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On the cuspidality of pullbacks of Siegel Eisenstein series to $\mathrm{Sp}_{2m} \times \mathrm{Sp}_{2n}$

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A B S T R A C T

In this paper we study the pullback of a Siegel Eisenstein series on Sp_{2m+2n} to $\mathrm{Sp}_{2m} \times \mathrm{Sp}_{2n}$. There is a well-established literature on such pullbacks. In the case that $m = n$ Garrett showed that the pullback is actually a cusp form in each variable separately. Here we generalize this result showing the pullback is cuspidal in the smaller variable in the case $m \neq n$. Such results have applications to producing congruences between Siegel modular forms.

1. Introduction

It is well known to anyone familiar with classical modular forms that Eisenstein series play a crucial role in the development of the theory. They provide us with modular forms we can get our hands on and understand. As such, it is not surprising that they occupy a similar place when developing the theory of automorphic forms on groups other than GL(2). In this paper we focus on a particular construction using Eisenstein series on symplectic groups that does not arise when considering Eisenstein series on GL(2). In particular, one can consider the pullback of an Eisenstein series to smaller symplectic spaces embedded into the symplectic space the Eisenstein series is originally defined on.

Let L be a totally real number field and let $E_{\mathfrak{f}}(g,s)$ be a Siegel Eisenstein series on $\mathrm{Sp}_{2N}(\mathbb{A}_L)$ attached to a section $\mathfrak{f}$. Let m and n be positive integers so that $N = m + n$. One can consider the pullback of $E_{\mathfrak{f}}$ to $\mathrm{Sp}_{2m} \times \mathrm{Sp}_{2n}$ via the embedding $\mathrm{Sp}_{2m} \times \mathrm{Sp}_{2n} \to \mathrm{Sp}_{2N}$ given by

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$$\left(\begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}, \begin{pmatrix} a_h & b_h \\ c_h & d_h \end{pmatrix}\right) \mapsto \begin{pmatrix} a_g & 0 & b_g & 0 \\ 0 & a_h & 0 & b_h \\ c_g & 0 & d_g & 0 \\ 0 & c_h & 0 & d_h \end{pmatrix}.$$

There is a well-established theory of pullbacks of Eisenstein series. For example, one can see this in [2,6,7,12].

In [7], it is shown that for a particular choice of section if $m = n$ then one has that the pullback of the Siegel Eisenstein series is actually a cusp form in each variable g and h separately. In this paper we consider the case that $m \neq n$ and more general sections defining our Eisenstein series, placing restrictions only at the infinite places and restricting the section for places $\upsilon | \mathfrak{n}$ so that it is only supported on $P(L_\upsilon)K_\upsilon(\mathfrak{n})$ for P the Siegel parabolic and $K_\upsilon(\mathfrak{n})$ a compact subgroup defined in Section 2 where $\mathfrak{n}$ is a proper non-trivial ideal of L. In particular, we show that the pullback of the Eisenstein series is cuspidal in the smaller variable.

The proof presented here follows the outline of the proof given in [7] in the case $m = n$. We show that the pullback of the Eisenstein series is supported only on the "big cell" by using the restriction on the sections $\mathfrak{f}_\upsilon$ for $\upsilon | \mathfrak{n}$. Once this has been shown, we use the restriction on the sections at the infinite places to prove the pullback is cuspidal in the small variable.

Pullbacks of Eisenstein series have been used for various applications. In [7], Garrett uses his pullback formula to study the algebraicity of certain ratios of inner products of Siegel modular forms. Shimura used a pullback formula in [12] to prove that the standard L-function associated to a cuspidal Siegel Hecke eigenform can be meromorphically continued to the entire complex plane with only finitely many poles. In recent work, Pitale–Schmidt [9] and Saha [10,11] have used pullback formulas to prove algebraicity of normalized convolution L-functions of Siegel eigenforms twisted by elliptic newforms. In his thesis, Agarwal used pullbacks to construct a p-adic L-function on $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ [1]. It is clear even from these examples that the study of pullbacks of Eisenstein series is a powerful tool in number theory.

The result found in [7] for $m = n$ that with the section given there that $E_{\mathfrak{f}}(Z, W)$ is cuspidal in Z and W was used in [3] to extend the results of [4]. In particular, knowing the pullback of this Eisenstein series is cuspidal in each variable allowed one to extend the result that (up to some technical hypotheses) if $p | L_{\mathrm{alg}}(k, f)$, then $p | Ш(k, f)$ to a result showing that $\mathrm{ord}_p(L_{\mathrm{alg}}(k, f)) \leqslant \mathrm{ord}_p(Ш(k, f))$ where $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ is a normalized eigenform. This gives evidence for the Bloch–Kato conjecture for elliptic modular forms. This evidence is obtained by constructing a congruence between a Saito–Kurokawa lift and a cuspidal Siegel eigenform with irreducible Galois representation. Roughly, one expands the pullback of the Eisenstein series as

$$E_{\mathfrak{f}}(Z, W) = \sum_{i,j} c(i, j) F_i(Z) G_j(W).$$

One can then use various inner product relations to study the coefficients $c(i, j)$ which turn out to be composed of L-functions. Showing that $E_{\mathfrak{f}}$ is cuspidal in each variable allows one to restrict the F_i and the G_j to a basis of cusp forms, which makes it much easier to apply the inner product relations to obtain all the $c(i, j)$. If one does not know cuspidality, various other tricks must be applied to obtain the desired congruence. This adds hypotheses as well as weakens the final result.

The reason we put fewer restrictions on the section used to define the Eisenstein series is to obtain more general results, as the sections used in [7,12] are not always correct for arithmetic applications. It is useful to have the cuspidality of the pullback in the smaller variable when $m \neq n$. For instance, one can use the pullback formula of [11] to produce analogous results to those found in [4] with less restrictive technical hypotheses. One can see [5] for an example of this. In addition, some current work with REU students is using this result along with the pullback formula found in [2] to compute average value formulas for standard L-functions associated to a basis of cuspidal Siegel eigenforms.

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2. Notation and set-up

Let R be a commutative ring with identity. Denote the set of m by n matrices with entries in R by $M_{m,n}(R)$. If $g \in M_{m,n}(R)$ and we need to keep track of the dimensions of g, we write $g_{m,n}$. If $m = n$ we simply write $M_m(R)$ and g_m for $g_{m,m}$. We write 1_n for the n by n identity matrix. Given $x \in M_n(R)$, we write ${}^t x$ for the transpose of x.

Let L be a totally real number field of degree d with ring of integers $\mathcal{O}$. Fix an ordering of the embeddings of L into $\mathbb{R}$ giving an identification

$$L \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^d.$$

We denote the adeles of L by $\mathbb{A}_L$ and the finite adeles by $\mathbb{A}_{L,f}$. For fixed $0 \leqslant r \leqslant n$ we block decompose an element $g \in M_n(\mathbb{A}_L)$ by

$$g = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix}$$

where a_1, b_1, c_1 and d_1 lie in $M_n(\mathbb{A}_L)$ and a_4, b_4, c_4 and d_4 lie in $M_{n-r}(\mathbb{A}_L)$. Write $a_i(g)$, $b_i(g)$, $c_i(g)$, and $d_i(g)$ if necessary, to indicate the dependence on g.

Let $n \geqslant 1$ be an integer and define $J_n = \left(\begin{smallmatrix} 0_n & -1_n \\ 1_n & 0_n \end{smallmatrix}\right)$. Let

$$\mathrm{GSp}_{2n} = \left\{ g \in \mathrm{GL}_{2n}\colon\ {}^t g J_n g = \mu_n(g) J_n,\ \mu(g) \in \mathrm{GL}_1 \right\}.$$

The homomorphism $\mu_n : \mathrm{GSp}_{2n} \to \mathrm{GL}_1$ is the similitude. Its kernel is Sp_{2n}. To ease the notation we denote Sp_{2n} by G_n.

The Siegel upper half-space is

$$\mathfrak{h}_n = \left\{ Z \in M_n(\mathbb{C})\colon\ {}^t Z = Z,\ \mathrm{Im}(Z) > 0 \right\}.$$

The group $G_n(\mathbb{R})$ acts on $\mathfrak{h}_n$ via linear fractional transformations, i.e.,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1}.$$

We set

$$j(g, Z) = \det(CZ + D)^{-1}$$

for $Z \in \mathfrak{h}_n$ and $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_n(\mathbb{R})$.

Our identification of $L \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^d$ gives

$$G_n(L \otimes_{\mathbb{Q}} \mathbb{R}) \cong G_n(\mathbb{R})^d$$

and so we view $G_n(L)$ as a subgroup of $G_n(\mathbb{R})^d$. Thus, we have an action of $G_n(L)$ on $\mathfrak{h}_n^d$.

For each $0 < r < n$ we have parabolic subgroups of G_n given by

$$P_{n,r} = \left\{ g \in G_n\colon\ a_2(g) = c_2(g) = 0,\ c_3(g) = d_3(g) = 0,\ c_4(g) = 0 \right\}.$$

The case of $r = 0$ gives the Siegel parabolic subgroup

$$P_n := P_{n,0} = \{g \in G_n\colon\ c(g) = 0\}$$

and for $r = n$ we have $P_{n,n} = G_n$. Recall the standard decomposition of P_n into a unipotent radical and Levi subgroup given by $P_n = U_{P_n} M_{P_n}$ where

$$U_{P_n} = \left\{ u(x) = \begin{pmatrix} 1_n & x \\ 0_n & 1_n \end{pmatrix}\colon\ x \in \mathbb{S}_n \right\}$$

and

$$M_{P_n} = \left\{ Q(A) = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}\colon\ A \in \mathrm{GL}_n \right\}$$

where

$$\mathbb{S}_n = \{x \in M_n\colon\ {}^tx = x\}.$$

The modulus character of the Siegel parabolic is given by

$$\delta_{P_n}\left(\begin{pmatrix} 1_n & x \\ 0_n & 1_n \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \right) = |\det A|^{4n}.$$

Let $\mathfrak{n}$ be a non-zero ideal of $\mathcal{O}$. For a prime $\upsilon | \mathfrak{n}$ define

$$K_\upsilon(\mathfrak{n}) = \{g \in G_n(L_\upsilon) \cap M_{2n}(\mathcal{O}_\upsilon)\colon\ g \equiv 1 \pmod{\mathfrak{n}}\}$$

and for a finite prime $\upsilon \nmid \mathfrak{n}$ set $K_\upsilon(\mathfrak{n}) = G_n(L_\upsilon) \cap M_{2n}(\mathcal{O}_\upsilon)$. Set

$$K_f(\mathfrak{n}) = \prod_{\upsilon \nmid \infty} K_\upsilon(\mathfrak{n}).$$

Define

$$K_\infty = \{g = (g_1, \ldots, g_d) \in G_n(\mathbb{R})^d\colon\ g_j(i_n) = i_n,\ j = 1, \ldots, d\}$$

where we denote $i1_n$ by i_n. Set

$$K(\mathfrak{n}) = K_\infty K_f(\mathfrak{n}).$$

We now define the Siegel Eisenstein series. Let $\chi = \bigotimes_\upsilon \chi_\upsilon$ be an idele class character. For $s \in \mathbb{C}$, consider the induced representation

$$I(\chi, s) = \bigotimes_\upsilon I_\upsilon(\chi_\upsilon, s) = \mathrm{Ind}_{P_n(\mathbb{A}_L)}^{G_n(\mathbb{A}_L)}\left(\chi |\cdot|^{2s}\right)$$

consisting of smooth functions $\mathfrak{f}$ on $G_n(\mathbb{A}_L)$ satisfying

$$\mathfrak{f}(pg, s) = \chi\left(\det(A)\right) |\det A|^{2s} \mathfrak{f}(g, s)$$

for $p = u(x)Q(A) \in P_n(\mathbb{A}_L)$ and $g \in G_n(\mathbb{A}_L)$. Given such a section, we define the associated Eisenstein series by

$$E_{\mathfrak{f}}(g,s) = \sum_{\gamma \in P_n(L)\backslash G_n(L)} \mathfrak{f}(\gamma g, s).$$

We put some minor restrictions on the Eisenstein series of interest. Let $\mathfrak{n}$ be a proper non-zero ideal of $\mathcal{O}$ and $\kappa > 2n+1$ an integer. At the infinite places we choose $\mathfrak{f}_\infty$ to be the unique vector in $I_\infty(\chi_\infty, s)$ so that

$$\mathfrak{f}_\infty(k_\infty, s) = \det\bigl(j(k_\infty, i_n)\bigr)^{-\kappa}$$

for all $k_\infty \in K_\infty$ where we use the multi-index convention that

$$j(\gamma, z)^\kappa = \prod_{i=1}^{d} j(\gamma_i, z_i)^\kappa$$

for

$$\gamma = (\gamma_1, \ldots, \gamma_d) \in G_n(\mathbb{R})^d$$

and

$$z = (z_1, \ldots, z_d) \in \mathfrak{h}_n^d.$$

Furthermore, for those $\upsilon | \mathfrak{n}$, we require that $\mathfrak{f}_\upsilon$ be $K_\upsilon(\mathfrak{n})$-fixed and vanish off $P_n(L_\upsilon)K_\upsilon(\mathfrak{n})$.

One should note here that the Eisenstein series considered in [4,7], and [12] all have this property. While the Siegel Eisenstein series used in [5] does not have the property that for $\upsilon | \mathfrak{n}$ $\mathfrak{f}_\upsilon$ vanishes off $P_n(L_\upsilon)K_\upsilon(\mathfrak{n})$, it is required that such $\mathfrak{f}_\upsilon$ vanish off $P_n(L_\upsilon)\mathcal{Q}_\upsilon K_\upsilon(\mathfrak{n})$ for a particular fixed element $\mathcal{Q}_\upsilon$. One can easily adapt the arguments given here to cover that case as well.

3. Pullbacks and restriction to the big cell

The focus of this paper is pullbacks of Siegel Eisenstein series. One can embed $G_m \times G_n$ into G_{m+n} via the map ι defined by

$$\iota(\alpha, \beta) = \begin{pmatrix} a_\alpha & 0 & b_\alpha & 0 \\ 0 & a_\beta & 0 & b_\beta \\ c_\alpha & 0 & d_\alpha & 0 \\ 0 & c_\beta & 0 & d_\beta \end{pmatrix}.$$

Given the Siegel Eisenstein series defined in Section 2 on $G_{m+n}(\mathbb{A}_L)$, we can consider the Eisenstein series restricted to $\iota(G_m(\mathbb{A}_L) \times G_n(\mathbb{A}_L))$. We refer to this restriction as the pullback of E to $G_m(\mathbb{A}_L) \times G_n(\mathbb{A}_L)$. It is known that the pullback of E is an automorphic form in each variable separately.

Let $\kappa > 2m + 2n + 1$ be an integer and $\mathfrak{n}$ a non-zero proper ideal of $\mathcal{O}$. Let $\chi = \bigotimes_\upsilon \chi_\upsilon$ be an idele class character of the same parity as κ at the infinite places. Let $\mathfrak{f} \in I(\chi, s)$ be as in Section 2.

Let N be a positive integer (we will later take it to be $m+n$). Let υ be a prime dividing $\mathfrak{n}$ and let $\mathrm{GL}_N(L_\upsilon)$ have the Haar measure normalized so that $\mathrm{GL}_N(\mathcal{O}_\upsilon)$ has measure 1. Let φ_υ be the characteristic function of

$$\bigl\{(AB) \in M_{N,2N}(L_\upsilon)\colon\ (AB) \equiv (0_N 1_N) \pmod{\mathfrak{n}}\bigr\}.$$

As in [7], we define

$$\mathcal{I}_\upsilon(g) = \int\limits_{\mathrm{GL}_N(L_\upsilon)} |\det t|^\kappa \chi(\det t)\varphi_\upsilon\big(t(0_N 1_N)g\big)\,dt.$$

By assumption, $\mathfrak{f}_\upsilon$ vanishes off $P(L_\upsilon)K_\upsilon(\mathfrak{n})$ and is $K_\upsilon(\mathfrak{n})$-fixed. Thus for $\upsilon|\mathfrak{n}$ and $g = pk \in P(L_\upsilon)K_\upsilon(\mathfrak{n})$ we have that

$$\mathfrak{f}_\upsilon(g, k/2) = \frac{\mathcal{I}_\upsilon(p)}{\mathcal{I}_\upsilon(1_{2N})}.$$

We now reduce to the case that $N = m + n$. We assume that $n \leqslant m$ from now on. We have the following result that follows immediately from the work of Garrett in [7] and Lemma 4.2 of [12].

Lemma 3.1. *For $0 \leqslant r \leqslant \min(m, n)$ set*

$$e_r = \begin{pmatrix} 0 & 0 \\ 0 & 1_r \end{pmatrix} \in M_{m,n}(L)$$

and

$$\tau_r = \begin{pmatrix} 0 & 0 & -1_m & 0 \\ 0 & 1_n & 0 & 0 \\ 1_m & e_r & 0 & 0 \\ 0 & 0 & -{}^t e_r & 1_n \end{pmatrix}.$$

Then the τ_r form a complete set of representatives of $X = P_N(L)\backslash G_N(L)/\iota(G_m(L) \times G_n(L))$.

We will work with a translate of the Eisenstein series defined in Section 2. Define $\sigma \in G_N(\mathbb{A}_{L,f})$ by setting

$$\sigma_\upsilon = \begin{cases} \tau_n & \text{if } \upsilon|\mathfrak{n}, \\ 1_N & \text{otherwise.} \end{cases}$$

Define

$$E_{\mathfrak{f}}^\sharp(g, s) = E_{\mathfrak{f}}\big(g\sigma^{-1}, s\big).$$

For $x \in X$, let I_x denote the isotropy group of $P_N(L)x$ in $P_N(L)\backslash G_N(L)$ under the right action of $G_m(L) \times G_n(L)$, i.e.,

$$I_x = \big\{(g, h) \in G_m(L) \times G_n(L)\colon\ P_N(L)x\iota(g, h) = P_N(L)x\big\}.$$

Write I_i for I_{τ_i} to ease notation. Set

$$\mathcal{E}_i(g, h) = \sum_{\gamma \in I_i\backslash G_m(L)\times G_n(L)} \mathfrak{f}_{k/2}\big(\tau_i\iota(\gamma)\iota(g, h)\sigma^{-1}\big).$$

Then we have

$$E_{\mathfrak{f}}^{\sharp}\big(\iota(g,h),k/2\big)=\sum_{i=0}^{n}\mathcal{E}_i(g,h).$$

Our goal is to show that $\mathcal{E}_i(g,h)=0$ unless $i=n$. This will show our Eisenstein series vanishes off the big cell. In order to show this, we will prove that $\mathfrak{f}_{\upsilon,k/2}(\tau_i\iota(\gamma)\iota(g,h))=0$ unless $i=n$.

Our work above shows that it is enough to show that $\mathcal{I}_{\upsilon}(\tau_i\iota(g,h)\sigma_{\upsilon}^{-1})=0$ for all $(g,h)\in G_m(L_{\upsilon})\times G_n(L_{\upsilon})$ and all $0\leqslant i<n$ for $\upsilon|\mathfrak{n}$. In turn, the definition of $\mathcal{I}_{\upsilon}$ shows it is enough to show that

$$\varphi_{\upsilon}\big(t(0_N 1_N)\tau_i\iota(g,h)\sigma_{\upsilon}^{-1}\big)=0$$

for all $(g,h)\in G_m(L_{\upsilon})\times G_n(L_{\upsilon})$ and all $t\in \mathrm{GL}_N(L_{\upsilon})$. Note that $\varphi_{\upsilon}(t(0_N 1_N)\tau_i\iota(g,h)\sigma_{\upsilon}^{-1})\neq 0$ if and only if $t(0_N 1_N)\tau_i\iota(g,h)\sigma_{\upsilon}^{-1}\equiv(0_N 1_N) \pmod{\mathfrak{n}}$, i.e.,

$$t(0_N 1_N)\tau_i\iota(g,h)\equiv(0_N 1_N)\sigma_{\upsilon}\pmod{\mathfrak{n}}.$$

Let

$$(AB)=\begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12}\\ a_{21} & a_{22} & b_{21} & b_{22}\end{pmatrix}\in M_{N,2N}.$$

Define $\Psi((AB))=\left(\begin{smallmatrix} a_{11} & b_{11}\\ a_{21} & b_{21}\end{smallmatrix}\right)$. Observing that

$$(0_N 1_N)\sigma_{\upsilon}=\begin{pmatrix} 1_m & e_n & 0 & 0\\ 0 & 0 & -{}^t e_n & 1_n\end{pmatrix},$$

we see that $\Psi((0_N 1_N)\sigma_{\upsilon})$ has rank $m+n$.

We now calculate $\Psi(t(0_N 1_N)\tau_i\iota(g,h))$. We have

$$(0_N 1_N)\tau_i=\begin{pmatrix} 1_m & e_i & 0 & 0\\ 0 & 0 & -{}^t e_i & 1_n\end{pmatrix}.$$

This gives that $\Psi((0_N 1_N)\tau_i)$ has rank $n+i$. Let $t=\left(\begin{smallmatrix} \alpha_m & \beta_{m,n}\\ \gamma_{n,m} & \delta_n\end{smallmatrix}\right)$, $g=\left(\begin{smallmatrix} a_m & b_m\\ c_m & d_m\end{smallmatrix}\right)$, and $h=\left(\begin{smallmatrix} u_n & v_n\\ s_n & t_n\end{smallmatrix}\right)$. Then we have

$$t(0_N 1_N)\tau_i\iota(g,h)=\begin{pmatrix} \alpha a-\beta\, {}^t e_i c & \alpha e_i u+\beta s & \alpha b-\beta\, {}^t e_i d & \alpha e_i v+\beta t\\ \gamma a-\delta\, {}^t e_i c & \gamma e_i u+\delta s & \gamma b-\delta\, {}^t e_i d & \gamma e_i v+\delta t\end{pmatrix}.$$

From this a straight-forward calculation gives that

$$\Psi\big(t(0_N 1_N)\tau_i\iota(g,h)\big)=t\Psi\big((0_N 1_N)\tau_i\big)g.$$

Thus, we see that $\Psi(t(0_N 1_N)\tau_i\iota(g,h))$ has rank $m+i$. However, we now use the fact that $\varphi_{\upsilon}(t(0_N 1_N)\tau_i\iota(g,h)\sigma_{\upsilon}^{-1})\neq 0$ if and only if

$$t(0_N 1_N)\tau_i\iota(g,h)\equiv(0_N 1_N)\sigma_{\upsilon}\ (\mathrm{mod}\ \mathfrak{n}).$$

This would give a matrix of rank $m+i$ congruent to a matrix of rank $m+n$ modulo a proper ideal. This is a contradiction unless $i=n$. Thus, we have shown that $\mathcal{E}_i(g,h)=0$ unless $i=n$ as desired. In particular, we have the following result.

Lemma 3.2. *Let $\kappa > 2m + 2n + 1$ be an integer and $\mathfrak{n}$ a non-zero proper ideal in $\mathcal{O}$. Let χ be an idele class character of conductor so that χ has the same parity as κ at the infinite places. Let $\mathfrak{f} \in I(\chi, s)$ be a section with the restriction that for $\upsilon|\mathfrak{n}$, $\mathfrak{f}_\upsilon$ vanishes off $P_\upsilon(L_\upsilon)K_\upsilon(\mathfrak{n})$ and is $K_\upsilon(\mathfrak{n})$-fixed. Then we have*

$$E_{\mathfrak{f}}^{\sharp}\big(\iota(g,h), k/2\big) = \sum_{\gamma \in I_n \backslash G_m(L) \times G_n(L)} \mathfrak{f}\big(\tau_n \iota(\gamma)\iota(g,h)\sigma^{-1}, k/2\big).$$

4. Cuspidality in the small variable

In this section we use the results of Section 3 to show that the pullback of the Eisenstein series considered there is cuspidal in the small variable. We again assume that $m \geqslant n$ in this section and set $N = m + n$.

The next step in simplifying the Eisenstein series is to study $I_n \backslash (G_m(L) \times G_n(L))$. In the case that $m = n$, it is shown in [7] that $I_n \cong G_n(L)$ and so one has in that case that $I_n \backslash (G_n(L) \times G_n(L)) \cong G_n(L)$. It is easy to recover this result from the following more general computation.

Let $(g,h) \in G_m(L) \times G_n(L)$ with $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $h = \left(\begin{smallmatrix} u & v \\ s & t \end{smallmatrix}\right)$ as in Section 3. Observe that we have

$$\tau_n \iota(g,h)\tau_n^{-1} = \begin{pmatrix} d & ce_n & -c & 0 \\ -{}^t e_n v & u & 0 & v \\ -b - e_n v\,{}^t e_n & -ae_n + e_n u & a & e_n v \\ {}^t e_n d - t\,{}^t e_n & {}^t e_n c e_n + s & -{}^t e_n c & t \end{pmatrix}.$$

We have that $(g,h) \in I_n$ if and only if $\tau_n \iota(g,h)\tau_n^{-1} \in P_N(L)$. In other words, we have $(g,h) \in I_n$ if and only if for $h \in G_n(L)$ we have

$$g = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & u & 0 & -v \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & -s & 0 & t \end{pmatrix}.$$

Set $\widetilde{P}_{m,n}$ to be the subset of G_N given by matrices of the form

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix}.$$

Thus, we have

$$I_n = \big\{\big(g, \Phi(g)\big)\colon\ g \in \widetilde{P}_{m,n}(L)\big\}$$

where $\Phi : \widetilde{P}_{m,n}(L) \to G_n(L)$ is defined by

$$\Phi\left(\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix}\right) = \begin{pmatrix} a_{22} & -a_{24} \\ -a_{42} & a_{44} \end{pmatrix}.$$

Note that one can easily check that for $g \in \widetilde{P}_{m,n}(L)$ one has $\Phi(g) \in G_n(L)$ and so the map is well defined. Thus, we see that $I_n(g,h) = I_n(g',h')$ if and only if there exists $p \in \widetilde{P}_{m,n}(L)$ so that $(pg, \Phi(p)h) = (g',h')$. Moreover, given $h = \left(\begin{smallmatrix} u & v \\ s & t \end{smallmatrix}\right) \in G_n(L)$, we see that

$$\Upsilon(h) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u & 0 & -v \\ 0 & 0 & 1 & 0 \\ 0 & -s & 0 & t \end{pmatrix} \in \widetilde{P}_{m,n}(L).$$

Let $\{g_i\}_{i\in I}$ be a set of coset representatives for $\widetilde{P}_{m,n}(L)\backslash G_m(L)$. We claim that $\{(g_i, h)\colon\ i \in I,\ h \in G_n(L)\}$ is a set of coset representatives for $I_n\backslash(G_m(L) \times G_n(L))$. Define $\Theta : G_m(L) \to \widetilde{P}_{m,n}(L)$ by setting $\Theta(g)$ to be the unique element in $\widetilde{P}_{m,n}(L)$ so that $g = \Theta(g)g_i$ for i so that $g \in \widetilde{P}_{m,n}(L)g_i$. Let $(g,h) \in G_m(L) \times G_n(L)$. Then we have

$$\begin{aligned} I_n(g,h) &= I_n\big(\Theta(g)g_i, h\big) = I_n\big(\Theta(g)g_i, \Phi\big(\Theta(g)\big)\big(\Phi\big(\Theta(g)\big)^{-1}h\big)\big) \\ &= I_n\big(g_i, \Phi\big(\Theta(g)\big)^{-1}h\big). \end{aligned}$$

Since $\Phi(\Theta(g)) \in G_n(L)$, we have that $\Phi(\Theta(g))^{-1}h \in G_n(L)$ so we see that every coset has a representative in $\{(g_i, h)\colon\ i \in I,\ h \in G_n(L)\}$. Conversely, let (g_i, h) be any element of $\{(g_i, h)\colon\ i \in I,\ h \in G_n(L)\}$. Choose a $g \in G_m(L)$ so that $\widetilde{P}_{m,n}(L)g = \widetilde{P}_{m,n}(L)g_i$. Then we have $(g, \Phi(g)h) \in G_m(L) \times G_n(L)$ and

$$I_n\big(g, \Phi\big(\Theta(g)\big)h\big) = I_n\big(\Theta(g)g_i, \Phi\big(\Theta(g)\big)h\big) = I_n(g_i, h).$$

Thus, we have that $\{(g_i, h)\colon\ i \in I,\ h \in G_n(L)\}$ is a set of coset representatives of $I_n\backslash(G_m(L) \times G_n(L))$. This is close, but not exactly what we need. However, using this set of representatives one sees that the set

$$\big\{\big(\Upsilon(y)g_i, u\big)\colon\ y \in \widetilde{P}_{m,n}(L)\backslash G_m(L),\ i \in I,\ u \in U_{P_n}(L)\big\}$$

also gives a set of coset representatives. This is the desired set of representatives.

We can now proceed with the same argument as given in [3,7]. Set $Y = \{\Upsilon(y)g_i\colon\ y \in \widetilde{P}_{m,n}(L)\backslash G_m(L),\ i \in I\}$. Then we can write our Eisenstein series as

$$E^{\sharp}_{\mathfrak{f}}\big(\iota(g,h), k/2\big) = \sum_{\gamma\in Y}\sum_{u\in U_{P_n}(L)} \mathfrak{f}\big(\tau_n\iota(\gamma g, uh)\sigma^{-1}, k/2\big).$$

We expand the sum

$$\sum_{u\in U_{P_n}(L)} \mathfrak{f}\big(\tau_n\iota(\gamma g, uh)\sigma^{-1}, k/2\big)$$

in its Fourier expansion in h along the unipotent radical. Recalling that our goal is to show that the Eisenstein series is cuspidal in the $G_n(L)$-variable, it is enough to show that the Tth Fourier coefficient is zero for all $(g,h) \in G_m(\mathbb{A}_L) \times G_n(\mathbb{A}_L)$ unless T is totally positive definite.

Let ψ be the standard additive character on $\mathbb{A}_L/L$ given by $x \mapsto e^{2\pi ix}$ on $\mathbb{R}$ and is trivial on L_υ for $\upsilon \nmid \infty$. The Fourier coefficient is given by

$$\int\limits_{U_{P_n}(L)\backslash U_{P_n}(\mathbb{A}_L)} \overline{\psi}\big(\mathrm{Tr}(Tx)\big) \sum_{u\in U_{P_n}(L)} \mathfrak{f}\big(\tau_n\iota\big(g, uu(x)h\big)\sigma^{-1}, k/2\big)\,dx.$$

Folding this integral we obtain

$$\int\limits_{U_{P_n}(\mathbb{A}_L)} \overline{\psi}\big(\mathrm{Tr}(Tx)\big)\mathfrak{f}\big(\tau_n\iota\big(g,u(x)h\big)\sigma^{-1},k/2\big)\,dx.$$

We now restrict to considering an infinite place υ. In particular, we show that the integral

$$\int\limits_{U_{P_n}(\mathbb{R})} \overline{\psi}\big(\mathrm{Tr}(Tx)\big)\mathfrak{f}_\upsilon\big(\tau_n\iota\big(g,u(x)h\big),k/2\big)\,dx$$

is zero unless T is totally positive definite. Note that $\sigma_\upsilon = 1$ for all $\upsilon|\infty$ so it drops out of the calculation. Observe that the above integral can be written as

$$\int\limits_{\mathbb{S}_n(\mathbb{R})} \overline{\psi}\big(\mathrm{Tr}(Tx)\big)\mathfrak{f}_\upsilon\big(\tau_n\iota\big(g,u(x)h\big),k/2\big)\,dx.$$

We reduce to considering g and h of the form $g = u(y)Q(A_1)$ and $h = Q(A_2)$ by using the Iwasawa decomposition and the right $(K(\mathfrak{n}), j^{-\kappa})$-equivariance of $\mathfrak{f}_\upsilon$.

We calculate

$$\tau_n\iota\big(g,u(x)h\big) = \begin{pmatrix} 0 & 0 & -{}^tA_1^{-1} & 0 \\ 0 & A_2 & 0 & x\,{}^tA_2^{-1} \\ A_1 & e_nA_2 & y\,{}^tA_1^{-1} & e_nx\,{}^tA_2^{-1} \\ 0 & 0 & -{}^te_n\,{}^tA_1^{-1} & {}^tA_2^{-1} \end{pmatrix}.$$

Given any $g \in G_N(\mathbb{R})$, we have that $\mathfrak{f}_\upsilon(g,k/2) = \chi_\upsilon(A_g)j(g,i)^{-\kappa}$ where $g = u(x_g)Q(A_g)k_g$. This follows immediately from the definition of $\mathfrak{f}_\upsilon$ for $\upsilon|\infty$ and the fact that $j(g,i)^{-\kappa} = |\det A_g|^{\kappa} j(k_g,i)^{-\kappa}$. Thus, in our situation we have

$$\mathfrak{f}_\upsilon\big(\tau_n\iota\big(g,u(x)h\big),k/2\big) = \chi_\upsilon(\det\widetilde{A})\det\left(i\begin{pmatrix} A_1 & e_nA_2 \\ 0 & 0\end{pmatrix} + \begin{pmatrix} y\,{}^tA_1^{-1} & e_nx\,{}^tA_2^{-1} \\ -{}^te_n\,{}^tA_1^{-1} & {}^tA_2^{-1}\end{pmatrix}\right)^{-\kappa}$$

where we write $\widetilde{A} = A_{\tau_n\iota(g,u(x)h)}$ to ease notation. Write

$$i\begin{pmatrix} A_1 & e_nA_2 \\ 0 & 0\end{pmatrix} + \begin{pmatrix} y\,{}^tA_1^{-1} & e_nx\,{}^tA_2^{-1} \\ -{}^te_n\,{}^tA_1^{-1} & {}^tA_2^{-1}\end{pmatrix} = \begin{pmatrix} iA_1 + y\,{}^tA_1^{-1} & ie_nA_2 + e_nx\,{}^tA_2^{-1} \\ -{}^te_n\,{}^tA_1^{-1} & {}^tA_2^{-1}\end{pmatrix}.$$

We can decompose this matrix as

$$\begin{pmatrix} iA_1\,{}^tA_1 + y & ie_nA_2\,{}^tA_2 + e_nx \\ -{}^te_n & 1\end{pmatrix}\begin{pmatrix} {}^tA_1^{-1} & 0 \\ 0 & {}^tA_2^{-1}\end{pmatrix}.$$

From this we see it is enough to show that

$$\int\limits_{\mathbb{S}_n(\mathbb{R})} \overline{\psi}\big(\mathrm{Tr}(Tx)\big)\det\begin{pmatrix} iA_1\,{}^tA_1 + y & ie_nA_2\,{}^tA_2 + e_nx \\ -{}^te_n & 1\end{pmatrix}^{-\kappa}dx = 0$$

unless T is totally positive definite.

Observe that given a block matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ so that D^{-1} exists we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ C & D \end{pmatrix}$$

and so

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D) \det\bigl(A - BD^{-1}C\bigr).$$

In our case we obtain

$$\begin{aligned} &\begin{pmatrix} iA_1\,{}^tA_1 + y & ie_nA_2\,{}^tA_2 + e_nx \\ -{}^te_n & 1 \end{pmatrix} \\ &\quad = \begin{pmatrix} 1 & ie_nA_2\,{}^tA_2 + e_nx \\ 0 & 1 \end{pmatrix} \begin{pmatrix} iA_1\,{}^tA_1 + y + ie_nA_2\,{}^tA_2 + e_nx & 0 \\ -{}^te_n & 1 \end{pmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} \det \begin{pmatrix} iA_1\,{}^tA_1 + y & ie_nA_2\,{}^tA_2 + e_nx \\ -{}^te_n & 1 \end{pmatrix} &= \det \begin{pmatrix} iA_1\,{}^tA_1 + y + ie_nA_2\,{}^tA_2\,{}^te_n + e_nx\,{}^te_n & 0 \\ -{}^te_n & 1 \end{pmatrix} \\ &= \det\bigl(iA_1\,{}^tA_1 + y + ie_nA_2\,{}^tA_2\,{}^te_n + e_nx\,{}^te_n\bigr). \end{aligned}$$

Observe that we can write

$$iA_1\,{}^tA_1 + y + ie_nA_2\,{}^tA_2\,{}^te_n + e_nx\,{}^te_n = Z + \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$$

for $Z = iA_1\,{}^tA_1 + y \in \mathfrak{h}_{m+n}$ and $A = iA_2\,{}^tA_2 + x$. Write $Z = \left(\begin{smallmatrix} z & u \\ {}^tu & t \end{smallmatrix}\right)$. Then we have

$$\begin{pmatrix} 1 & 0 \\ -{}^tuz^{-1} & 1 \end{pmatrix} \begin{pmatrix} z & u \\ {}^tu & t \end{pmatrix} \begin{pmatrix} 1 & -z^{-1}u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & t - {}^tuz^{-1}u \end{pmatrix}.$$

Using this, we have reduced the problem to showing the following integral vanishes unless T is totally positive definite

$$\begin{aligned} &\int_{\mathbb{S}_n(\mathbb{R})} \overline{\psi}\bigl(\mathrm{Tr}(Tx)\bigr) \det\left(Z + \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}\right)^{-\kappa} dx \\ &\quad = \int_{\mathbb{S}_n(\mathbb{R})} \overline{\psi}\bigl(\mathrm{Tr}(Tx)\bigr) \det \begin{pmatrix} z & 0 \\ 0 & t - {}^tuz^{-1}u \end{pmatrix}^{-\kappa} dx \\ &\quad = (\det z)^{-\kappa} \int_{\mathbb{S}_n(\mathbb{R})} \overline{\psi}\bigl(\mathrm{Tr}(Tx)\bigr) \det\bigl(x + \bigl(t - {}^tuz^{-1}u\bigr)\bigr)^{-\kappa} dx. \end{aligned}$$

However, it is a classical result due to Siegel that the integral

$$\int_{\mathbb{S}_n(\mathbb{R})} \overline{\psi}\bigl(\mathrm{Tr}(Tx)\bigr) \det\bigl(x + \bigl(t - {}^tuz^{-1}u\bigr)\bigr)^{-\kappa} dx$$

vanishes unless T is totally positive definite. As this seems to be a difficult result to find in the literature, we include a proof here graciously provided to the author by Paul Garrett.

Proposition 4.1. *(See [8].) Let $C_n(\mathbb{R})$ be the cone of positive definite symmetric real n by n matrices. For $y \in C_n(\mathbb{R})$ we have*

$$\int\limits_{\mathbb{S}_n(\mathbb{R})} e^{i\,\mathrm{Tr}(x\xi)} \det(y-ix)^{-s}\,dx = \begin{cases} c(s)e^{-\mathrm{Tr}(y\xi)}(\det\xi)^{s-\frac{n+1}{2}} & (\text{for } \xi \in C_n(\mathbb{R})),\\ 0 & (\text{for } \xi \notin C_n(\mathbb{R})),\end{cases}$$

where dx is the product of usual Lebesgue measures on the coordinates x_{ij} with $i \geqslant j$ and

$$c(s) = \frac{1}{\Gamma(s)\Gamma(s-1/2)\Gamma(s-3/2)\cdots\Gamma(s-(n-1)/2)(2\pi)^n\pi^{n(n-1)}}.$$

Proof. Recall that the gamma function attached to $C_n(\mathbb{R})$ is given by

$$\Gamma_n(s) = \int\limits_{C_n(\mathbb{R})} e^{-\mathrm{Tr}(\xi)}(\det\xi)^s \frac{d\xi}{(\det\xi)^{\frac{n+1}{2}}}$$

with $d\xi$ the product of the usual Lebesgue measures on the coordinates ξ_{ij} with $i \leqslant j$. Observe that the measure $d\xi/(\det\xi)^{\frac{n+1}{2}}$ is invariant under the action of $\mathrm{GL}_n(\mathbb{R})$ on C_n given by $A\cdot\xi = A\xi{}^tA$.

Let $y \in C_n(\mathbb{R})$. Then y has a unique square root in $C_n(\mathbb{R})$, which we denote $\sqrt{y}$. We have

$$\mathrm{Tr}(\sqrt{y}\xi\sqrt{y}) = \mathrm{Tr}(y\xi).$$

We can use the invariance of the measure under the action as mentioned above to replace ξ by $\sqrt{y}\xi\sqrt{y}$ in the integral defining $\Gamma_n(s)$ to obtain

$$\Gamma_n(s) = (\det y)^s \int\limits_{C_n(\mathbb{R})} e^{-\mathrm{Tr}(y\xi)}(\det\xi)^s \frac{d\xi}{(\det\xi)^{\frac{n+1}{2}}}.$$

Using the analytic continuation of $\Gamma_n(s)$, we have for $x \in \mathbb{S}_n(\mathbb{R})$

$$\Gamma_n(s) = \bigl(\det(y-ix)\bigr)^s \int\limits_{C_n(\mathbb{R})} e^{-\mathrm{Tr}((y-ix)\xi)}(\det\xi)^s \frac{d\xi}{(\det\xi)^{\frac{n+1}{2}}},$$

i.e.,

$$\frac{\Gamma_n(s)}{(\det(y-ix))^s} = \int\limits_{C_n(\mathbb{R})} e^{i\,\mathrm{Tr}(x\xi)}e^{-\mathrm{Tr}(y\xi)}(\det\xi)^s \frac{d\xi}{(\det\xi)^{\frac{n+1}{2}}}.$$

We can view this integral as an inverse Fourier transform on $\mathbb{S}_n(\mathbb{R})$ of the function

$$\varphi_y(\xi) = \begin{cases} e^{-\mathrm{Tr}(y\xi)}(\det\xi)^{s-\frac{n+1}{2}} & (\text{for } \xi \in C_n(\mathbb{R})),\\ 0 & (\text{for } \xi \notin C_n(\mathbb{R})),\end{cases}$$

where the Fourier transform on $\mathbb{S}_n(\mathbb{R})$ is normalized to

$$\widehat{f}(\xi) = \int\limits_{\mathbb{S}_n(\mathbb{R})} e^{-i\,\mathrm{Tr}(x\xi)} f(x)\,dx$$

with inverse transform

$$f^{\vee}(x) = \int\limits_{\mathbb{S}_n(\mathbb{R})} e^{i\,\mathrm{Tr}(x\xi)} f(\xi)\,d\xi.$$

The Fourier inversion constant is given by

$$(2\pi)^{-n}\pi^{-\frac{n(n-1)}{2}} f(x) = \int\limits_{\mathbb{S}_n(\mathbb{R})} e^{i\,\mathrm{Tr}(x\xi)}\widehat{f}(x)\,dx = \int\limits_{\mathbb{S}_n(\mathbb{R})} e^{-\,\mathrm{Tr}(x\xi)} f^{\vee}(\xi)\,d\xi.$$

Since we have

$$\widehat{\varphi}_y(x) = \frac{\Gamma_n(s)}{(\det(y-x))^s},$$

Fourier inversion gives

$$\widehat{\left(\frac{\Gamma_n(s)}{(\det(y-x))^s}\right)}(\xi) = (2\pi)^{-n}\pi^{-\frac{n(n-1)}{2}}\varphi_y(\xi),$$

i.e.,

$$\int\limits_{\mathbb{S}_n(\mathbb{R})} e^{-i\,\mathrm{Tr}(x\xi)}\big(\det(y-ix)\big)^{-s}\,dx = \frac{1}{\Gamma_n(s)(2\pi)^n\pi^{\frac{n(n-1)}{2}}}\varphi_y(\xi).$$

To finish the proof we need to relate $\Gamma_n(s)$ to the classical gamma function given by

$$\Gamma(s) = \int\limits_0^{\infty} e^{-t}t^s\frac{dt}{t}.$$

This is accomplished as follows. Define

$$f : C_{n-1}(\mathbb{R}) \times \mathbb{R}^{n-1} \times C_1(\mathbb{R}) \to C_n(\mathbb{R})$$

by setting

$$f(y,v,t) = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ {}^t v & 1 \end{pmatrix} = \begin{pmatrix} y + vt\,{}^t v & tv \\ t\,{}^t v & t \end{pmatrix}$$

where we view $\mathbb{R}^{n-1}$ as column vectors. Thus, we have

$$\begin{aligned}\Gamma_n(s) &= \int\limits_{C_{n-1}(\mathbb{R})\times\mathbb{R}^{n-1}\times C_1(\mathbb{R})} e^{-\operatorname{Tr}(y+vt^tv+t)}(\det y)^s t^s \frac{dy\, t^{n-1}\, dv\, dt}{(\det y)^{\frac{n+1}{2}} t^{\frac{n+1}{2}}} \\ &= \int\limits_{\mathbb{R}^{n-1}} e^{-{}^tvv}\, dv \cdot \int\limits_{C_{n-1}(\mathbb{R})} e^{-\operatorname{Tr}(y)}(\det y)^{s-\frac{1}{2}} \frac{dy}{(\det y)^{\frac{(n-1)+1}{2}}} \cdot \int\limits_0^\infty e^{-t} t^{s+(n-1)-\frac{n+1}{2}-\frac{n-1}{2}+1} \frac{dt}{t} \\ &= \pi^{(n-1)/2}\Gamma(s)\Gamma_{n-1}(s-1/2).\end{aligned}$$

We now apply induction to obtain

$$\Gamma_n(s) = \pi^{n(n-1)/2}\Gamma(s)\Gamma(s-1/2)\Gamma(s-3/2)\cdots\Gamma\big(s-(n-2)/2\big)\Gamma\big(s-(n-1)/2\big),$$

which gives the result. □

References

[1] M. Agarwal, p-Adic L-functions for $\mathrm{GSp}(4)\times\mathrm{GL}(2)$, PhD thesis, University of Michigan, 2007.

[2] S. Böcherer, Über die Funktionalgleichung automorpher L-Funktionen zur Siegelscher Modulgruppe, J. Reine Angew. Math. 362 (1985) 146–168.

[3] J. Brown, On the cuspidality of pullbacks of Siegel Eisenstein series and applications to the Bloch–Kato conjecture, Int. Math. Res. Not. IMRN, doi:10.1093/imrn/rnq135, in press.

[4] J. Brown, Saito–Kurokawa lifts and applications to the Bloch–Kato conjecture, Compos. Math. 143 (2) (2007) 290–322.

[5] J. Brown, Pullbacks of Eisenstein series on $U(3,3)$ and nonvanishing of Selmer groups, 2009, preprint, pp. 1–46.

[6] P. Garrett, Pullbacks of Eisenstein series applications, in: Automorphic Forms of Several Variables, Katata, 1983, in: Progr. Math., vol. 46, Birkhäuser, Boston, MA, 1984, pp. 114–137.

[7] P. Garrett, On the arithmetic of Siegel–Hilbert cuspforms: Petersson inner products and Fourier coefficients, Invent. Math. 107 (1992) 453–481.

[8] P. Garrett, Siegel's integral (personal correspondence), 2010, pp. 1–2.

[9] A. Pitale, R. Schmidt, Integral representations for L-functions for $\mathrm{GSp}_4\times\mathrm{GL}_2$, J. Number Theory 129 (2009) 1272–1324.

[10] A. Saha, L-functions for holomorphic forms on $\mathrm{GSp}_4\times\mathrm{GL}_2$ and their special values, Int. Math. Res. Not. IMRN 10 (2009) 1773–1837.

[11] A. Saha, Pullbacks of Eisenstein series and critical values for $\mathrm{GSp}_4\times\mathrm{GL}_2$, Pacific J. Math. 246 (2010).

[12] G. Shimura, Eisenstein series and zeta functions on symplectic groups, Invent. Math. 119 (1995) 539–584.