$L^M(s, F_f, \text{spin}) = \zeta^M(s-\kappa+1)\zeta^M(s-\kappa+2)L^M(s, f).$$

If $\mathcal{O}$ is a ring that can be embedded into $\mathbb{C}$ and f has Fourier coefficients in $\mathcal{O}$, the lift F_f can be normalized to have Fourier coefficients in $\mathcal{O}$. If $\mathcal{O}$ is a DVR, F_f can be normalized to have Fourier coefficients in $\mathcal{O}$ with at least one Fourier coefficient in $\mathcal{O}^\times$.

Definition 5.2 Let $\kappa \geq 2$ be an even integer, M an odd square-free integer, and $f \in S_{2\kappa-2}^{\text{new}}(\Gamma_0(M); \mathcal{O})$ a newform. Then we call $F_f \in S_\kappa(\Gamma_0^{(2)}(M))$ the Saito–Kurokawa lift of f. We will always assume F_f has been normalized to have Fourier coefficients in $\mathcal{O}$ with at least one Fourier coefficient in $\mathcal{O}^\times$.

Definition 5.3 Let Σ be a set of finite primes containing the primes dividing M. Let $S_\kappa^{\text{SK}}(\Gamma_0^{(2)}(M))$ denote the subspace of $S_\kappa(\Gamma_0^{(2)}(M))$ spanned by common eigenforms F of $\mathbb{T}_\mathbb{Z}^{S,\Sigma}$ such that

$$L^\Sigma(s, F, \text{spin}) = \zeta^\Sigma(s-\kappa+1)\zeta^\Sigma(s-\kappa+2)L^\Sigma(s, g)$$

for some $g \in S_{2\kappa-2}(\Gamma_0(M))$. Write $\text{SK}(f)$ for the subspace of $S_\kappa^{\text{SK}}(\Gamma_0^{(2)}(M))$ spanned by common eigenforms F of $\mathbb{T}_\mathbb{Z}^{S,\Sigma}$ such that

$$L^\Sigma(s, F, \text{spin}) = \zeta^\Sigma(s-\kappa+1)\zeta^\Sigma(s-\kappa+2)L^\Sigma(s, f).$$

We set $S_\kappa^{\text{N-SK}}(\Gamma_0^{(2)}(M))$ be the orthogonal complement of $S_\kappa^{\text{SK}}(\Gamma_0^{(2)}(M))$ in $S_\kappa(\Gamma_0^{(2)}(M))$.

Observe that Theorem 5.1 immediately gives for $f \in S_{2\kappa-2}(\Gamma_0(M))$ a newform with κ even and M odd square-free we have $\dim_\mathbb{C} \text{SK}(f) \geq 1$. Moreover, we have the following multiplicity one result.

Theorem 5.4 [33, Theorem 5.2] *Let $\kappa \geq 6$, M be odd and square-free. Let $f \in S_{2\kappa-2}^{\text{new}}(\Gamma_0(M))$ be a newform. Then the space $\text{SK}(f)$ is one dimensional.*

One should note that it is straightforward to check that $S_\kappa^{\text{SK}}(\Gamma_0^{(2)}(M)) \subset S_\kappa^*(\Gamma_0^{(2)}(M))$ where $S_\kappa^*(\Gamma_0(M))$ is the space of Maass spezialschar. In the case $M = 1$ these spaces coincide, but in general this is not known. We will not have need of the space of Maass spezialschar in this paper.

Corollary 5.5 *Let ε be the ℓ-adic cyclotomic character and let $F_f \in \text{SK}(f)$. Then*

$$\rho_{F_f,\lambda} \simeq \varepsilon^{2-\kappa} \oplus \rho_{f,\lambda} \oplus \varepsilon^{1-\kappa}.$$

Proof Follows from Theorems 4.1 and 5.1 along with the fact that our Galois representations are taken with geometric conventions. □

Corollary 5.6 *Let $\mathbb{T}_\mathbb{Z}^S$ and $\mathbb{T}_\mathbb{Z}$ be the standard Hecke algebras acting on the space of cusp forms $S_\kappa(\Gamma_0^2(M))$ and on the space of cusp forms $S_{2\kappa-2}(\Gamma_0(M))$ respectively. There is a surjection $\phi : \mathbb{T}_\mathbb{Z}^S \to \mathbb{T}_\mathbb{Z}$ which commutes with the Saito Kurokawa lift i.e.*

$$T(F_f) = F_{(\phi(T)f)}$$

We will also make use of the following result on the standard L-function of F_f.

Theorem 5.7 [9, Theorem 2.9] *Let M and N be positive integers with M odd and square-free and $M \mid N$. Let χ be a Dirichlet character of conductor N. Let $f \in S^{\text{new}}_{2\kappa-2}(\Gamma_0(M))$ be a newform and F_f the Saito–Kurokawa lift of f. The standard L-function of F_f factors as*

$$L^N(2s, F_f, \chi, \text{st}) = L^N(2s-2, \chi)L^N(2s+\kappa-3, f, \chi)L^N(2s+\kappa-4, f, \chi).$$

Theorem 5.8 [3, Corollary 5.7] *Let $\kappa \geq 2$ be an even integer, M an odd square-free integer, and $f \in S^{\text{new}}_{2\kappa-2}(\Gamma_0(M))$ a newform. Then we have*

$$\langle F_f, F_f\rangle = \mathcal{A}_{\kappa,M}\frac{a_{\zeta_D^* f}(|D|)^2}{|D|^{\kappa-3/2}}\frac{L(\kappa, f)}{\pi L(\kappa-1, f, \chi_D)}\langle f, f\rangle$$

where

$$\mathcal{A}_{\kappa,M} = \frac{M^\kappa \zeta_M(4)\zeta_M(1)^2(\kappa-1)\left(\prod_{p\mid M}(1+p^2)(1+p^{-1})\right)}{2^{\nu(M)+3}[\Gamma_0(M):\Gamma_0(4M)][\mathrm{Sp}_4(\mathbb{Z}):\Gamma_0^{(2)}(M)]},$$

$D<0$ is a fundamental discriminant, $\nu(M)$ is the number of primes dividing M, χ_D is the quadratic character associated to it and $a_{\zeta_D^* f}(|D|)$ is the Dth Fourier coefficient of the Shintani lifting $\zeta_D^* f$ of f [39].

6 Siegel Eisenstein series and construction of the congruence

In this section we define a Siegel Eisenstein series associated to a character and then make a suitable choice of a section defining the Siegel Eisenstein series. Using this series we then construct the desired congruence.

6.1 Siegel Eisenstein series: general set up

Let P_n be the Siegel parabolic subgroup of G_n given by $P_n = \{g \in G_n : c_g = 0\}$. We have that P_n factors as $P_n = N_{P_n}M_{P_n}$ where N_{P_n} is the unipotent radical

$$N_{P_n} = \left\{ n(x) = \begin{pmatrix} 1_n & x \\ 0_n & 1_n \end{pmatrix} : {}^t x = x, \quad x \in \mathrm{Mat}_n \right\}$$

and M_{P_n} is the Levi subgroup

$$M_{P_n} = \left\{ \begin{pmatrix} A & 0_n \\ 0_n & \alpha({}^tA)^{-1} \end{pmatrix} : A \in \mathrm{GL}_n, \quad \alpha \in \mathrm{GL}_1 \right\}.$$

Fix an idele class character χ and consider the induced representation

$$I(\chi) = \mathrm{Ind}_{P_n(\mathbb{A})}^{G_n(\mathbb{A})}(\chi) = \bigotimes_{\upsilon} I_\upsilon(\chi_\upsilon)$$

consisting of smooth functions $\mathfrak{f}$ on $G_n(\mathbb{A})$ that satisfy

$$\mathfrak{f}(pg) = \chi(\det(A_p))\mathfrak{f}(g)$$

for $p = \begin{pmatrix} A_p & B_p \\ 0 & D_p \end{pmatrix} \in P_n(\mathbb{A})$ and $g \in G_n(\mathbb{A})$. For $s \in \mathbb{C}$ and $\mathfrak{f} \in I(\chi)$ define

$$\mathfrak{f}(pg, s) = \chi(\det(A_p))|\det(A_pD_p^{-1})|^s\mathfrak{f}(g)$$

For υ a place of $\mathbb{Q}$ we define $I_\upsilon(\chi_\upsilon)$ and $\mathfrak{f}_\upsilon(pg, s)$ analogously. We associate to such a section the Siegel Eisenstein series

$$E_{\mathfrak{f}}(g, s) = \sum_{\gamma \in P_n(\mathbb{Q}) \backslash G_n(\mathbb{Q})} \mathfrak{f}(\gamma g, s).$$

6.2 Siegel Eisenstein series: a choice of section

For our applications we need to restrict the possible χ and pick a particular section $\mathfrak{f}$. Let $\kappa > \max\{3, n+1\}$ and $N > 1$ be integers. Let $\chi = \otimes_\upsilon \chi_\upsilon$ be an idele class character that satisfies

$$\chi_\infty(x) = \left(\frac{x}{|x|}\right)^\kappa$$
$$\chi_p(x) = 1 \quad \text{if } \ p \nmid \infty, x \in \mathbb{Z}_p^\times, \text{ and } \ x \equiv 1 \pmod{N}.$$

We choose our section $\mathfrak{f} = \otimes_\upsilon \mathfrak{f}_\upsilon$ as follows:

(1) We set $\mathfrak{f}_\infty$ to be the unique vector in $I_\infty(\chi_\infty, s)$ so that

$$\mathfrak{f}_\infty(k, \kappa) = j(k, i)^{-\kappa}$$

for all $k \in K_\infty^{(n)}$.

(2) If $p \nmid N$ we set $\mathfrak{f}_p$ to be the unique $K_{0,p}^{(n)}(N)$-fixed vector so that

$$\mathfrak{f}_p(1) = 1.$$

(3) If $p \mid N$ we set $\mathfrak{f}_p$ to be the vector given by

$$\mathfrak{f}_p(k) = \chi_p(\det(a_k))$$

for all $k \in K_{0,p}^{(n)}(N)$ with $k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$ and

$$\mathfrak{f}_p(g) = 0$$

for all $g \notin P_n(\mathbb{Q}_p) K_{0,p}^{(n)}(N)$.

This Eisenstein series is the Eisenstein series studied by Shimura [37,38]. In particular, one knows that $E_{\mathfrak{f}}$ converges absolutely and uniformly for (g, s) on compact subsets of $G_n(\mathbb{A}) \times \{s \in \mathbb{C} : \mathrm{Re}(s) > (n+1)/2\}$, it defines an automorphic form on $G_n(\mathbb{A})$ and a holomorphic function on $\{s \in \mathbb{C} : \mathrm{Re}(s) > (n+1)/2\}$ that has meromorphic continuation to $\mathbb{C}$ with at most finitely many poles. Furthermore, Langlands [23] gives a functional equation for $E_{\mathfrak{f}}$ relating the value at $(n+1)/2 - s$ to the value at s.

We can associate an Eisenstein series on $\mathfrak{h}^n \times \mathbb{C}$ to $E_{\mathfrak{f}}(g, s)$ by setting

$$E_{\mathfrak{f}}(Z, s) = \mu_n(g_\infty)^{-n\kappa/2} j(g_\infty, i_n)^\kappa E_{\mathfrak{f}}(g_\infty, s)$$

where $g_\infty \in G_n^+(\mathbb{R})$ is such that $g_\infty(i_n) = Z$. If we set $s = \kappa/2$, then $E_{\mathfrak{f}}(Z, (n+1)/2 - \kappa/2)$ is a Siegel modular form of weight κ and level $\Gamma_0^{(n)}(N)$ [36].

We follow Shimura [38] and consider

$$E_{\mathfrak{f}}^\sharp(g, s) = E_{\mathfrak{f}}(g\iota_n^{-1}, s)$$

where we recall $\iota_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. Let $E^{\sharp}_{\mathfrak{f}}(Z,s)$ be the corresponding classical Eisenstein series. Set $\mathbb{S} = \{\mathfrak{s} \in \mathrm{Mat}_n : {}^t\mathfrak{s} = \mathfrak{s}\}$, $L = \mathbb{S}(\mathbb{Q}) \cap \mathrm{Mat}_n(\mathbb{Z})$, $L' = \{\mathfrak{s} \in \mathbb{S}(\mathbb{Q}) : \mathrm{Tr}(\mathfrak{s}L) \subset \mathbb{Z}\}$, and $\mathcal{N} = N^{-1}L'$. We have a Fourier expansion for $E^{\sharp}_{\mathfrak{f}}(Z,s)$ of the form

$$E^{\sharp}_{\mathfrak{f}}(Z,s) = \sum_{h \in \mathcal{N}} a(h, Y, s) e(\mathrm{Tr}(hX))$$

for $Z = X + iY \in \mathfrak{h}^n$. Normalize $E^{\sharp}_{\mathfrak{f}}(Z,s)$ by setting

$$D_{E^{\sharp}_{\mathfrak{f}}}(Z,s) = \pi^{-n(n+2)/4} L^N(2s,\chi) \left(\prod_{j=1}^{[n/2]} L^N(4s-j,\chi^2) \right) E^{\sharp}_{\mathfrak{f}}(Z,s).$$

Theorem 6.1 *Let $\ell \geq n+1$ be an odd prime with $\ell \nmid N$. Then*

$$D_{E^{\sharp}_{\mathfrak{f}}}(Z,(n+1)/2 - \kappa/2) \in M_\kappa(\Gamma_0^{(n)}(N), \mathbb{Z}_\ell[\chi, i^{n\kappa}]).$$

Proof One can see [1] or [7] for this fact. □

The main tool we use in producing our congruence is a pullback formula. These formulas are well-known in our case due to work of Böcherer, Garrett, and Shimura. We consider the embedding

$$\mathfrak{h}^2 \times \mathfrak{h}^2 \to \mathfrak{h}^4$$
$$(z,w) \mapsto \mathrm{diag}[z,w] = \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}.$$

Define $\sigma \in G_4(\mathbb{A})$ by

$$\sigma_\upsilon = \begin{cases} 1_8 & \text{if } \upsilon \nmid N \text{ or } \upsilon = \infty \\ \begin{pmatrix} 1_4 & 0_4 \\ \begin{pmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \end{pmatrix} & 1_4 \end{pmatrix} & \text{if } \upsilon \mid N. \end{cases}$$

Applying strong approximation gives an element $\rho \in \mathrm{Sp}_8(\mathbb{Z}) \cap K_0^{(4)}(N)\sigma$ with the property that $N_\upsilon \mid a(\sigma\rho^{-1})_\upsilon - 1_4$ for every $\upsilon \mid N$. Thus, we have that $E_{\mathfrak{f}}|_\rho$ corresponds to $E_{\mathfrak{f}}(g\sigma^{-1})$. Set

$$\mathcal{E}(z,w) = D_{E_{\mathfrak{f}}|_\rho(1\times\iota_2^{-1})}(\mathrm{diag}[z,w], (5-\kappa)/2). \tag{2}$$

We can now apply [7, Theorem 4.5] and the results of [11] to conclude the following theorem.

Theorem 6.2 *Let $\kappa \geq 6$ and $N > 1$ be integers. The Eisenstein series $\mathcal{E}(z,w)$ is a holomorphic cuspform of weight κ and level $\Gamma_0^{(2)}(N)$ in each variable. Moreover, if $\ell \geq 5$ and $\ell \nmid N$ then $\mathcal{E}(z,w)$ has Fourier coefficients in $\mathbb{Z}_\ell[\chi]$. Let $F \in S_\kappa(\Gamma_0^{(2)}(N))$ be an eigenform. Then*

$$\langle \mathcal{E}(z,w), F^c(w) \rangle = \pi^{-3} \mathcal{B}_{\kappa,N} L^N(5-k, F^c, \chi, \mathrm{st}) F(z)$$

where

$$\mathcal{B}_{\kappa,N} = \frac{(-1)^\kappa 2^{2\kappa-3} \nu_N}{3[\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(N)]},$$

and $\nu_N = \pm 1$.

6.3 Constructing a congruence

We fix the following notation throughout this subsection. Let κ, M, and N be positive integers with $\kappa \geq 6$ even, M odd and square-free and $N > 1$ so that $M \mid N$. Let $f \in S_{2\kappa-2}^{\text{new}}(\Gamma_0(M))$ be a newform and $\mathbb{Q}(f)$ the number field obtained by adjoining the eigenvalues of f to $\mathbb{Q}$. Let λ be a prime of $\mathcal{O}_{\mathbb{Q}(f)}$ of residue characteristic ℓ so that $\ell > 2\kappa - 2$, $\ell \nmid N$, $\overline{\rho}_{f,\lambda}$ is irreducible, and $\lambda \mid L_{\text{alg}}(\kappa, f)$. Write $\mathcal{O}$ for the completion of $\mathcal{O}_{\mathbb{Q}(f)}$ at λ.

Let $\mathcal{E}(z, w)$ be the normalized Eisenstein series given in Eq. (2). We begin by replacing $\mathcal{E}(z, w)$ by a form of level $\Gamma_0^{(2)}(M)$ in each variable. To accomplish this, we take the trace:

$$\mathcal{E}_M(z, w) = \sum_{\gamma\times\delta\in(\Gamma_0^{(2)}(M)\times\Gamma_0^{(2)}(M))/(\Gamma_0^{(2)}(N)\times\Gamma_0^{(2)}(N))} \mathcal{E}(z, w)|_{\gamma\times\delta}.$$

It is easy to check that $\mathcal{E}_M(Z, W)$ is a Siegel modular of weight κ and level $\Gamma_0^{(2)}(M)$ in each variable separately. The q-expansion principle for Siegel modular forms [12, Prop. 1.5] gives that the Fourier coefficients of $\mathcal{E}_M(z, w)$ lie in $\mathbb{Z}_\ell[\chi]$ in light of Theorem 6.2. Observe that for $F \in S_\kappa(\Gamma_0^{(2)}(M))$ one has

$$\langle \mathcal{E}(z, w), F(w)\rangle = \langle \mathcal{E}_M(z, w), F(w)\rangle.$$

Let $F_0 = F_f, F_1, \ldots, F_m$ be an orthogonal basis of $S_\kappa^{\text{SK}}(\Gamma_0^{(2)}(M))$ and $F_{m+1}, \ldots, F_{m+r}$ be an orthogonal basis of $S_\kappa^{\text{N-SK}}(\Gamma_0^{(2)}(M))$ both consisting of eigenforms away from M. Enlarge $\mathcal{O}$ if necessary so that these bases are all defined over $\mathcal{O}$ and $\mathbb{Z}_\ell[\chi] \subset \mathcal{O}$. Using that $\mathcal{E}_M(z, w)$ is cuspidal in each variable, we can write

$$\mathcal{E}_M(z, w) = \sum_{i,j} c_{i,j} F_i(z) F_j^c(w) \tag{3}$$

for some $c_{i,j} \in \mathbb{C}$.

Lemma 6.3 *One has*

$$\mathcal{E}_M(z, w) = \sum_{i=0}^{m+r} c_{i,i} F_i(z) F_i^c(w) \tag{4}$$

where

$$c_{i,i} = \mathcal{B}_{\kappa,N} \frac{L^N(5-\kappa, F_i^c, \chi, \text{st})}{\pi^3 \langle F_i^c, F_i^c\rangle}.$$

Proof Combine Theorem 6.2 with the expansion of $\mathcal{E}(z, w)$ given in Eq. (3). □

We write $c_i := c_{i,i}$ from now on. We will use the expansion in Eq. (4) to produce our congruence. However, before doing this we kill off $F_1, \ldots, F_m \in S_\kappa^{\text{SK}}(\Gamma_0^{(2)}(M))$. This is necessary to be sure that we are not producing a congruence between two Saito–Kurokawa lifts. We use the following theorem in this regard.

Theorem 6.4 *Let $f \in S_{2\kappa-2}^{\text{new}}(\Gamma_0(M), \mathcal{O})$ a newform with κ, ℓ, M and $\mathcal{O}$ as above. Furthermore, assume that $\overline{\rho}_{f,\lambda}$ is irreducible. If $M = 1, 3$ we further assume that f is ordinary at ℓ. Then there exists $t_f^S \in \mathbb{T}_{\mathcal{O}}^S$ so that $t_f^S F_f = \alpha_f F_f$ with $\alpha_f = u_f \frac{\langle f, f\rangle}{\Omega_f^+ \Omega_f^-}$ for $u_f \in \mathcal{O}^\times$ and $t_f F_i = 0$ for $i = 1, \ldots, m$.*

Proof Given a newform $f \in S_{2\kappa-2}(\Gamma_0(M))$, we can view $f \in S_{2\kappa-2}(\Gamma_1(M))$ and obtain the Hecke operator t_f described in Theorem 3.4. We apply Corollary 5.6 to obtain t_f^S so that $t_f^S F_f = \alpha_f F_f$. In addition, using Theorem 5.4 we note that the forms $F_1, \dots, F_m$ are not in $\mathrm{SK}(f)$ and so t_f^S necessarily kills each of these forms. Thus, we have $t_f^S F_f = \alpha_f F_f$ and $t_f^S F_i = 0$ for all $i = 1, \dots, m$ and so t_f^S is the desired Hecke operator. □

We now apply t_f^S to $\mathcal{E}_M(z, w)$ in the z-variable to give

$$t_f^S \mathcal{E}_M(z, w) = c_0 \alpha_f F_f(z) F_f(w) + \sum_{i=1}^{r} c_{m+i} t_f^S F_{m+i}(z) F_{m+i}^c(w), \tag{5}$$

where we have used that $F_f^c = F_f$. In order to produce our congruence it is necessary to study the λ-divisibility of $c_0 \alpha_f$. We can apply Theorems 5.7 and Corollary 5.8 to write

$$\alpha_f c_0 = \mathcal{C}_{\kappa,M,N} \frac{L^N(3-\kappa, \chi) L(\kappa-1, f, \chi_D) L^N(1, f, \chi) L^N(2, f, \chi)}{\pi^2 L(\kappa, f) \Omega_f^+ \Omega_f^-}$$

where

$$\mathcal{C}_{\kappa,M,N} = \frac{\nu_N 2^{2\kappa+n+1} u_f |D|^{\kappa-3/2} \left(\prod_{i=1}^{n} (1+p_i^2)(1+p_i)\right)^{-1}}{3 M^{\kappa-1} (\kappa-1) \zeta_M(1)^2 \zeta_M(4) |a_{\zeta_D^* f}(|D|)|^2 [\Gamma_0^{(2)}(M) : \Gamma_0^{(2)}(N)]}$$

where $M = p_1 \cdots p_n$ and we assume D can be chosen so that $a_{\zeta_D^* f}(|D|) \neq 0$.

We begin with $\mathcal{C}_{\kappa,M,N}$. We have that $a_{\zeta_D^* f}(|D|)$ is in $\mathcal{O}$ by Stevens [42, Prop. 2.31]. Thus, everything in the numerator (respectively the denominator) of $\mathcal{C}_{\kappa,M,N}$ lies in $\mathcal{O}$. As long as we choose D so that $\ell \nmid D$, λ will not divide anything in the numerator as we assumed ℓ is odd so λ cannot divide 2. Thus, we have that

$$\mathrm{ord}_\lambda(\mathcal{C}_{\kappa,M,N}) \leq 0.$$

We can write

$$\frac{L^N(1, f, \chi) L^N(2, f, \chi)}{\Omega_f^+ \Omega_f^-} = \frac{\tau(\chi)^2 (2\pi i)^3 L_{\mathrm{alg}}(1, f, \chi) L_{\mathrm{alg}}(2, f, \chi)}{L_N(1, f, \chi) L_N(2, f, \chi)}.$$

We also know that if $\ell \nmid N$ then $L^N(3-\kappa, \chi) \in \mathbb{Z}_\ell[\chi]$. If we require that $\chi_D(-1) = -1$, then we have

$$\frac{L(\kappa-1, f, \chi_D)}{L(\kappa, f)} = \frac{\tau(\chi_D) L_{\mathrm{alg}}(\kappa-1, f, \chi_D)}{(2\pi i) L_{\mathrm{alg}}(\kappa, f)}.$$

Since $\ell \nmid DN$ we have that $\tau(\chi)$ and $\tau(\chi_D)$ are in $\mathcal{O}^\times$. To ease notation we define

$$\mathcal{L}(\kappa, \chi, D, f, N) := L^N(3-\kappa, \chi) L_{\mathrm{alg}}(\kappa-1, f, \chi_D) L_{\mathrm{alg}}(1, f, \chi) L_{\mathrm{alg}}(2, f, \chi)$$

Thus, we have that if

$$\mathrm{ord}_\lambda \left(\frac{\mathcal{L}(\kappa, \chi, D, f, N)}{L_{\mathrm{alg}}(\kappa, f)} \right) < 0,$$

then we will have $\mathrm{ord}_\lambda(\alpha_f c_0) < 0$. For example, this certainly is the case if we can choose D and χ so that $\lambda \nmid \mathcal{L}(\kappa, \chi, D, f, N)$ since by assumption we have $\lambda \mid L_{\mathrm{alg}}(\kappa, f)$. We now work under the assumption that

$$0 > \mathrm{ord}_\lambda(c_0 \alpha_f) \geq \mathrm{ord}_\lambda(c_0 \alpha_f C_{\kappa,M,N}^{-1}) = -b$$

We apply Theorem 5.1 to pick a T_0 so that $a_{F_f}(T_0)$ is in $\mathcal{O}^\times$. Since t_f^S is defined over $\mathcal{O}$ and $\mathcal{E}_M(z, w)$ has Fourier coefficients in $\mathcal{O}$, we have that $t_f^S \mathcal{E}_M(z, w)$ has Fourier coefficients in $\mathcal{O}$. Write $c_0 \alpha_f = \lambda^{-b} u^{-1}$ for $u \in \mathcal{O}^\times$. We can rewrite Eq. (5) as

$$\lambda^b u t_f^S \mathcal{E}_M(z, w) = F_f(z) F_f(w) + \lambda^b u \sum_{i=1}^{r} c_{m+i} t_f^S F_{m+i}(z) F_{m+i}^c(w).$$

We now expand each side of this equation in the z-variable, reduce modulo λ^b, and equate the T_0th Fourier coefficients to obtain

$$F_f(w) \equiv a_{F_f}(T_0)^{-1} u \lambda^b \sum_{i=1}^{r} c_{m+i} a_{t_f^S F_{m+i}}(T_0) F_{m+i}^c(w) \pmod{\lambda^b}.$$

Thus, we have proven the following theorem.

Theorem 6.5 *Let κ and M be positive integers with $\kappa \geq 6$ even and M odd and square-free. Let $f \in S_{2\kappa-2}^{\text{new}}(\Gamma_0(M))$ be a newform. Let ℓ be an odd prime with $\ell > 2\kappa - 2$, $\ell \nmid M$, $\mathcal{O}$ a sufficiently large extension of $\mathbb{Z}_\ell$, λ the prime of $\mathcal{O}$ over ℓ, $\overline{\rho}_{f,\lambda}$ irreducible, and $\lambda \mid L_{\text{alg}}(\kappa, f)$. If there exists a fundamental discriminant $D < 0$ so that $\gcd(\ell M, D) = 1$, $\chi_D(-1) = -1$, and an integer $N > 1$ with $M \mid N$, $\ell \nmid N$, and an even Dirichlet character χ of conductor N so that*

$$\operatorname{ord}_\lambda \left(\frac{\mathcal{L}(\kappa, \chi, D, f, N)}{L_{\text{alg}}(\kappa, f)} \right) = -b < 0,$$

then there exists a nonzero $G \in S_\kappa^{\text{N-SK}}(\Gamma_0^{(2)}(M))$ so that

$$F_f \equiv G \pmod{\lambda^b}.$$

7 CAP forms and weak endoscopic lifts

In order to apply our congruence results to the Bloch–Kato conjecture, it is essential to know that the congruence we produce is not to an eigenform with reducible Galois representation. It is known that if $F \in S_\kappa(\Gamma_0^{(2)}(M))$ has reducible Galois representation then F is necessarily CAP of Saito–Kurokawa type or a weak endoscopic lift. One can see [40, Theorem 3.2.1] or [47] for this result. Moreover, if F is CAP then it must be CAP with respect to the Siegel parabolic. Such CAP forms have been studied by Piatetski-Shapiro [27]. We briefly recall the notion of an automorphic representation being CAP or a weak endoscopic lift.

Definition 7.1 Let G be a reductive group defined over $\mathbb{Q}$.

(1) Let $\Pi = \otimes \Pi_v$ and $\Pi' = \otimes \Pi'_v$ be two irreducible automorphic representations of $G(\mathbb{A})$. We say Π and Π' are *nearly equivalent* if $\Pi_v \cong \Pi'_v$ for almost all places v.
(2) Let P be a parabolic subgroup of G with Levi decomposition $P = M_P N_P$. We say an irreducible cuspidal automorphic representation Π of $G(\mathbb{A})$ is CAP with respect to P if there is an irreducible cuspidal automorphic representation π of $M_P(\mathbb{A})$ so that Π is nearly equivalent to an irreducible component of $\operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi$.

Definition 7.2 A unitary irreducible cuspidal representation Π of $\mathrm{GSp}_4(\mathbb{A})$ is called a *weak endoscopic lift*, if there exist two unitary irreducible cuspidal automorphic representations π_1, π_2 of $\mathrm{GL}_2(\mathbb{A})$ with central characters $\omega_{\pi_1} = \omega_{\pi_2}$ such that

$$L_v(\Pi, s) = L_v(\pi_1, s)L_v(\pi_2, s)$$

holds for almost all places. Here $L_v(\Pi, s)$ denotes the local L-factor of the spinor L-series.

7.1 CAP forms

We are interested in the case that $G = G_2 = \mathrm{GSp}(4)$. Let $F \in S_\kappa(\Gamma_0^{(2)}(M))$. We say F is a CAP form if the representation Π_F generated by $F_{\mathbb{A}}$ is a CAP representation. It is known [28, Corl. 4.5] that if F is CAP with $\kappa > 2$ then it must be CAP to the Siegel parabolic subgroup. Moreover, Π_F is CAP with respect to the Siegel parabolic if and only if it is either a theta lift from an irreducible cuspidal automorphic representation $\tilde{\sigma}$ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ or is a twist of a theta lift by a non-trivial quadratic character ω where $\widetilde{\mathrm{SL}}_2$ is the metaplectic cover of SL_2 and ω is a one-dimensional representation of $G(\mathbb{A})$ via $g \mapsto \omega(\mu_2(g))$ [27, Theorem 2.1]. The case of a theta lift gives rise to the classical Saito–Kurokawa lifting, and in Theorem 7.3 we show that in the case that M is square-free Π_F cannot be the nontrivial twist of a theta lift. The following result was communicated to us by Ralf Schmidt.

Theorem 7.3 *Let $F \in S_\kappa(\Gamma_0^{(2)}(M))$ be an eigenform so that $F_{\mathbb{A}}$ generates a CAP automorphic representation. If M is square-free then necessarily $F \in S_\kappa^{\mathrm{SK}}(\Gamma_0^{(2)}(M))$.*

Proof Suppose F is a CAP form and it generates a CAP representation of the form $\Pi \otimes \sigma$ where Π is a theta lift of $\widetilde{\mathrm{SL}}_2$ and σ is a non-trivial quadratic character. Since M is square free, every local representation $\Pi_v \otimes \sigma_v$ is Iwahori spherical (i.e. the representation has a non-trivial Iwahori invariant vector) where we recall the Iwahori subgroup is given by

$$I_v = \left\{ g \in \mathrm{GSp}_4(\mathbb{Z}_v) : g \equiv \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \pmod{v} \right\}.$$

The Iwahori spherical representations are constituents of representations parabolically induced from an unramified character of the minimal parabolic. Since $\Pi_v \otimes \sigma_v$ is Iwahori spherical, Π_v must be a constituent of something induced from the Borel on $\mathrm{GSp}(4)$. But M is square free so the representation π (on GL_2) is Steinberg, so there are two possible lifts giving Π, namely, $\Pi(\mathrm{St} \otimes 1)$ and $\Pi(\mathrm{St} \otimes \mathrm{St})$. The representation $\Pi(\mathrm{St} \otimes 1)$ is the Langland's quotient $L((\nu^{1/2}\pi, \nu^{-1/2}))$ while $\Pi(\mathrm{St} \otimes \mathrm{St}) = \tau(T, \nu^{-1/2})$, from [32]. But in either case one can see that if σ_v is ramified then $\Pi_v \otimes \sigma_v$ is not spherical by consulting the tables of Iwahori spherical representations given in [31]. But since $N > 1$, σ_v must be ramified at some place. Hence F cannot generate a CAP representation of the form $\Pi \otimes \sigma$. Hence $F \in S_\kappa^{\mathrm{SK}}(\Gamma_0^{(2)}(M))$. □

7.2 Weak endoscopic lifts

We now prove that under an additional mild assumption the F_f constructed in Theorem 6.5 cannot satisfy $F_f \equiv_{\mathrm{ev},\Sigma} G \pmod{\lambda}$ for some finite set of places Σ if G is a weak endoscopic lift.

Theorem 7.4 *Let κ be an even integer, M an odd square-free integer, and let $\ell > 2\kappa - 2$ be a prime with $\ell \nmid M$, $\ell \nmid p^2 - 1$ for all $p \mid M$. Let $f \in S^{\text{new}}_{2\kappa-2}(\Gamma_0(M))$ be a newform and $F_f \in S_\kappa(\Gamma_0^{(2)}(M))$ be the Saito–Kurokawa lift of f. If $G \in S_\kappa(\Gamma_0^{(2)}(M))$ is an eigenform so that $F_f \equiv_{\text{ev},\Sigma} G \pmod{\lambda}$ for some finite set of primes Σ, then G is not a weak endoscopic lift.*

Proof Suppose that the eigenvalues of F_f are congruent to those of a weak endoscopic lift G for all Hecke operators in $\mathbb{T}^{S,\Sigma}_{\mathcal{O}}$ for some finite set of places Σ. Write $\overline{\rho}_{F_f,\lambda}$ and $\overline{\rho}_{G,\lambda}$ for the reductions modulo λ of $\rho_{F_f,\lambda}$ and $\rho_{G,\lambda}$. If the eigenvalues of F_f and G are congruent modulo λ for all Hecke operators in $\mathbb{T}^{S,\Sigma}_{\mathcal{O}}$, we have that the characteristic polynomials of $\overline{\rho}^{\text{ss}}_{F_f,\lambda}(\text{Frob}_p)$ and $\overline{\rho}^{\text{ss}}_{G,\lambda}(\text{Frob}_p)$ are equal for all $p \notin \Sigma \cup \{\ell, M\}$. We apply the Chebotarev Density Theorem to conclude that the characteristic polynomials must agree on all of $G_{\mathbb{Q}}$ and so the semi-simplifications of $\overline{\rho}_{F_f,\lambda}$ and $\overline{\rho}_{G,\lambda}$ are isomorphic. Thus, after possibly rearranging g_1 and g_2, we have that $\overline{\rho}_{g_1,\lambda} \cong \overline{\rho}_{f,\lambda}$ and $\overline{\rho}^{\text{ss}}_{g_2,\lambda} \cong \omega^{\kappa-2} \oplus \omega^{\kappa-1}$ where ω is the reduction of the ℓ-adic cyclotomic character. In particular, we have $\overline{\rho}^{\text{ss}}_{g_2,\lambda}(\kappa-2) \cong \omega^{-1} \oplus 1$. We now apply [29, Prop. 2.1] to conclude that there is a lattice so that $\overline{\rho}_{g_2,\lambda}(\kappa-2)$ is of the form $\begin{pmatrix} \omega^{-1} & * \\ 0 & 1 \end{pmatrix}$ and is not split. We now show this is impossible by using this Galois representation to construct a piece of the ω^{-1}-isotypic piece of the class group of $\mathbb{Q}(\zeta_\ell)$ which by Herbrand's theorem gives the contradiction that $\ell \mid B_2 = \frac{1}{30}$.

To ease notation write $\rho = \overline{\rho}_{g_2,\lambda}(\kappa-2)$. We have that $\rho|_{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_\ell))} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, and so $h := * : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_\ell)) \to \mathbb{F}$ is a non-trivial homomorphism where $\mathbb{F}$ is a finite field of characteristic ℓ. Set $K = \overline{\mathbb{Q}}^{\ker h}$. We have that $\text{Gal}(K/\mathbb{Q}(\zeta_\ell))$ is abelian of ℓ-power order because

$$\begin{aligned} \text{Gal}(K/\mathbb{Q}(\zeta_\ell)) &\cong \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_\ell))/\,\text{Gal}(\overline{\mathbb{Q}}/K) \\ &= \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_\ell))/\ker h \\ &\cong \text{Image}(h) \end{aligned}$$

and $\text{Image}(h)$ is a subgroup of $\mathbb{F}$, which is abelian of ℓ-power order.

It remains to show that $K/\mathbb{Q}(\zeta_\ell)$ is unramified and $\text{Gal}(K/\mathbb{Q})$ acts on $\text{Gal}(K/\mathbb{Q}(\zeta_\ell))$ by ω^{-1}. Most of this follows directly as in [7, §8] except for the $p \mid N$ arguments. However, we repeat the other arguments here for completeness. Observe that for $\sigma \in \text{Gal}(K/\mathbb{Q}(\zeta_\ell))$ and $\tau \in \text{Gal}(K/\mathbb{Q})$, we have

$$\rho(\tau\sigma\tau^{-1}) = \rho(\tau)\rho(\sigma)\rho(\tau)^{-1},$$

i.e.,

$$h(\tau\sigma\tau^{-1}) = \omega^{-1}(\tau)h(\sigma).$$

This gives that $\text{Gal}(K/\mathbb{Q})$ acts on $\text{Gal}(K/\mathbb{Q}(\zeta_\ell))$ by ω^{-1}.

Let p be a prime so that $p \nmid \ell M$. We know that ρ is unramified at p and so $h(I_p) = 0$ for all $p \nmid \ell M$. In particular, we have $h(I_p(K/\mathbb{Q}(\zeta_\ell)) = 0$ for all $p \nmid \ell M$. Since we have that h is an isomorphism of $\text{Gal}(K/\mathbb{Q}(\zeta_\ell))$ to the subgroup $\text{Image}(h)$ in $\mathbb{F}$, we must have $I_p(K/\mathbb{Q}(\zeta_\ell)) = 1$. Thus, $K/\mathbb{Q}(\zeta_\ell)$ is unramified at all $p \nmid \ell M$.

Our next step is to show that the extension $K/\mathbb{Q}(\zeta_\ell)$ that we have constructed is unramified at ℓ. We have that $h|_{D_\ell} \in \mathrm{H}^1(\mathbb{Q}_\ell, \mathbb{F}(-1))$. Therefore, we have that h gives an extension X of $\mathcal{O}/\lambda\mathcal{O}$ by $\mathbb{F}(-1)$:

$$0 \longrightarrow \mathbb{F}(-1) \longrightarrow X \longrightarrow \mathcal{O}/\lambda\mathcal{O} \longrightarrow 0.$$

Applying Lemma 8.6 and Theorem 8.1 we have that $h|_{D_\ell} \in \mathrm{H}^1_f(\mathbb{Q}_\ell, \mathbb{F}(-1))$. A calculation in [5] shows that $\mathrm{H}^1_f(\mathbb{Q}_\ell, E(-1)) = 0$ where E is the field of definition for $\rho_{G,\lambda}$. Actually, it is shown that $\mathrm{H}^1_f(\mathbb{Q}_\ell, \mathbb{Q}_\ell(r)) = 0$ for every $r < 0$; this implies $\mathrm{H}^1_f(\mathbb{Q}_\ell, E(-1)) = 0$ since E is a finite extension [5, Example 3.9]. Since we define $\mathrm{H}^1_f(\mathbb{Q}_\ell, E/\mathcal{O}(-1))$ to be the image of the $\mathrm{H}^1_f(\mathbb{Q}_\ell, E(-1))$, we have $\mathrm{H}^1_f(\mathbb{Q}_\ell, E/\mathcal{O}(-1)) = 0$. Using that $h|_{D_\ell} \in \mathrm{H}^1_f(\mathbb{Q}_\ell, \mathbb{F}(-1))$, Proposition 8.7 gives that $h|_{D_\ell} \in \mathrm{H}^1_f(\mathbb{Q}_\ell, E/\mathcal{O}(-1))$ and hence is 0. Thus we have that h vanishes on the entire decomposition group D_ℓ; in particular, it must be unramified at ℓ as claimed.

Let $p \mid M$. Note that we have $\rho(I_p) \subset \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$ and so $\rho|_{I_p}$ factors through the tame quotient I_p^t of I_p. Recall we have the exact sequence

$$0 \longrightarrow I_p \longrightarrow D_p \longrightarrow \widehat{\mathbb{Z}} \longrightarrow 0.$$

Let $\sigma_p \in D_p$ be any lift of Frob_p. Then we know that σ_p acts on the tame inertia via raising to the pth power, i.e., if $x \in I_p^t$, then $\sigma_p \cdot x = \sigma_p x \sigma_p^{-1} = x^p$. Now suppose there exists $x \in I_p^t(K/\mathbb{Q}(\zeta_\ell))$ so that $\rho(x) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b \neq 0$. Then we have

$$\begin{aligned} \begin{pmatrix} 1 & pb \\ 0 & 1 \end{pmatrix} &= \rho(x)^p \\ &= \rho(x^p) \\ &= \rho(\sigma_p x \sigma_p^{-1}) \\ &= \rho(\sigma_p)\rho(x)\rho(\sigma_p)^{-1} \\ &= \begin{pmatrix} 1 & p^{-1}b \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

This gives that $pb = p^{-1}b$ as elements of $\mathbb{F}$. However, since $b \neq 0$, we must have $p^2 \equiv 1 \pmod{\ell}$. This contradicts our assumption that $\ell \nmid p^2 - 1$ for all $p \nmid M$ and so it must be that $h(I_p(K/\mathbb{Q}(\zeta_\ell))) = 0$ for all $p \mid M$ as claimed. This completes the proof. □

7.3 CAP ideal

We are now in a position to refine Theorem 6.5.

Corollary 7.5 *With the setup as in Theorem 6.5 with the additional assumption that $\ell \nmid (p^2 - 1)$ for all $p \mid M$, there exists an eigenform $F \in S_\kappa^{\text{N-SK}}(\Gamma_0^{(2)}(M))$ so that*

$$F_f \equiv_{\mathrm{ev},\Sigma} F \pmod{\lambda}$$

for Σ a finite set of places. Moreover, the Galois representation associated to F is irreducible.

Proof Let G be as in Theorem 6.5. Let Σ be a finite set of places containing all the primes dividing M. Recall that

$$\mathbb{T}_{\mathcal{O}}^{S,\Sigma} \cong \prod \mathbb{T}_{\mathfrak{m}}^{S,\Sigma}$$

where the product runs over the maximal ideals of $\mathbb{T}_{\mathcal{O}}^{S,\Sigma}$. There is a maximal ideal $\mathfrak{m}_{F_f}$ of $\mathbb{T}_{\mathcal{O}}^{S,\Sigma}$ associated to F_f. It is the kernel of the map $\mathbb{T}_{\mathcal{O}}^{S,\Sigma} \to \mathcal{O} \to \mathbb{F}$ given by sending t to $\overline{\lambda_{F_f}(t)}$. This decomposition gives a Hecke operator $t \in \mathbb{T}_{\mathcal{O}}^{S,\Sigma}$ so that $tF = F$ if $\mathfrak{m}_F = \mathfrak{m}_{F_f}$ and $tF = 0$ if $\mathfrak{m}_F \neq \mathfrak{m}_{F_f}$. In other words, $tF = F$ if $F \equiv_{\text{ev},\Sigma} F_f \pmod{\lambda}$ and $tF = 0$ if $F \not\equiv_{\text{ev},\Sigma} F_f \pmod{\lambda}$.

Write $G = \sum a_i F_i$. By construction we know that the F_i all lie in $S_\kappa^{\text{N-SK}}(\Gamma_0^{(2)}(M))$. We apply t to G and note that $tG \not\equiv 0 \pmod{\lambda}$ since $tG \equiv F_f \pmod{\lambda}$. Thus, there is an eigenform $F \in S_\kappa^{\text{N-SK}}(\Gamma_0^{(2)}(M))$ with $F \equiv_{\text{ev},\Sigma} F_f \pmod{\lambda}$. Moreover, we know that F must have irreducible Galois representation by Theorems 5.4, 7.3 and 7.4. □

One should note that even though we have a congruence $F_f \equiv G \pmod{\lambda^b}$, the congruence given in the corollary is only modulo λ. We work around this by introducing the CAP ideal associated to F_f. Let $\mathbb{T}_{\mathcal{O}}^{\text{N-SK},\Sigma}$ denote the image of $\mathbb{T}_{\mathcal{O}}^{S,\Sigma}$ in $\text{End}_{\mathbb{C}}(S_\kappa^{\text{N-SK}}(\Gamma_0^{(2)}(M)))$. Let $\phi : \mathbb{T}_{\mathcal{O}}^{S,\Sigma} \to \mathbb{T}_{\mathcal{O}}^{\text{N-SK},\Sigma}$ denote the canonical $\mathcal{O}$-algebra surjection. Write $\text{Ann}(F_f)$ for the annihilator of F_f in $\mathbb{T}_{\mathcal{O}}^{S,\Sigma}$ and observe we have an isomorphism

$$\mathbb{T}_{\mathcal{O}}^{S,\Sigma} / \text{Ann}(F_f) \cong \mathcal{O}.$$

We have that $\phi(\text{Ann}(F_f))$ is an ideal in $\mathbb{T}_{\mathcal{O}}^{\text{N-SK},\Sigma}$ since ϕ is surjective. We refer to the ideal $\phi(\text{Ann}(F_f))$ as the CAP ideal associated to F_f. We have seen above that if F is an eigenform of weight κ and level $\Gamma_0^{(2)}(M)$ with reducible Galois representation so that $F \equiv_{\text{ev},\Sigma} F_f \pmod{\lambda}$ for some finite set of places Σ, then $F \in S_\kappa^{\text{SK}}(\Gamma_0^{(2)}(M))$. As such, we can view the CAP ideal associated to F_f as measuring congruences between F_f and eigenforms with irreducible Galois representations. Thus, the CAP ideal plays much the same role as the Eisenstein ideal in classical theory. We make this more precise as follows.

One has that there exists an $r \in \mathbb{Z}_{\geq 0}$ so that the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{T}_{\mathcal{O}}^{S,\Sigma} & \xrightarrow{\phi} & \mathbb{T}_{\mathcal{O}}^{\text{N-SK},\Sigma} \\
\downarrow & & \downarrow \\
\mathbb{T}_{\mathcal{O}}^{S,\Sigma} / \text{Ann}(F_f) & \xrightarrow{\phi} & \mathbb{T}_{\mathcal{O}}^{\text{N-SK},\Sigma} / \phi(\text{Ann}(F_f)) \\
\downarrow & & \downarrow \simeq \\
\mathcal{O} & \longrightarrow & \mathcal{O}/\lambda^r\mathcal{O}.
\end{array}$$

Note that all of the maps in the above diagram are $\mathcal{O}$-algebra surjections.

Corollary 7.6 *With r as in the above diagram and b as in Theorem 6.5, we have $r \geq b$.*

Proof Assume that $b > r$. Let G be as in Theorem 6.5. Choose $t \in \phi^{-1}(\lambda^r) \subset \mathbb{T}_{\mathcal{O}}^{S,\Sigma}$. Thus, $tG = \lambda^r G$. Using that the diagram commutes we have $t \in \text{Ann}(F_f)$ and so the congruence in Theorem 6.5 gives

$$\lambda^r G \equiv 0 \pmod{\lambda^b},$$

i.e.,

$$G \equiv 0 \pmod{\lambda^{b-r}}.$$

However, since $b - r > 0$, we must have

$$\begin{aligned} F_f &\equiv G \pmod{\lambda} \\ &\equiv 0 \pmod{\lambda}, \end{aligned}$$

which is a contradiction. □

8 Selmer groups

In this section we define the relevant Selmer group and give a lower bound for the size of the Selmer group using the congruence we constructed in Sect. 7.3, hence proving the main theorem.

8.1 Definition of the appropriate Selmer group

In this section we define the relevant Selmer group following [5]. For a number field K and a topological $G_K = \mathrm{Gal}(\bar{K}/K)$-module $\mathscr{M}$ with a continuous action of G_K on $\mathscr{M}$, we consider the group $H^1_{\mathrm{cont}}(G_K, \mathscr{M})$ of cohomology classes of continuous cocycles $G_K \to \mathscr{M}$. To ease notation we simply write $H^1(K, \mathscr{M})$ when we mean $H^1_{\mathrm{cont}}(G_K, \mathscr{M})$.

Let $\Sigma \supset \{\ell, p \mid M\}$ be a finite set of primes of $\mathbb{Q}$ and denote by G_Σ the Galois group of the maximal Galois extension $\mathbb{Q}_\Sigma$ of $\mathbb{Q}$ unramified outside of Σ. Let E be a finite extension of $\mathbb{Q}_\ell$ and $\mathcal{O}$ be its ring of integers. Let V be a finite dimensional E-vector space with a continuous G_Σ-action. We will find it convenient to write $\rho : G_\Sigma \to \mathrm{GL}_n(E)$ to denote this action when $\dim_E(V) = n$. Let $T \subset V$ be a G_Σ-stable $\mathcal{O}$-lattice, i.e., T is G_Σ-stable and $T \otimes_{\mathcal{O}} E \cong V$. Set $W := V/T \cong T \otimes_{\mathcal{O}} E/\mathcal{O}$.

We write B_{crys} for the ring of ℓ-adic periods [18]. Set

$$D = (V \otimes_{\mathbb{Q}_\ell} B_{\mathrm{crys}})^{D_\ell}$$

and

$$\mathrm{Crys}(V) = H^0(\mathbb{Q}_\ell, V \otimes_{\mathbb{Q}_\ell} B_{\mathrm{crys}}).$$

We say the representation V is *crystalline* if $\dim_{\mathbb{Q}_\ell} V = \dim_{\mathbb{Q}_\ell} \mathrm{Crys}(V)$. Let $\mathrm{Fil}^i D$ be a decreasing filtration of D. If V is crystalline, we say V is *short* if $\mathrm{Fil}^0 D = D$, $\mathrm{Fil}^\ell D = 0$, and if whenever V' is a nonzero quotient of V, then $V' \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(\ell - 1)$ is ramified. Note that $\mathbb{Q}_\ell(n)$ is the 1-dimensional space over $\mathbb{Q}_\ell$ on which $\mathrm{G}_{\mathbb{Q}}$ acts via the nth power of the ℓ-adic cyclotomic character.

The following theorem gives us examples of the crystalline and short representations of interest to us in the work.

Theorem 8.1 [16,44] *Let $F \in S_\kappa(\Gamma_0^{(2)}(M))$ be an eigenform. The restriction of $\rho_{F,\lambda}$ to D_ℓ is crystalline at ℓ. If $\ell > 2\kappa - 2$ then $\rho_{F,\lambda}$ is short.*

For every $p \in \Sigma$ and a G_Σ-module $\mathscr{M}$ define

$$H^1_{\mathrm{un}}(\mathbb{Q}_p, \mathscr{M}) := \ker\{H^1(\mathbb{Q}_p, \mathscr{M}) \xrightarrow{\mathrm{res}} H^1(I_p, \mathscr{M})\}.$$

We define the local p-Selmer group for V as

$$H^1_f(\mathbb{Q}_p, V) := \begin{cases} H^1_{\mathrm{un}}(\mathbb{Q}_p, V) & p \in \Sigma \setminus \ell \\ \ker\{H^1(\mathbb{Q}_\ell, V) \to H^1(\mathbb{Q}_\ell, V \otimes B_{\mathrm{crys}})\} & p = \ell. \end{cases}$$

For every p, define $H^1_f(\mathbb{Q}_p, W)$ to be the image of $H^1_f(\mathbb{Q}_p, V)$ under the natural map $H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, W)$. Using the fact, that $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$ has cohomological dimension one, one has that if W is unramified at p and $p \neq \ell$, then $H^1_f(\mathbb{Q}_p, W) = H^1_{\mathrm{un}}(\mathbb{Q}_p, W)$.

We are now in a position to define the Selmer group of interest to us. For any set $\Sigma' \subset \Sigma \smallsetminus \ell$, let

$$\mathrm{Sel}_\Sigma(\Sigma', W) := \ker\left\{ H^1(G_\Sigma, W) \xrightarrow{\mathrm{res}} \bigoplus_{p \in \Sigma' \cup \{\ell\}} \frac{H^1(\mathbb{Q}_p, W)}{H^1_f(\mathbb{Q}_p, W)} \right\}.$$

In the case that $\Sigma' = \emptyset$, we write $\mathrm{Sel}_\Sigma(W)$ for $\mathrm{Sel}_\Sigma(\emptyset, W)$.

For a $\mathbb{Z}_p$ module $\mathscr{M}$, let $\mathscr{M}^\vee$ denote the Pontryagin dual of $\mathscr{M}$ defined as

$$\mathscr{M}^\vee = \mathrm{Hom}_{\mathrm{cont}}(\mathscr{M}, \mathbb{Q}_p/\mathbb{Z}_p).$$

We denote the Pontryagin dual of $\mathrm{Sel}_\Sigma(\Sigma', W)$ by $X_\Sigma(\Sigma', W)$ i.e.

$$X_\Sigma(\Sigma', W) = (\mathrm{Sel}_\Sigma(\Sigma', W))^\vee.$$

The following lemma follows from [30] and [41, §3].

Lemma 8.2 *$X_\Sigma(\Sigma', W)$ is a finitely generated $\mathcal{O}$-module and if the mod λ reduction $\bar{\rho}$ of ρ is absolutely irreducible, then the length of $X_\Sigma(\Sigma', W)$ as an $\mathcal{O}$-module is independent of the choice of the lattice T.*

Remark 8.3 For an $\mathcal{O}$-module $\mathscr{M}$, $\mathrm{ord}_\ell(\#\mathscr{M}) = [O/\lambda : \mathbb{F}_\ell]\mathrm{length}_{\mathcal{O}}(\mathscr{M})$.

Example 8.4 Let $f \in S_{2\kappa-2}(\Gamma_0(M))$ be a cuspidal eigenform and $(\rho_{f,\lambda}, V_{f,\lambda})$ be the λ-adic Galois representation associated to it. Let $\Sigma = \{p \mid M\} \cup \{\ell\}$ and let $V_{f,\lambda}(\kappa-2)$ denote the representation space of $\rho = \rho_{f,\lambda} \otimes \varepsilon^{\kappa-2}$ of $G_\mathbb{Q}$. Let $T_{f,\lambda}(\kappa-2) \subset V_{f,\lambda}(\kappa-2)$ be some choice of a $G_\mathbb{Q}$-stable lattice. Set $W_{f,\lambda}(\kappa-2) = V_{f,\lambda}(\kappa-2)/T_{f,\lambda}(\kappa-2)$. Note that the action of $G_\mathbb{Q}$ on $V_{f,\lambda}(\kappa-2)$ factors through G_Σ. Since the mod λ reduction of ρ is absolutely irreducible by assumption, $\mathrm{val}_\ell(X_\Sigma(\{p \mid M\}, W_{f,\lambda}(\kappa-2)))$ is independent of the choice of $T_{f,\lambda}(\kappa-2)$.

We will also need the notation of degree n Selmer groups. In fact, we have already made use of these in Sect. 7.2. We begin by reviewing the relationship between extensions of modules and the first cohomology group. Let G be a group, R a ring, and let $\mathscr{M}$ and $\mathscr{N}$ be $R[G]$-modules. An extension of $\mathscr{M}$ by $\mathscr{N}$ is a short exact sequence

$$0 \longrightarrow \mathscr{N} \xrightarrow{\alpha} \mathscr{X} \xrightarrow{\beta} \mathscr{M} \longrightarrow 0$$

where $\mathscr{X}$ is a $R[G]$-module and α and β are $R[G]$-homomorphisms. We sometimes refer to such an extension as the extension $\mathscr{X}$. We say two extensions $\mathscr{X}$ and $\mathscr{Y}$ are equivalent if there is a $R[G]$-isomorphism γ making the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{N} & \xrightarrow{\alpha_\mathscr{X}} & \mathscr{X} & \xrightarrow{\beta_\mathscr{X}} & \mathscr{M} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_\mathscr{N}} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{id}_\mathscr{M}} & & \\ 0 & \longrightarrow & \mathscr{N} & \xrightarrow{\alpha_\mathscr{Y}} & \mathscr{Y} & \xrightarrow{\beta_\mathscr{Y}} & \mathscr{M} & \longrightarrow & 0. \end{array}$$

Let $\mathrm{Ext}^1_{R[G]}(\mathscr{M}, \mathscr{N})$ denote the set of equivalence classes of $R[G]$-extensions of $\mathscr{M}$ by $\mathscr{N}$ which split as extensions of R-modules.

We have the following result that will be used to define the degree n Selmer groups. The case that $\mathscr{M} = \mathscr{N}$ is given in [46], and the argument given in [8] to prove the result below is a simple modification of this argument.

Theorem 8.5 [8, Theorem 9.2] *Let $\mathscr{M}$ and $\mathscr{N}$ be $R[G]$-modules. There is a one-one correspondence between the sets $H^1(G, \mathrm{Hom}_R(\mathscr{M}, \mathscr{N}))$ and $\mathrm{Ext}^1_{R[G]}(\mathscr{M}, \mathscr{N})$.*

The map from $\mathrm{Ext}^1_{R[G]}(\mathscr{M}, \mathscr{N})$ to $H^1(G, \mathrm{Hom}_R(\mathscr{M}, \mathscr{N}))$ is given as follows. Let

$$0 \longrightarrow \mathscr{N} \xrightarrow{\alpha} \mathscr{X} \underset{\curvearrowleft s_{\mathscr{X}}}{\xrightarrow{\beta}} \mathscr{M} \longrightarrow 0$$

be an extension with $s_{\mathscr{X}}$ a R-section of $\mathscr{X}$. This extension is mapped to the cohomology class $g \mapsto \mathfrak{c}_g$ where $\mathfrak{c}_g : \mathscr{M} \to \mathscr{N}$ is defined by

$$\mathfrak{c}_g(m) = \alpha^{-1}(\rho(g)s_{\mathscr{X}}(\rho_{\mathscr{M}}(g^{-1})m) - s_{\mathscr{X}}(m))$$

where ρ denotes the G-action on $\mathscr{X}$ and $\rho_{\mathscr{M}}$ the G-action on $\mathscr{M}$.

Let V, T, and W be as above. Let $W[n]$ be the λ^n torsion elements in W. The previous theorem gives a bijection between $\mathrm{Ext}^1_{(\mathcal{O}/\lambda^n)[D_\ell]}(\mathcal{O}/\lambda^n, W[n])$ and $H^1(D_\ell, W[n])$. For $p \neq \ell$, we define the local degree n Selmer groups by $H^1_f(\mathbb{Q}_p, W[n]) = H^1_{\mathrm{un}}(\mathbb{Q}_p, W[n])$. At the prime ℓ we define the local degree n Selmer group to be the subset of classes of extensions of D_ℓ-modules

$$0 \longrightarrow W[n] \longrightarrow \mathscr{X} \longrightarrow \mathcal{O}/\lambda^n \longrightarrow 0$$

where $\mathscr{X}$ lies in the essential image of the functor $\mathbb{V}$ defined in §1.1 of [14]. The precise definition of $\mathbb{V}$ is technical and is not needed here. We content ourselves with stating that this essential image is stable under direct sums, subobjects, and quotients [14, §2.1]. For our purposes the following two propositions are what is needed.

Proposition 8.6 [14, p. 670] *If V is a short crystalline representation at ℓ, T a D_ℓ-stable lattice, and $\mathscr{X}$ a subquotient of $T/\lambda^n T$ that gives an extension of D_ℓ-modules as above, then the class of this extension is in $H^1_f(\mathbb{Q}_\ell, W[n])$.*

Proposition 8.7 [7, Prop. 7.9] *Assume that $T/\lambda T$ is irreducible and let h be a non-zero cocycle in $\mathrm{Sel}_\Sigma(\Sigma', W[1])$. If $h|_{D_p} \in H^1_f(\mathbb{Q}_p, W[1])$ is non-zero, then $h|_{D_p}$ gives a non-zero λ-torsion element of $H^1_f(\mathbb{Q}_p, W)$. If $h|_{D_p} \in H^1_f(\mathbb{Q}_p, W[1])$ for every prime p, then h is a non-zero λ-torsion element of $\mathrm{Sel}_\Sigma(\Sigma', W)$.*

Using the notation in Sect. 7.3, let $I_f = \phi(\mathrm{Ann}(F_f))$ be the CAP ideal associated to F_f in $\mathbb{T}^{\text{N-SK},\Sigma}_{\mathcal{O}}$. We now state our main theorem.

Theorem 8.8 *Let Σ and $W_{f,\lambda}(\kappa - 2)$ be as in example 8.4. Then we have*

$$\mathrm{ord}_\ell(\#X_\Sigma(\{p \mid M\}, W_{f,\lambda}(\kappa - 2))) \geq \mathrm{ord}_\ell(\#\mathbb{T}^{\text{N-SK}}_{\mathfrak{m}_{F_f}}/I_f).$$

We will give a proof of Theorem 8.8 in the next section.

Corollary 8.9 *Let κ and M be positive integers with $\kappa > 5$ even and M odd and square-free. Let $f \in S^{\mathrm{new}}_{2\kappa-2}(\Gamma_0(M))$ be a newform. Let ℓ be an odd prime with $\ell > 2\kappa - 2$, $\ell \nmid M$, $\ell \nmid (p^2 - 1)$ for all $p \mid M$, $\mathcal{O}$ a sufficiently large extension of $\mathbb{Z}_\ell$, λ the prime of $\mathcal{O}$, $\overline{\rho}_{f,\lambda}$*

irreducible, and $\lambda \mid L_{\text{alg}}(\kappa, f)$*. Let* $\Sigma = \{p \mid M\} \cup \{\ell\}$*. If there exists a fundamental discriminant* $D < 0$ *so that* $\gcd(\ell M, D) = 1$*,* $\chi_D(-1) = -1$*, and an integer* $N > 1$ *with* $M \mid N$*,* $\ell \nmid N$*, and an even Dirichlet character* χ *of conductor* N *so that*

$$\text{ord}_\lambda \left(\frac{\mathcal{L}(\kappa, \chi, D, f, N)}{L_{\text{alg}}(\kappa, f)} \right) = -b < 0,$$

then we have

$$\text{ord}_\ell(\# X_\Sigma(\{p \mid M\}, W_{f,\lambda}(\kappa - 2))) \geq b.$$

In particular, if N*,* χ*, and* D *can be chosen so that*

$$\text{ord}_\lambda(\mathcal{L}(\kappa, \chi, D, f, N)) = 0,$$

then we have

$$\text{ord}_\ell(\# X_\Sigma(\{p \mid M\}, W_{f,\lambda}(\kappa - 2))) \geq \text{ord}_\ell(\# \mathcal{O}/L_{\text{alg}}(\kappa, f)).$$

Proof This corollary is an immediate consequence of Theorem 8.8 and Corollary 7.6. □

8.2 A lower bound on the Selmer group

We now recall the results from [10] which involve the construction of a lattice which we will need to give a lower bound on the size of the desired Selmer group. In [10] the second author generalizes a result in [43] wherein Urban presents the case where the Galois representation $\bar{\rho}^{\text{ss}}$ splits into two irreducible representations. We will limit ourselves to recalling the results here and refer the reader to [10] for more details.

Let $E/\mathbb{Q}_\ell$ be a suitably large finite extension, $\mathcal{O}$ the ring of integers of E, and λ the prime of $\mathcal{O}$. Set $\mathbb{F} = \mathcal{O}/\lambda$. Let $n = n_1 + n_2 + n_3$ with $n_i \geq 1$ and let $\Sigma \supset \{\ell\}$ be a finite set of primes of $\mathbb{Q}$. Let V_i be E vector spaces of dimension n_i affording continuous absolutely irreducible representations $\rho_i : G_\Sigma \to \text{Aut}_E(V_i)$ for $1 \leq i \leq 3$. Assume the residual representations ρ_i are irreducible and non-isomorphic for $1 \leq i \leq 3$. Let $\mathcal{V}_1, \ldots, \mathcal{V}_m$ be n-dimensional E vector spaces affording absolutely irreducible continuous representations $\varrho_i : G_\Sigma \to \text{Aut}_E(\mathcal{V}_i)$ for $1 \leq i \leq m$. Further assume that the modulo λ reductions of ϱ_i satisfy $\overline{\varrho}^{\text{ss}} = \overline{\varrho}_1 \oplus \overline{\varrho}_2 \oplus \overline{\varrho}_3$ for some G_Σ-stable lattice in $\mathcal{V}_i$ (and hence for all such lattices.)

For each $\sigma \in G_\Sigma$, let

$$\sum_{j=0}^{n} a_j(\sigma) X^j \in \mathcal{O}[X]$$

be the characteristic polynomial of $(\rho_1 \oplus \rho_2 \oplus \rho_3)(\sigma)$ and

$$\sum_{j=0}^{n} c_j(i, \sigma) X^j \in \mathcal{O}[X]$$

be the characteristic polynomial of $\varrho_i(\sigma)$. Set

$$c_j(\sigma) = \begin{pmatrix} c_j(1, \sigma) \\ \vdots \\ c_j(m, \sigma) \end{pmatrix} \in \mathcal{O}^m$$

for $0 \leq j \leq n-1$. Let $\mathbb{T} \subset \mathcal{O}^m$ be the $\mathcal{O}$-subalgebra generated by the set $\{c_j(\sigma) : \sigma \in G_\Sigma, 0 \leq j \leq n-1\}$.

By the continuity of the ϱ_i, note that this is the same as the $\mathcal{O}$-subalgebra of $\mathcal{O}^m$ generated by

$$\{c_j(\mathrm{Frob}_p) | 0 \leq j \leq n-1, p \notin \Sigma\}.$$

Also, note that $\mathbb{T}$ is a finite $\mathcal{O}$-algebra. Let $I \subset \mathbb{T}$ be the ideal generated by the set $\{c_j(\mathrm{Frob}_p) - a_j(\mathrm{Frob}_p) | 0 \leq j \leq n-1, p \notin \Sigma\}$.

Then we have the following theorem which we state here without proof, for more details we refer the reader to [10].

Theorem 8.10 *Suppose $\mathbb{F}^\times$ contains at least n distinct elements. Then there exists a G_Σ-stable $\mathbb{T}$-submodule $\mathcal{L} \subset \oplus_{i=1}^m \mathcal{V}_i$, $\mathbb{T}$-submodules $\mathcal{L}_1, \mathcal{L}_2$, and $\mathcal{L}_3$ contained in $\mathcal{L}$ and finitely generated $\mathbb{T}$-modules $\mathcal{T}_1$ and $\mathcal{T}_2$ such that*

(1) as $\mathbb{T}$-modules we have $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ and $\mathcal{L}_i \simeq \mathbb{T}^{n_i}$ for $1 \leq i \leq 3$;
(2) $\mathcal{L}$ has no $\mathbb{T}[G_\Sigma]$-quotient isomorphic to $\overline{\rho}'$ where $\overline{\rho}'^{\,\mathrm{ss}} = \overline{\rho}_1 \oplus \overline{\rho}_3$;
(3) $(\mathcal{L}_1 \oplus \mathcal{L}_2)/I(\mathcal{L}_1 \oplus \mathcal{L}_2)$ is G_Σ-stable and there exists a $\mathbb{T}[G_\Sigma]$-isomorphism

$$\mathcal{L}/(\mathcal{L} + I(\mathcal{L}_1 \oplus \mathcal{L}_2)) \simeq \mathscr{M}_3 \otimes_{\mathcal{O}} \mathbb{T}/I$$

for any G_Σ-stable $\mathcal{O}$-lattice $\mathscr{M}_3 \subset \mathcal{V}_3$;
(4) one has either

$$\mathrm{Hom}_{\mathbb{T}/I}(\mathscr{M}_1 \otimes_{\mathcal{O}} \mathcal{T}_1/I\mathcal{T}_1, \mathscr{M}_2 \otimes_{\mathcal{O}} \mathcal{T}_2/I\mathcal{T}_2) = 0$$

or

$$\mathrm{Hom}_{\mathbb{T}/I}(\mathscr{M}_2 \otimes_{\mathcal{O}} \mathcal{T}_2/I\mathcal{T}_2, \mathscr{M}_1 \otimes_{\mathcal{O}} \mathcal{T}_1/I\mathcal{T}_1) = 0$$

for any G_Σ-stable $\mathcal{O}$-lattices $\mathscr{M}_i \subset \mathcal{V}_i$ for $i = 1, 2$;
(5) $\mathrm{Fitt}_{\mathbb{T}}(\mathcal{T}_i) = 0$ for $i = 1, 2$ and there exists a $\mathbb{T}[G_\Sigma]$-isomorphism

$$\mathcal{L}_i/I\mathcal{L}_i \simeq \mathscr{M}_i \otimes_{\mathcal{O}} \mathcal{T}_i/I\mathcal{T}_i$$

for any G_Σ-stable $\mathcal{O}$-lattice $\mathscr{M}_i \subset \mathcal{V}_i$ for $i = 1, 2$.

To specialize to our situation, we make the following choices:

- $n_1 = n_3 = 1, n_2 = 2$;
- $\rho_1 = \varepsilon^{-1}$, $\rho_2 = \rho_{f,\lambda} \otimes \varepsilon^{\kappa-2}$, $\rho_3 = \mathrm{id}$. Note that these are the components of $\rho_{F_f,\lambda} \otimes \varepsilon^{\kappa-2}$.
- $\mathbb{T} = \mathbb{T}^{\text{N-SK},\Sigma}_{\mathfrak{m}_{F_f}}$;
- $G_1, \ldots, G_m$ for the elements in an orthogonal eigenbasis of $S_\kappa^{\mathrm{N-SK}}(\Gamma_0^{(2)}(M))$ such that $\phi^{-1}(\mathfrak{m}^{\mathrm{N-SK}}_{G_i}) = \mathfrak{m}_{F_f}$ where $\phi : \mathbb{T}^{S,\Sigma}_{\mathcal{O}} \to \mathbb{T}^{\text{N-SK},\Sigma}_{\mathcal{O}}$ is the canonical $\mathcal{O}$-algebra surjection as given in Sect. 7.3.
- $I =$ the ideal of $\mathbb{T}$ generated by $\phi_{\mathfrak{m}_{F_f}}(\mathrm{Ann}\, F_f)$;
- $(\mathcal{V}_i, \rho_i) =$ the representation $\rho_{G_i,\lambda}$ for $1 \leq i \leq m$.

Let $\mathscr{M}_i$ be a $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice inside V_i for $1 \leq i \leq 3$. Using the matrix notation introduced in [10], we can break the situation into two cases depending on whether $\mathrm{Hom}_{\mathbb{T}/I}(\mathscr{M}_1 \otimes_{\mathcal{O}} \mathcal{T}_1/I\mathcal{T}_1, \mathscr{M}_2 \otimes_{\mathcal{O}} \mathcal{T}_2/I\mathcal{T}_2) = 0$ or $\mathrm{Hom}_{\mathbb{T}/I}(\mathscr{M}_2 \otimes_{\mathcal{O}} \mathcal{T}_2/I\mathcal{T}_2, \mathscr{M}_1 \otimes_{\mathcal{O}} \mathcal{T}_1/I\mathcal{T}_1) = 0$. We limit ourselves to outlining the steps in the first case as the details of these arguments can be found in [10]. Suppose $\mathrm{Hom}_{\mathbb{T}/I}(\mathscr{M}_1 \otimes_{\mathcal{O}} \mathcal{T}_1/I\mathcal{T}_1, \mathscr{M}_2 \otimes_{\mathcal{O}} \mathcal{T}_2/I\mathcal{T}_2) = 0$. Then we have a short exact sequence of $(\mathbb{T}/I)[G_\Sigma]$-modules

$$0 \longrightarrow \mathcal{N}_1 \oplus \mathcal{N}_2 \longrightarrow \mathcal{L} \otimes \mathbb{T}/I \longrightarrow \rho_3 \otimes \mathbb{T}/I \to 0$$

which splits as a sequence of $\mathbb{T}/I$-modules where $\mathcal{N}_i = \mathscr{M}_i \otimes \mathcal{T}_i/I\mathcal{T}_i$ for $i = 1, 2$. This gives rise to a cocycle

$$\mathfrak{c}_2 \in H^1(G_\Sigma, \mathrm{Hom}_{\mathbb{T}/I}(M_3 \otimes_{\mathcal{O}} \mathbb{T}/I, \mathscr{M}_2 \otimes_{\mathcal{O}} \mathcal{T}_2/I\mathcal{T}_2)).$$

Since

$$\mathrm{Hom}_{\mathbb{T}/I}(\mathscr{M}_3 \otimes_{\mathcal{O}} \mathbb{T}/I, \mathscr{M}_2 \otimes_{\mathcal{O}} \mathcal{T}_2/I\mathcal{T}_2) \simeq \mathrm{Hom}_{\mathcal{O}}(\mathscr{M}_3, \mathscr{M}_2) \otimes_{\mathcal{O}} \mathcal{T}_2/I\mathcal{T}_2$$

we can regard $\mathfrak{c}_2$ as a cocycle in $H^1(G_\Sigma, \mathrm{Hom}_{\mathcal{O}}(\mathscr{M}_3, \mathscr{M}_2) \otimes_{\mathcal{O}} \mathcal{T}_2/I\mathcal{T}_2)$. Define a map

$$\iota_2 : \mathrm{Hom}_{\mathcal{O}}(\mathcal{T}_2/I\mathcal{T}_2, E/\mathcal{O}) \to H^1(G_\Sigma, \mathrm{Hom}_{\mathcal{O}}(\mathscr{M}_3, \mathscr{M}_2) \otimes_{\mathcal{O}} E/\mathcal{O})$$
$$f \mapsto (1 \otimes f)(\mathfrak{c}_2).$$

Our assumption that $\mathrm{Hom}_{\mathbb{T}/I}(\mathscr{M}_1 \otimes_{\mathcal{O}} \mathcal{T}_1/I\mathcal{T}_1, \mathscr{M}_2 \otimes_{\mathcal{O}} \mathcal{T}_2/I\mathcal{T}_2) = 0$ and the fact that the modulo λ reduction of $\rho_{f,\lambda} \otimes \varepsilon^{\kappa-2}$ is absolutely irreducible give that we can choose $T = \mathrm{Hom}_{\mathcal{O}}(\mathscr{M}_3, \mathscr{M}_2)$ and we have

$$W = \mathrm{Hom}_{\mathcal{O}}(\mathscr{M}_3, \mathscr{M}_2) \otimes_{\mathcal{O}} E/\mathcal{O}.$$

We state two lemma's and we refer the reader to [10] for their proof. One should note that in proving Lemma 8.12 one does need the argument given in Sect. 7.2 showing there is no non-split extension of the form $\begin{pmatrix} \omega^{-1} & * \\ 0 & 1 \end{pmatrix}$, but one can substitute that argument into the one given in [10] and the rest of the proof remains unchanged.

Lemma 8.11 $\mathrm{Image}(\iota_2) \subseteq \mathrm{Sel}_\Sigma(\{p|M\}, W)$.

Lemma 8.12 $(\ker(\iota_2))^\vee = 0$.

Proof of Theorem 8.8 Set $W = W_{f,\lambda}(\kappa - 2)$. Lemma 8.11 implies that

$$\mathrm{ord}_\ell(\# X_\Sigma(\{p|M\}, W)) \geq \mathrm{ord}_\ell(\#(\mathrm{Im}\, \iota_2)^\vee)$$

and from Lemma 8.12 it follows that

$$\mathrm{ord}_\ell(\# \mathrm{Hom}_{\mathcal{O}}(\mathcal{T}_2/I\mathcal{T}_2, E/\mathcal{O})^\vee) = \mathrm{ord}_\ell(\#(\mathrm{Im}\, \iota_2)^\vee).$$

But we know that $\mathrm{Hom}_{\mathcal{O}}(\mathcal{T}_2/I\mathcal{T}_2, E/\mathcal{O})^\vee \cong (\mathcal{T}_2/I\mathcal{T}_2)^{\vee\vee} \cong \mathcal{T}_2/I\mathcal{T}_2$, hence

$$\mathrm{ord}_\ell(\#(\mathrm{Im}\, \iota_2)^\vee) = \mathrm{ord}_\ell(\# \mathcal{T}_2/I\mathcal{T}_2).$$

But now by Theorem 8.10, $\mathrm{Fitt}(\mathcal{T}_2) = 0$ so $\mathrm{Fitt}_{\mathbb{T}}(\mathcal{T}_2 \otimes_{\mathbb{T}} \mathbb{T}/I) \subset I$, hence

$$\mathrm{ord}_\ell(\# \mathcal{T}_2 \otimes_{\mathbb{T}} \mathbb{T}/I) \geq \mathrm{ord}_\ell(\# \mathbb{T}/I).$$

Noting that $\mathrm{ord}_\ell(\# \mathcal{T}_2/I\mathcal{T}_2) = \mathrm{ord}_\ell(\# \mathcal{T}_2 \otimes_{\mathbb{T}} \mathbb{T}/I)$ we get

$$\mathrm{ord}_\ell(\# S_\Sigma(\{p|M\}, W)) \geq \mathrm{ord}_\ell(\# \mathbb{T}/I).$$

□

References

1. Agarwal, M.: p-adic L-functions for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$. Ph.D. thesis, University of Michigan, Ann Arbor, MI (2007)
2. Agarwal, M., Brown, J.: Computational evidence for the Bloch–Kato conjecture for elliptic modular forms of square-free level. http://www.ces.clemson.edu/~jimlb/ResearchPapers/BlochKatoCompEvid.pdf. Accessed 17 July 2013
3. Agarwal, M., Brown, J.: Saito–Kurokawa lifts of square-free level and multiplicity one theorem, 1–21, preprint (2013)
4. Agarwal, M., Klosin, K.: Yoshida lifts and the Bloch–Kato conjecture for the convolution L-function, 1–49, preprint (2011)
5. Bloch, S., Kato, K.: L-functions and Tamagawa numbers of motives. In: Cartier, P., et al. (eds.), The Grothendieck Festschrift, vol. 1 of Progress in Mathematics, pp. 333–400. Birkhäuser, Boston, MA (1990)
6. Böcherer, S., Dummigan, N., Schulze-Pillot, R.: Yoshida lifts and Selmer groups 1–37, preprint (2011)
7. Brown, J.: Saito–Kurokawa lifts and applications to the Bloch–Kato conjecture. Comput. Math. **143**(2), 290–322 (2007)
8. Brown, J.: L-functions on $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ and the Bloch–Kato conjecture. Int. J. Number Theory **6**(8), 1901–1926 (2010)
9. Brown, J.: On the congruence primes of Saito–Kurokawa lifts of odd square-free level. Math. Res. Lett **17**(5), 977–991 (2011)
10. Brown, J.: On the cuspidality of pullbacks of Siegel Eisenstein series and applications to the Bloch–Kato conjecture. Int. Math. Res. Notices **7**, 1706–1756 (2011)
11. Brown, J.: On the cuspidality of pullbacks of Siegel Eisenstein series to $\mathrm{Sp}(2m) \times \mathrm{Sp}(2n)$. J. Number Theory **131**, 106–119 (2011)
12. Chai, C., Faltings, G.: Degeneration of Abelian Varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge Band 22, A Series of Modern Surveys in Mathematics. Springer, Berlin (1980)
13. Darmon, H., Diamond, F., Taylor, R.: Fermat's Last Theorem, vol. 1 of Current Developments in Mathematics. International Press, Cambridge (1995)
14. Diamond, F., Flach, M., Guo, L.: The Tamagawa number conjecture of adjoint motives of modular forms. Ann. Sc. École Norm. Sup. **37**(4), 663–727 (2004)
15. Dummigan, N., Stein, W., Watkins, M.: Constructing elements in Shafarevich–Tate groups of modular motives, vol. 303 of London Mathematical Society Lecture Note Series. Cambridge University Press (2003)
16. Faltings, G.: Crystalline cohomology and p-adic Galois representations. In: Proceedings of JAMI Inaugural Conference. John Hopkins University Press (1989)
17. Flach, M.: A generalisation of the Cassels–Tate pairing. J. Reine Angew. Math. **412**, 113–127 (1990)
18. Fontaine, J.M.: Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti–Tate. Ann. Math. **115**, 529–577 (1982)
19. Fontaine, J.-M., Perrin-Riou, B.: Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L. In: Motives (Seattle, WA, 1991), vol. 55 of Proceedings of Symposium on Pure Mathematics, pp. 599–706, Providence, RI, American Mathematical Society (1994)
20. Hida, H.: Theory of p-adic Hecke algebras and Galois representations. Sugaku Expo. **2–3**, 75–102 (1989)
21. Kato, K.: p-adic Hodge theory and values of zeta functions of modular forms. Asterisque **295**, 117–290 (2004)
22. Klosin, K.: Congruences among modular forms on $U(2, 2)$ and the Bloch–Kato conjecture. Ann. Inst. Fourier **59**(1), 81–166 (2009)
23. Langlands, R.: On the functional equations satisfied by Eisenstein series, vol. 544 of Lecture Notes in Mathematics. Springer, Berlin (1976)
24. Manickam, M., Ramakrishnan, B.: On Shimura, Shintani and Eichler–Zagier correspondences. Trans. Am. Math. Soc. **352**, 2601–2617 (2000)
25. Manickam, M., Ramakrishnan, B.: On Saito–Kurokawa correspondence of degree two for arbitrary level. J. Ramanujan Math. Soc. **17**(3), 149–160 (2002)
26. Manickam, M., Ramakrishnan, B., Vasudevan, T.C.: On Saito–Kurokawa descent for congruence subgroups. Manuscripta Math. **81**, 161–182 (1993)
27. Piatetski-Shapiro, I.I.: On the Saito–Kurokawa lifting. Invent. Math. **71**, 309–338 (1983)
28. Pitale, A., Schmidt, R.: Ramanujan type results for Siegel cusp forms of degree 2. J. Ramanujan Math. Soc. **24**(1), 87–111 (2009)
29. Ribet, K.: A modular construction of unramified p-extensions of $\mathbb{Q}(\mu_p)$. Invent. Math. **34**, 151–162 (1976)

30. Rubin, K.: Euler Systems, vol. 147 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ (2000)
31. Schmidt, R.: Iwahori-spherical representations of GSp(4) and Siegel modular forms of degree 2 with square-free level. J. Math. Soc. Jpn. **57**(1), 259–293 (2005)
32. Schmidt, R.: The Saito–Kurokawa lifting and functoriality. Am. J. Math. **127**, 209–240 (2005)
33. Schmidt, R.: On classical Saito–Kurokawa liftings. J. Reine Angew. Math. **604**, 211–236 (2007)
34. Scholl, A.J.: Motives for modular forms. Invent. Math. **100**, 419–430 (1990)
35. Shimura, G.: The special values of the zeta functions associated to cusp forms. Commun. Pure Appl. Math. **XXIX**, 783–804 (1976)
36. Shimura, G.: On Eisenstein series. Duke Math. J. **50**, 417–476 (1983)
37. Shimura, G.: Eisenstein series and zeta functions on symplectic groups. Invent. Math. **119**, 539–584 (1995)
38. Shimura, G.: Euler products and Eisenstein series, vol. 93 of CBMS. Regional Conference Series in Mathematics. AMS, Providence (1997)
39. Shintani, T.: On construction of holomorphic cusp forms of half integral weight. Nagoya Math. J. **58**, 83–126 (1975)
40. Skinner, C., Urban, E.: Sur les déformations p-adiques de certaines représentations automorphes. J. Inst. Math. Jussieu **5**, 629–698 (2006)
41. Skinner, C., Urban, E.: The Iwasawa main conjectures for GL_2. Invent. Math. (2013). doi:10.1007/s00222-013-0448-1
42. Stevens, G.: Λ-adic modular forms of half-integral weight and a Λ-adic Shintani lifting. Contemp. Math. **174**, 129–151 (1994)
43. Urban, E.: Selmer groups and the Eisenstein–Klingen ideal. Duke Math. J. **106**(3), 485–525 (2001)
44. Urban, E.: Sur les représentations p-adiques associées aux représentations cuspidales de $GSp_{4/\mathbb{Q}}$. Number 302 in Asterique. Société Mathématique de France, Inst. Henri Poincaré (2005)
45. Vatsal, V.: Canonical periods and congruence formulae. Duke Math. J. **98**(2), 397–419 (1999)
46. Washington, L.: Galois cohomology. In: Cornell, G., Silverman, J., Stevens, G. (eds.) Modular Forms and Fermat's Last Theorem (Boston, MA, 1995), pp. 101–120. Springer, New York (1997)
47. Weissauer, R.: Four dimensional Galois representations. Formes automorphes. II. Le cas du groupe GSp(4). Asterique **302**, 67–150 (2005)
48. Wiles, A.: The Iwasawa conjecture for totally real fields. Ann. Math. **131**(3), 493–540 (1990)