COMPUTATIONAL EVIDENCE FOR THE BLOCH-KATO CONJECTURE FOR ELLIPTIC MODULAR FORMS OF SQUARE-FREE LEVEL

MAHESH AGARWAL AND JIM BROWN

ABSTRACT. In this short note we provide computational evidence for the main result found in *On the Bloch-Kato conjecture for elliptic modular forms of square-free level*.

1. INTRODUCTION

Let $\kappa \geq 6$ be an even integer, M an odd square-free integer, and $f \in S_{2\kappa-2}(\Gamma_0(M))$ a newform. Let $\mathcal{V}$ be the motive associated to f and let $\mathcal{V}_\lambda$ be the λ-adic realization of $\mathcal{V}$ where λ is a prime of residue characteristic ℓ. In [1] we prove that under some reasonable assumptions that one direction of the λ-part of the Bloch-Kato conjecture for $\mathcal{V}_\lambda(\kappa-2)$ is true by bounding the ℓ-valuation of the order of the Bloch-Kato Selmer group of $\mathcal{V}_\lambda(\kappa-2)$ below by the ℓ-valuation of relevant special value of the L-function of f. We prove this by constructing a congruence between the Saito-Kurokawa lift of f and a cuspidal Siegel modular form. The two main hypotheses used in this result are non-vanishing modulo λ results about certain special values of L-functions associated to f. In this note we conjecture that one can always satisfy these hypotheses and provide computational evidence to support this conjecture.

2. MAIN HYPOTHESES AND CONJECTURES

In this section we discuss the two main hypotheses in used in the main theorem found in [1], namely, that there exists a fundamental discriminant $D < 0$ with $\gcd(\ell M, D) = 1$, $\chi_D(-1) = -1$ so that

$$\mathrm{ord}_\lambda(L_{\mathrm{alg}}(\kappa-1, f, \chi_D)) = 0$$

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and an there exists an integer $N > 1$ with $M \mid N$, $\ell \nmid N$, and an even Dirichlet character χ of conductor N so that

$$\mathrm{ord}_\lambda(L(3-\kappa,\chi)L_{\mathrm{alg}}(1,f,\chi)L_{\mathrm{alg}}(2,f,\chi)) = 0.$$

Consider first the central critical value $L_{\mathrm{alg}}(\kappa-1,f,\chi_D)$. There have been several results on the λ-divisibility of this particular special value due to its relation with the Fourier coefficients of the half-integral weight modular form $\mathcal{SH}_D(f)$. For example, it is shown in [3, Corollary 3] that for non-exceptional primes $\ell \nmid M$ there is a period Ω of f so that for infinitely many fundamental discriminants $D < 0$ one has

$$\mathrm{ord}_\lambda\left(\frac{D^{\kappa-3/2}L_{\mathrm{alg}}(\kappa-1,f,\chi_D)}{\Omega}\right) = 0.$$

As we are assuming $\overline{\rho}_{f,\lambda}$ is irreducible, ℓ is automatically a non-exceptional prime for f (see [5, Corollary 2] for example.) However, we are unable to apply this result in our situation as the period Ω used is not the canonical period Ω_f^+ that we are using to normalize the L-value. We are unaware of any known period relation between Ω and Ω_f^+.

We next consider $L(3-\kappa,\chi)$. It is well known that $L(3-\kappa,\chi) = -B_{\kappa-2,\chi}/(\kappa-2)$, which means that the λ-adic valuation of $L(3-\kappa,\chi)$ is given by that of $B_{\kappa-2,\chi}$ and so can be related to class numbers. For instance, let p be a prime with $p \neq \ell$, $n \geq 1$ and φ be a Dirichlet character. In this setting Washington proves ([6]) that for all but finitely many Dirichlet characters ψ of p-power conductor with $\varphi\psi(-1) = (-1)^n$ one has

$$\mathrm{ord}_\lambda(L(1-n,\varphi\psi)/2) = 0.$$

In our set-up we can take $n = \kappa-2$, $\chi = \varphi\psi$, and observe that $\chi(-1) = (-1)^\kappa = (-1)^{\kappa-2}$ to see there are infinitely many χ so that

$$\mathrm{ord}_\lambda(L(3-\kappa,\chi)) = 0.$$

If this were the only L-value controlled by χ we would be able to remove the hypothesis regarding this L-value. However, we need for $L_{\mathrm{alg}}(1,f,\chi)L_{\mathrm{alg}}(2,f,\chi)$ to be a λ-adic unit, which means χ must be chosen so these L-values are simultaneously λ-adic units. This leads to two conjectures on the ℓ-divisibility of twists of special values of L-functions. The first conjecture is given as follows.

Conjecture 1. Let M be an odd square-free integer and $\kappa \geq 6$ an even integer. Let $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M),\mathcal{O})$ be a newform with $\mathcal{O}$ a suitably large finite extension of $\mathbb{Z}_\ell$ and $\ell > 2\kappa-2$. Let λ be a prime of $\mathcal{O}$ dividing $L_{\mathrm{alg}}(\kappa,f)$. Then there exists a Dirichlet character χ of conductor N with $M \mid N$ so that $\lambda \nmid L_{\mathrm{alg}}(1,f,\chi)L_{\mathrm{alg}}(2,f,\chi)$.

Our second conjecture is a strengthening of the first conjecture. Note we present these as two separate conjectures due to the fact that we can provide slightly more computational evidence for Conjecture 1 as will be discussed below.

Conjecture 2. Let M be an odd square-free integer and $\kappa \geq 6$ an even integer. Let $f \in S^{\text{new}}_{2\kappa-2}(\Gamma_0(M), \mathcal{O})$ be a newform with $\mathcal{O}$ a suitably large finite extension of $\mathbb{Z}_\ell$ and $\ell > 2\kappa - 2$. Let λ be a prime of $\mathcal{O}$ dividing $L_{\text{alg}}(\kappa, f)$. Then there exists a Dirichlet character χ of conductor N with $M \mid N$ so that $\lambda \nmid L(3-\kappa, \chi)L_{\text{alg}}(1, f, \chi)L_{\text{alg}}(2, f, \chi)$.

3. Evidence

We now provide some evidence for these conjectures in the case that M and κ are relatively small. We restrict ourselves to computing using MAGMA ([2]) for the values $L_{\text{alg}}(1, f, \chi)L_{\text{alg}}(2, f, \chi)$ and Sage ([4]) for $L(3-\kappa, \chi)$. We are restricted to the cases of $M = 1, 3, 5$ due to the memory involved in doing such computations. This restriction is forced due to large sizes of the spaces of newforms as κ and M increase. We are also restricted to quadratic characters due to our computational methods used in MAGMA.

For Conjecture 1 every relevant example we checked worked with the simplest possible character. However, Conjecture 2 is significantly more difficult to compute examples for. One difficulty with the second conjecture is due to the fact that since κ is assumed to be even, $L(3-\kappa, \chi) = 0$ unless χ is an even character. As mentioned above we are also forced to use quadratic characters. Combining these restrictions with the fact that the conductors must remain small for computational purposes and we are essentially forced to use $\chi = \chi_5$. The only case where Conjecture 2 was computable and did not work is the case of level $\Gamma_0(5)$ and weight 18. In this case $19 \mid L_{\text{alg}}(10, f)$ and $19 \mid L(-7, \chi_5)$. We now briefly explain our calculations.

Let $f^{(1)}, \ldots, f^{(r)}$ denote a $\operatorname{Gal}\mathbb{Q}$-conjugacy class of newforms in $S^{\text{new}}_{2\kappa-2}(\Gamma_0(M))$. This determines an irreducible space of cuspidal modular symbols defined over $\mathbb{Q}$. We denote this space by T. For $j \in \{1, 2, \ldots, 2\kappa - 3\}$, we define

$$L_{\text{alg}}(j, T) = \prod_{i=1}^{r} L_{\text{alg}}(j, f^{(i)}).$$

MAGMA allows one to compute the rational number $L_{\text{alg}}(j, T)$. We now give a detailed account of a particular example, and then content ourselves to list the rest of the examples.

The following detailed example provides evidence for Conjecture 2.

Example 3. We let $2\kappa - 2 = 22$ and $M = 5$ for this example. Let $S = S^{\text{new}}_{22}(\Gamma_0(5))$. Working in MAGMA notation, we consider the subspaces given by $S[1]$ and $S[2]$. The space $S[1]$ has dimension 6 and corresponds to the newform

$$f_1^{(1)} = q + \alpha q^2 + \frac{1}{23}(-2\alpha^2 - 196\alpha + 5169498)q^3 + (\alpha^2 - 2097152)q^4 + \cdots$$

where α is a root of the polynomial $x^3 + 1312x^2 - 2780624x - 2939762688$ and the space $S[2]$ has dimension 8 and corresponds to the newform

$$f_2^{(1)} = q+\beta q^2+\frac{1}{2592}(-\beta^3+430\beta^2+5624840\beta-1581976320)q^3+(\beta^2-2097152)q^4+\cdots$$

where β is a root of the polynomial $x^4-2910x^3-4542888x^2+15642931840x-4309053579264$. Write $f_1^{(1)},\dots,f_1^{(6)}$ for the Galois conjugates of $f_1^{(1)}$ and $f_2^{(1)},\dots,f_2^{(8)}$ for the Galois conjugates of $f_2^{(1)}$. Then we have

$$L_{\text{alg}}(j,S[1])=\prod_{i=1}^{6}L_{\text{alg}}(j,f_1^{(i)})$$
$$L_{\text{alg}}(j,S[2])=\prod_{i=1}^{8}L_{\text{alg}}(j,f_2^{(i)}).$$

For $n=1,2$ we use MAGMA to determine that $643\mid L_{\text{alg}}(12,S[1])$ and $5747117\mid L_{\text{alg}}(12,S[2])$.

We now focus on $S[1]$ as the other case is handled in exactly the same way. Let K be the number field given by adjoining α to $\mathbb{Q}$. Let $\mathcal{O}_K$ be the ring of integers of K. Thus, there is a prime $\lambda\subset\mathcal{O}_K$ over 643 so that $\lambda\mid L_{\text{alg}}(12,f_1^{(1)})$. We now show that if we set $\chi=\chi_5$ our conjecture is true. In order to calculate $L_{\text{alg}}(1,f,\chi)$, we observe that $L_{\text{alg}}(1,f,\chi)=L_{\text{alg}}(1,f_\chi)$ for $f_\chi\in S_{22}^{\text{new}}(\Gamma_0(25),\chi^2)$ the newform obtained by twisting f by χ. We use MAGMA to create the space of modular symbols corresponding to this space, call it S_χ. One now just calculates the L-values $L_{\text{alg}}(1,S_\chi[n])$ and $L_{\text{alg}}(2,S_\chi[n])$ for $n=1,\dots,9$ to see that they are relatively prime to 643 as desired where 9 is the number of Galois conjugacy classes of newforms in S_χ.

We now give tables for the primes $\ell>2\kappa-2$ with ℓ dividing the appropriate L-value, along with the character used to satisfy the conjecture. Note that for level $\mathrm{SL}_2(\mathbb{Z})$ there is always only one Galois conjugacy class of newforms so we omit that from the table. For level $\Gamma_0(M)$ with $M>1$, we have a separate column listing n where n is the conjugacy class $S[n]$ where the form is found using the above notation for S.

The first table for level $\mathrm{SL}_2(\mathbb{Z})$ is relevant for both conjectures where $\chi=\chi_5$, but is only relevant for Conjecture 1 when $\chi=\chi_{-3}$. This is because $L(3-\kappa,\chi_{-3})=0$. As such, the table for level $\Gamma_0(3)$ is relevant only for Conjecture 1 where the table for $\Gamma_0(5)$ applies to both conjectures. The table for $\Gamma_0(3)$ is all with $\chi=\chi_{-3}$ and the table for $\Gamma_0(5)$ is all with $\chi=\chi_5$, so we omit the characters from the table. It should also be noted that when the character is χ_5, we have checked that Conjecture 2 is valid, where if the character is χ_{-3} the table only indicates where Conjecture 1 has been checked.

Table 1: Forms of level $\mathrm{SL}_2(\mathbb{Z})$

$2\kappa - 2$	p	χ
42	1423	χ_5
46	83	χ_5
	157	χ_5
54	516223	χ_5
58	12457	χ_5
	70457	χ_5
62	149	χ_{-3}
	27706741	χ_{-3}
66	116639	χ_{-3}
	14605602473899	χ_{-3}
70	22549	χ_{-3}
	1869909325769	χ_{-3}
74	211	χ_{-3}
	58613	χ_{-3}
	2523187	χ_{-3}
	44047987	χ_{-3}
78	547	χ_{-3}
	4425713	χ_{-3}
	1620600215209	χ_{-3}
	4121958671029649	χ_{-3}
82	439	χ_{-3}
	2799323278130215509378285376179150224240 9	χ_{-3}
86	220369	χ_{-3}
	420303476230608718280882472590477248 01	χ_{-3}
90	4873723027	χ_{-3}
	76072832455593285107519	χ_{-3}
	166234420403638612424126983	χ_{-3}
94	107	χ_{-3}
	293	χ_{-3}
	634681	χ_{-3}
	3892369040821801	χ_{-3}
	356725786558457817292127829989891675270677	χ_{-3}
98	905929130295004255323221710842184575206546879031874772606874 1	χ_{-3}
102	821	χ_{-3}
	6659826527	χ_{-3}
	4040743568189	χ_{-3}
	2483164814870114539812997726590348996714535518516569	χ_{-3}
106	145723	χ_{-3}
	1185855274189	χ_{-3}
	10324153484994367	χ_{-3}
	1342779441459373807	χ_{-3}

	4061787151237311599	χ_{-3}
	5158972363737149254575899	χ_{-3}

Table 2: Forms of level $\Gamma_0(3)$

$2\kappa - 2$	n	p
22	2	59
26	1	31
	2	1987
30	1	59
	2	8803
34	1	173
		1262893
	2	28137589
38	1	67
		5413
	2	5639
		1478280899
42	1	8674292309
	2	1587996761303
46	1	257
		48450529855090691
	2	67
		761
		63697
		221687857
50	1	2731
		5869
		19553
		60017
	2	113
		2566431039463
		20282395300337
54	1	182431
		182379239015315248421
	2	9087839
		8467292699
		8077851743053
58	1	79
		5533963
		20048689
		211085779657745477227O1
	2	9005743
		5941844304708688142671044249523

62	1	131
		706535743289846291366720003502598093747
62	2	239
		683
		769
		51673691443395602572890553163719258308799181
66	1	79
		367
		3691
		4561
		48247
		477154729
		2447505820819
	2	449
		7937
		14723
		226189
		3401465858983921
		13415063357419692383
70	1	22933560736008655510317391
		18872507689228115498600821646303009
	2	3570869
		4292759
		4002570937312008602889068909029626649472666011168483
74	1	97
		2823782483
		3247814366740389167750977464219804454672395910843996351
	2	2861
		15307
		977447906509122866984325509134428216106126790433835596596041669291325666771
78	1	475567730893442471551387498063600201329146259788806270454211843850798674527
	2	13297
		38113
		790481720293
		75938431261972977691967
		2509540592987488173577843598008452099443

Table 3: Forms of level $\Gamma_0(5)$

$2\kappa - 2$	n	p
22	1	643
	2	5747117
26	1	211
	2	932457391
30	1	1125063119
	2	220256171797
34	1	16229
		1575631
		22394087
	2	887
		976062821
		2928363533
38	1	199
		30305373187485315061039
	2	36383
		68699
		719223
		7743683
		17356170869
42	1	67741
		238781
		688759026572423
	2	271
		1237
		33809
		13096189
		566670938148211607
46	1	53
		179
		529
		1220599
		6338531
		20294558749069638633359
	2	67
		127
		157
		24618526540095105171397507491553617934953191
50	1	13331
		1120377270386712655804673055968742017507738517667
	2	4099
		311228684644799

		3657959022480707140006824948242484931529344268666641
54	1	15641
		74101
		9606026484276778800028639
		14146625075080508937462873089
	2	67
		5147
		697096694940722689
		1210061460886656457944560995963319131602423445745528453
58	1	15541741351
		472346356950959479355161
		228877808464406868525493918115986639507019879749
	2	101
		119932931985149378733306187728588657087358307307626725440 $\cdots$ 8516068913114892925546404698424100493063

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Department of Mathematics and Statistics, University of Michigan, Dearborn, Dearborn, MI 48128

E-mail address: mkagarwa@umd.umich.edu

Department of Mathematical Sciences, Clemson University, Clemson, SC 29634

E-mail address: jimlb@clemson.edu