Weinstein
5-18-11

Nonabelian local class field theory, and the geometry of Lubin-Tate spaces

Lecture 1: Formal groups and formal modules:

Class field theory circa 1930:

local / global field F

- classify ab. ext. of F.

If F is local then

$$\left\{\begin{matrix}\text{fin. ab. ext. of} \\ F\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finite index} \\ \text{subgrps of } F^\times\end{matrix}\right\}$$

Restatement: $\exists$ Artin map.

$$\mathrm{rec}_F : F^\times \longrightarrow \mathrm{Gal}(F^{ab}/F)$$

cont., dense image, and if F non-arch it is inj.

At this time the construction was completely global in nature. There was also no explicit construction of F^{ab}. We deal with this second problem.

F nonarch. $\varpi \in F^\times$ unif.

$$\mathrm{Gal}(F^{ab}/F) \simeq \hat{F}^\times \simeq \mathcal{O}_F^\times \times \varpi^{\hat{\mathbb{Z}}}$$

$$\begin{tikzcd} & F^{ab} \arrow[dl, dash, "\mathcal{O}_F^\times"'] \arrow[dr, dash, "\pi^{\hat{\mathbb{Z}}}"] & \\ F^{nr} \arrow[dr, dash] & & F_\varpi \arrow[dl, dash] \\ & F & \end{tikzcd}$$

$$F^{ab} = F^{nr} F_\varpi .$$

Case : $F = \mathbb{Q}_p$ $\varpi = p$.

$$F^{ab} = F^{cycl}$$
$$= F^{nr} \underbrace{F(\zeta_{p^\infty})}_{F_p}.$$

$\mathbb{G}_m$: p-power torsion generates F_p.

Want some object : $G \circlearrowleft \mathcal{O}_F^*$,
$$G[\varpi^n] \circlearrowleft \mathcal{O}_F / \varpi^n \mathcal{O}_F$$
free rk 1

Lubin-Tate formal group laws:

Let A = comm. ring.
A 1-dimensional formal group law G/A is a power series $G(x,y) \in A[[x,y]]$ s.t.

- $G(x,y) = x + y + O((x,y)^2)$
- $G(x,y) = G(y,x)$
- $G(G(x,y),z) = G(x, G(y,z))$
- $\exists\, i(x) \in A[[x]]$ s.t $G(i(x), x) = 0$.

We write
$$x +_G y = G(x,y).$$

Examples: 1) $x +_{\hat{\mathbb{G}}_a} y = x + y$

2) ~~$x+_G$~~
$$x +_{\hat{\mathbb{G}}_m} y = (1+x)(1+y) - 1 = x + y + xy.$$
(ones deal w/ origin issue)

A <u>homomorphism</u> $f: G' \to G$ is a power series $f(X) \in A[[X]]$ s.t. $f(X +_{G'} Y) = f(X) +_G f(Y)$.

Then $\mathrm{End}(G)$ is a ring. So there is a map
$$\mathbb{Z} \longrightarrow \mathrm{End}\, G$$
$$n \longmapsto [n]_G(X) = nX + O(X^2).$$

F = nonarch. local field, $i: \mathcal{O}_F \to A$ ← $\mathcal{O}_F$-alg.
A formal $\mathcal{O}_F$-module law G/A is a formal group law together with $\mathcal{O}_F \to \mathrm{End}\, G$, $a \longmapsto [a]_G(X) = aX + O(X^2)$.

<u>Examples:</u> 1) $\hat{\mathbb{G}}_a/\mathcal{O}_F$ $\quad [a]_{\hat{\mathbb{G}}_a}(X) = aX$.

2) $\hat{\mathbb{G}}_m/\mathbb{Z}_p$ is a formal $\mathbb{Z}_p$-module law.
$$[a]_{\hat{\mathbb{G}}_m}(X) = (1+X)^a - 1 = \sum_{n \geq 1} \binom{a}{n} X^n$$
$$= aX + O(X^2).$$

$$[p]_{\hat{\mathbb{G}}_m}(X) \equiv X^p \pmod{p}.$$

G is a Lubin-Tate formal $\mathcal{O}_F$-module law if
$$[\varpi]_G(X) \equiv X^q \pmod{\varpi}$$
where $q = \#(\mathcal{O}_F/\varpi\mathcal{O}_F)$.

<u>Thm (L-T, '66):</u> Let $f(X) \in \mathcal{O}_F[[X]]$ be s.t.

- $f(X) = \varpi X + O(X^2)$
- $f(X) \equiv X^q \pmod{\varpi}$

Then $\exists!$ Lubin-Tate $G/\mathcal{O}_F$ s.t. $[\varpi]_G(X) = f(X)$.

To construct F_ϖ: Let $f(X) = \varpi X + X^q$. Let G be as in the theorem.

$$G[\varpi^n] = \{ x \in m_{\bar{F}} : [\varpi^n]_G(x) = 0 \}.$$

__Observations:__

- $G[\varpi^n]$ is an $\mathcal{O}_F/\varpi^n\mathcal{O}_F$-module under $+_G$, $[a]_G$.
- $\# G[\varpi^n] = q^n$ ($[\varpi^n]_G(x) = X^{q^n} \pmod{\varpi}$, use Weierstrass prep. thm)
- $G[\varpi] = \ker(G[\varpi^n] \xrightarrow{\varpi} G[\varpi^{n-1}])$.

$\Rightarrow$ $G[\varpi^n]$ = free $\mathcal{O}_F/\varpi^n\mathcal{O}_F$ - module of rank 1

$$F_\varpi = F\Big(\bigcup_{n\geq 1} G(\varpi^n)\Big)$$

$$T_\varpi G = \varprojlim G[\varpi^n], \text{ free } \mathcal{O}_F\text{-module rk } 1$$

$$\rho : \mathrm{Gal}(F_\varpi/F) \longrightarrow \mathrm{Aut}(T_\varpi G) = \mathcal{O}_F^\times.$$

__Thm (L-T):__ ρ is an isomorphism and $F^{nr} F_\varpi = F^{ab}$
(local Kronecker-Weber thm).

__The Invariant Differential Form:__

Let G/A formal 1-dim. group law.

$\Omega^1_{G/A}$ = A-module of formal differentials. $P(T)\,dT$, $P(T) \in A[\![T]\!]$.

Let $\Sigma : A[\![T]\!] \longrightarrow A[\![X,Y]\!]$.

$$T \longmapsto G(X,Y).$$

Let $pr_1, pr_2 : A[\![T]\!] \rightrightarrows A[\![X,Y]\!]$

$$T \mapsto X, \quad T \mapsto Y$$

An invariant differential form $\omega \in \Omega^1_{G/A}$ is one satisfying

$$\Sigma_*(\omega) = (pr_1)_*(\omega) + (pr_2)_*(\omega).$$

{inv. diffs on G/A} is a free A-module of rk 1, spanned by $\omega = G_X(0,T)^{-1}dT$. (where $G_X = \frac{\partial G}{\partial X}$.)

If A is $\mathbb{Z}$-flat, $K = A \otimes_{\mathbb{Z}} \mathbb{Q} \supset$ ~~A~~ A.

Let

$$\log_G(T) = \int \omega \in \cancel{\quad} \; TK[\![T]\!]$$

- $\hat{\mathbb{G}}_a$: $\omega = dT$, $\log_{\hat{\mathbb{G}}_a}(T) = T$.
- $\hat{\mathbb{G}}_m$: $\omega = \frac{dT}{1+T}$ $\log_{\hat{\mathbb{G}}_m}(T) = \log(1+T)$.

(turns the operation into addition)

In fact, $\log_G : G_K \xrightarrow{\sim} (\hat{\mathbb{G}}_a)_K$

So there is nothing interesting in this case.

<u>Formal groups: Categorical definition:</u>

(Katz' paper on crystalline cohomology is good reference)

If G = formal grp law $/A$ and A' = "adic" A-algebra (complete for the I-adic top. for some $I \subset A'$).

$G(A') = \{$top. nilpotent elts of $A'\}$. It has group structure under $+_G$.

G is functor

$$\left\{\begin{matrix} \text{adic } A\text{-} \\ \text{algebras}\end{matrix}\right\} \longrightarrow \text{Ab Grps.}$$

A homomorphism $G' \to G$ is a natural trans. of functors.

An n-dimensional <u>formal Lie variety</u> $/A$ is a functor

$$V : \{\text{adic } A\text{-algs}\} \longrightarrow \text{Sets}.$$

which is isomorphic to the functor

$$A' \longrightarrow \{n\text{-tuples of top. nilp. elts of } A'\}.$$

$$= \mathrm{Hom}_{\substack{\text{cts} \\ A\text{-als}}}(A[\![x_1,\dots,x_n]\!], A')$$

This functor is $\mathrm{Spf}\, A[\![x_1,\dots,x_n]\!]$.

$$\mathrm{Lie}\, V = V\left(\frac{A[x]}{x^2}\right) \approx \mathrm{Hom}\left(A[\![x_1,\dots,x_n]\!], \frac{A[x]}{x^2}\right)$$

$$= \bigoplus_{i=1}^{n} A \frac{\partial}{\partial x_i} \qquad \left(\frac{\partial}{\partial x_i} : \begin{matrix} x_i \mapsto x \\ x_j \mapsto 0. \end{matrix}\right)$$

A formal group G/A is a group object in the

Category of formal Lie varieties /A.

If R is a ring, $\iota : R \to A$, a formal R-module /A is a formal group G/A together with $R \to \mathrm{End}(G)$ s.t. the derivative
$$\begin{tikzcd} R \arrow[r] \arrow[dr] & \mathrm{End}_A \mathrm{Lie}\, G \\ & A \arrow[u] \end{tikzcd}$$
factors through ι.

__Example:__ $V = \mathrm{Spf}\, A[[x_1, \dots, x_n]]$.

Each $f \in A[[x_1, \dots, x_n]]$ gives a natural transformation

$$\begin{tikzcd} \text{adic } A\text{-alss} \arrow[d] & V \arrow[r, "f"] & \text{Forgetful} \\ \text{Sets} & & \end{tikzcd}$$

$$\begin{array}{ccc} V(A') & \longrightarrow & A' \\ \shortparallel & & \\ \{(x_1, \dots, x_n)\} & \longmapsto & f(x_1, \dots, x_n) \end{array}$$

Given V a general formal Lie variety, define

$A(V)$ = set of natural transformations $V \to$ Forgetful.

$\approx A[[x_1, \dots, x_n]]$

If p is prime and G/A is a formal group

$$[p]_G^* : A(G) \longrightarrow A(G)$$

G is "p-divisible" if this map is finite.

__Example:__ $\hat{G}_a$ $\quad [p]_{\hat{G}_a}(x) = px$

$$[p]^*_{\hat{\mathbb{G}}_a} : A[\![X]\!] \to A[\![X]\!]$$
$$X \mapsto pX.$$

$\hat{\mathbb{G}}_a$ is not p-divisible if $p \notin A^*$

<u>Example:</u> $[p]_{\hat{\mathbb{G}}_m} : X \mapsto (1+X)^p - 1 = pX + \dots + X^p$
$$A[\![X]\!] \longrightarrow A[\![X]\!].$$

Presents $A[\![X]\!]$ as $A[\![X]\!]$-module free of rank p.

~~Genera~~

Generally, if G is p-divisible then rk is p^h, $h = \text{height}(G)$.

<u>Example:</u> $F/\mathbb{Q}_p$ finite, $\#$res. field $= q$.

LT formal $\mathcal{O}_F$-module G

$$[\varpi]_G(X) = \varpi X + \dots$$
$$\equiv X^q \pmod{\varpi}.$$

$q = p^f$

$e =$ ram. degree so $(\varpi^e) = (p)$.

$$[p]_G(X) \equiv X^{p^{ef}} + \dots \pmod{\varpi}$$

$fe = [F:\mathbb{Q}_p]$ so $ht(G) = [F:\mathbb{Q}_p]$.

The deRham complex:

If V/A is n-dim formal Lie variety (think $\hat{\mathbb{A}}^n_A$)

$$A(V) \cong A[\![X_1, \dots, X_n]\!]$$
$$\Omega^1_{V/A} \cong \hat{\bigoplus_{i=1}} A(V)\,dX_i$$
$$\Omega^k_{V/A} = \Lambda^k_{A(V)} \Omega^1_{V/A}$$

$$\Omega^\bullet_{V/A} \qquad A(V) \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to \cdots$$
$$\rightsquigarrow H^i_{dR}(V/A).$$

If A is a $\mathbb{Q}$-alg. then $H^i_{dR}(V/A) = 0$ for $i \geq 1$.
$\dim V = 1$

Let G/A be a formal 1-dim. group. We defined

$$\omega_G = \{\omega \in \Omega^1_{G/A} : \Sigma^*(\omega) = (pr_1)^*(\omega) + (pr_2)^*(\omega)\}.$$

$$G \times G \overset{pr_1}{\underset{pr_2}{\overset{\Sigma}{\rightrightarrows}}} G$$

$$\mathbb{D}(G/A) = \{[\omega] \in H^1_{dR}(G/A) : \Sigma^*(\omega) - (pr_1)^*(\omega) - (pr_2)^*(\omega) \text{ is exact}\}$$

Prop.: If A is $\mathbb{Z}$-flat, $K = A \otimes_{\mathbb{Z}} \mathbb{Q}$.

$$0 \to XA[\![X]\!] \to \{f \in XK[\![X]\!] : df \text{ integral}\} \xrightarrow{d} H^1_{dR}(G/A) \to 0$$
$$\cup| \qquad\qquad\qquad \cup|$$
$$0 \to XA[\![X]\!] \to \{f \in XK[\![X]\!] : f(x+y) - f(x) - f(y) \in A[\![x,y]\!]\} \to \mathbb{D}(G/A) \to 0$$

Example: • If $G = \hat{\mathbb{G}}_m$, then $\mathbb{D}(G/A) = A\omega$

$\uparrow$ inv. diff $= \frac{dT}{1+T}$

• If G is p-divisible and A is a p-adic ring, then generally $\mathbb{D}(G/A)$ is free A-module of rank $h = ht(G)$ and there exists a s.e.s.

$$0 \to \omega_G \to \mathbb{D}(G/A) \to \mathrm{Lie}(G^\vee) \to 0$$

Cartier dual

"$\mathbb{D}(G/A)$ only depends on G mod p".

Lemma: Let A be a $\mathbb{Z}_p$-flat p-adic ring. $f_1, f_2 \in XA[\![X]\!]$, $\omega \in A[\![X]\!]\,dX$. If $f_1 \equiv f_2 \pmod p$, then $f_1^*(\omega) - f_2^*(\omega)$ is exact.

Proof: $\int f_1^*(\omega) - f_2^*(\omega) = g(f_1) - g(f_2)$

(Let $\omega = dg$, $g \in K[\![X]\!]$, $K = A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$)

$= g(f_2 + p\Delta) - g(f_2)$ $\qquad \Delta \in XA[\![X]\!]$.

$= \sum_{n=1}^{\infty} \frac{p^n \Delta^n}{n!} g^{(n)}(f_2)$ $\qquad$ but $\frac{p^n}{n!} \in p\mathbb{Z}_p$

$\in A[\![X]\!]$. (b/c $g^{(1)} = \omega$ is integral, and so are higher derivatives). ∎

Thm: A as before, and $G, G'/A$ 1-dim formal groups.

1) $f: G' \to G$ morphism of Lie varieties
$f(0) = 0$

s.t. f mod p is a group homom. of formal groups

Then $f: \mathbb{D}(G/A) \to \mathbb{D}(G'/A)$

$\qquad \cap \qquad\qquad \cap$

$H^1_{dR}(G/A) \to H^1_{dR}(G'/A)$

2) If f_1, f_2, f_3 are 3 such maps and $f_3 \equiv f_1 + f_2 \pmod p$ then $(f_3)^* = (f_1)^* + (f_2)^*$ in $\mathrm{Hom}(\mathbb{D}(G/A), \mathbb{D}(G'/A))$.

Proof: See notes ~~provided~~ provided. ■

~~[illegible]~~

Let k be a perfect field of char. p, G_0/k p-adic formal group of height h. Define

$$M(G_0) = \mathbb{D}(G/W(k)) \qquad \text{(free of mod of rk } h/W(k))$$

G — any lift of G_0 to $W(k)$.

If G' is another lift, let $f: G' \to G$ be an isomorphism of underlying Lie varieties with $f(0)=0$, $f \equiv 1_G \pmod p$.

Get
$$f: \mathbb{D}(G/W(k)) \xrightarrow{\sim} \mathbb{D}(G'/W(k)).$$

This isomorphism does depend on $f: G' \to G$. Thus, $M(G_0)$ does not depend on the choice of lift G used.

These modules come with an action of Frobenius. Let $\sigma: k \xrightarrow{\sim} k$ is $x \mapsto x^p$ (k perfect so isom). So we get $\sigma: W(k) \xrightarrow{\sim} W(k)$.

$$F: G_0 \to G_0^{(\sigma)} \qquad G_0^{(\sigma)}(x,y) = [G_0(x,y)]^\sigma$$
$$F(x) = x^p.$$

This extends to
$$F: \mathbb{D}(G_0^{(\sigma)}/W(k)) \to \mathbb{D}(G_0/W(k)).$$
$$\parallel$$
$$\mathbb{D}(G_0/W(k)) \otimes_{W(k),\sigma} W(k) \to \mathbb{D}(G_0^{\sigma}/W(k))$$

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$$F: \mathbb{D}(G_0/W(k)) \longrightarrow \mathbb{D}(G_0/W(k))$$

$$F(\alpha x) = \alpha^{\sigma} F(x) \qquad x \in \mathbb{D}, \ \alpha \in W(k).$$

There is also $V \in \operatorname{End} \mathbb{D}$

$$FV = p.$$

Example: ~~[illegible]~~

1) $M((\hat{\mathbb{G}}_m)_{\mathbb{F}_p}) = $ rk 1 module $/ W(\mathbb{F}_p) = \mathbb{Z}_p$

$$= \mathbb{Z}_p \omega$$

$$F(\omega) = p\omega \quad \left(\begin{array}{c} F: (\hat{\mathbb{G}}_m)_{\mathbb{Z}_p} \longrightarrow (\hat{\mathbb{G}}_m)_{\mathbb{Z}_p} \\ F(X) = X^p \\ [p](X) = (1+X)^p - 1 \\ \equiv F(X) \pmod p \end{array} \right)$$

2) $E_0/\mathbb{F}_p$ formal group from super-singular elliptic curve

Can choose X so that

$$[p]_{E_0}(X) \equiv X^{p^2} \pmod p$$
$$\equiv F^2(X).$$

$M(E_0) = \mathbb{D}(E/\mathbb{Z}_p)$ $\qquad$ ω inv. diff on E

$$= \mathbb{Z}_p \omega \oplus \mathbb{Z}_p F(\omega)$$

$F^2 = p$

$F = \begin{pmatrix} 0 & p \\ 0 & 1 \end{pmatrix}.$

$\mathbb{Z}_p \omega \subset M(E_0)$

does not depend on E $\qquad$ does not depend on E.

A p-adic

$$\begin{array}{ccc} (\text{formal groups}/A) & & (\text{p-div. grps}/A) \\ \cup\!\shortmid & & \cup\!\shortmid \\ \left(\begin{array}{c}\text{formal groups which}\\ \text{are p-div}/A\end{array}\right) & \xrightarrow{\sim} & \left(\begin{array}{c}\text{connected}\\ \text{p-div grps}/A\end{array}\right) \end{array}$$

F acts nilp. here

$$G \longmapsto \{G[p^n]\}_{n\geq 1}.$$

Let A = flat $\mathbb{Z}_p$-alg. $I \subset A$ is a P.D. ideal, top. nilpotent.
(P.D. means $\forall x \in I$, $x^n/n! \in I$ $\forall n \geq 1$).
$A \longrightarrow A_0 = A/I$ is a nilpotent P.D. thickening
$(\mathbb{Z}_p \longrightarrow \mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p)$

__Earlier:__

$$\begin{array}{ccc} G/W(k) & \longrightarrow & \mathbb{D}(G/W(k)) \\ \big\uparrow \text{lift} & & \text{only depends} \\ G_0/k & & \text{on } G_0 \end{array}$$

$$\begin{array}{cccc} G/A & \longrightarrow & \mathbb{D}(G/A) & A\text{-module} \\ \big\uparrow & & \Big| & \text{does not depend on } G \\ G_0/A_0 & & \underbrace{\mathbb{D}(G_0)}_{\text{crystal } \mathcal{E}}(A \to A_0) & \end{array}$$

(Grothendieck - Messing)

The Lubin-Tate tower:

Set $k = \overline{\mathbb{F}}_p$, $W = W(k)$, $K_0 = W[1/p]$ ($h \geq 1$ fixed)

G_0/k 1-dim. formal group $/k$ of height h.

$$\begin{pmatrix} h=1 & \hat{\mathbb{G}}_m \\ h=2 & E \text{ s.s.} \\ \vdots & \end{pmatrix}$$

ART as in Howard's talks.

$M_0 : \text{ART} \longrightarrow \text{Sets}$

$$A \longmapsto \{(\underset{A}{G}, \iota)\}/\simeq$$

$$\iota : G_0 \xrightarrow{\sim} G \otimes_A k.$$

$\text{End}(G_0) = \mathcal{O}_B$, $B/\mathbb{Q}_p$ div. alg. dim h^2 (inv. $\frac{1}{h}$).

$\mathcal{O}_B^\times$ acts on M_0 $\qquad b \in \mathcal{O}_B^\times$

$$(G, \iota) \longmapsto (G, \iota \circ b)$$

Theorem (Lubin-Tate): M_0 is pro-representable by

$$\text{Spf } W[[u_1, \dots, u_{n-1}]] . \circlearrowleft \mathcal{O}_B^\times$$

$$M_0(A) = m_A^{h-1} \qquad (m_A = \text{max ideal of } A)$$

That gives the space without level.

With level structures:

~~M_A~~

$M_n : \text{ART} \longrightarrow \text{Sets}$

$$A \longmapsto \{(G, \iota, \alpha)\}/\simeq$$

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$$\alpha : \left(\mathbb{Z}/p^n\mathbb{Z}\right)^{\oplus h} \longrightarrow G[p^n](A)$$ Drinfeld level structure

$$\begin{array}{c} \vdots \\ \downarrow \\ M_2 \\ \downarrow \\ M_1 \\ \big\downarrow {\scriptstyle GL_h(\mathbb{F}_p)} \\ M_0 = \text{open ball} \\ \circlearrowleft \\ \mathcal{O}_B^\times \end{array} \qquad GL_h(\mathbb{Z}/p^2\mathbb{Z}) \text{ acting on } M_2 \to M_0$$

comes w/ valuation

$M = M_\infty \qquad K \supset K_0 = W(\overline{\mathbb{F}}_p)[\tfrac{1}{p}] \qquad \mathcal{O}_K \subset K$

$M(K) = \{(G, \iota, \alpha)\}$

- $G/\mathcal{O}_K$ formal group
- $\iota : G_0 \otimes_k \mathcal{O}_K/p\mathcal{O}_K \longrightarrow G \otimes_{\mathcal{O}_K} \mathcal{O}_K/p\mathcal{O}_K$ is a quasi-isogeny $\in \mathrm{Hom}(\ ,\ ) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.
- $\alpha : \mathbb{Q}_p^h \xrightarrow{\sim} V_p G = \varprojlim G[p^n] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is required to be $\mathrm{Gal}(\overline{K}/K)$-equiv.

Then $M \circlearrowleft GL_h(\mathbb{Q}_p) \times J \times W_{\mathbb{Q}_p}$ up to isogeny.

$J = B^\times$

Nonabelian LT theory:

$$\underset{GL_h(\mathbb{Q}_p) \times J \times W_{\mathbb{Q}_p}}{\overset{}{H_c^i(M, \mathbb{Q}_\ell)}} = \varinjlim H_c^i(M_n, \mathbb{Q}_\ell) \qquad (\ell \neq p)$$

Thm (Harris-Taylor, '02): Let π be a supercuspidal rep. of $GL_n(\mathbb{Q}_p)$. Then

$$\mathrm{Hom}_{GL_n(\mathbb{Q}_p)}(\pi, H^*(M, \bar{\mathbb{Q}}_\ell))$$

$$\parallel$$

$$\underset{\left(\substack{\text{rep of}\\ J}\right)}{JL(\pi)} \otimes \underset{\left(\substack{\text{rep. of}\\ W_{\mathbb{Q}_p}}\right)}{\mathrm{rec}(\pi)}$$

$JL(\pi)$ = Jacquet-Langlands

rec = Local-Langlands corres.

Period Maps:

$h \geq 1$, $k = \overline{\mathbb{F}}_p$, $W = W(k)$, $K_0 = W[\frac{1}{p}]$

G_0 = unique formal group, dim 1, ht h / k

(think: $h=1$, $G_0 = \hat{\mathbb{G}}_m$, $h=2$, $G_0 = \hat{E}_{s.s}$)

$\mathcal{E} = \mathbb{D}(G_0)$ rule

$$\mathbb{D}(G_0)(A \to k) = A\text{-module}$$

$\uparrow$ nilp. PD thickenings

$M(G_0) = \mathbb{D}(G/_W) = \mathcal{E}(W \to k)$

The Lubin-Tate tower

For $K \supset K_0$ valued field.

$M(K) = \{(G, \iota, \alpha)\}/\sim$

- $G/\mathcal{O}_K$ formal group
- $\iota : G_0 \otimes_k \mathcal{O}_K/p\mathcal{O}_K \to G \otimes_{\mathcal{O}_K} \mathcal{O}_K/p\mathcal{O}_K$ q-isog.
- $\alpha : \mathbb{Q}_p^h \to V_p(G) = T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Let $\mathcal{E} = \mathbb{D}(G \otimes \mathcal{O}_K/p\mathcal{O}_K)$, $\mathcal{E}_0 = \mathbb{D}(G_0 \otimes \mathcal{O}_K/p\mathcal{O}_K)$.

We construct the Gross-Hopkins map $M \to \mathbb{P}^{h-1}$.

$\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K$ is a PD-thickening.

Step 1: $\iota^*: \mathcal{E}(\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K)[\frac{1}{p}] \xrightarrow{\sim} \mathcal{E}_0(\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K)[\frac{1}{p}]$

$\nearrow$ K v.s of dim h.

$= \mathbb{D}(G_0)(W \to k) \otimes_W K$.

$= M(G_0) \otimes_W K$.

Step 2: Hodge s.e.s.

$$0 \to \omega_G \to \mathbb{D}(G/\mathcal{O}_K) \to \mathrm{Lie}\, G^\vee \to 0$$

dim 1   dim h dim $h-1$.

$$\omega_G[\tfrac{1}{p}] \hookrightarrow \mathbb{D}(G/\mathcal{O}_K)[\tfrac{1}{p}] = \mathcal{E}(\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K)[\tfrac{1}{p}].$$

Fil_G

Combining the two steps we have

$$\underset{\substack{\uparrow \\ \text{line}}}{\mathrm{Fil}_G} \subset M(G_0) \otimes_W K. \simeq K^h. \rightsquigarrow pt \in \mathbb{P}^{h-1}(K).$$

Thus, we have

$$\begin{array}{ccc} M(K) & \longrightarrow & \mathbb{P}^{h-1}(K) \\ (G, \iota, \alpha) & \longmapsto & \mathrm{Fil}_G. \end{array}$$

(depends only on G and ι).

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M_0 \arrow[dl, "\pi_{GH}"] \\ \mathbb{P}^{h-1} & \end{tikzcd}$$

$M_0 \simeq$ open ball of dim $h-1$

Theorem (Gross-Hopkins): The period map $M_0 \to \mathbb{P}^{h-1}$ is a morphism of rigid spaces that is étale and surjective. The fibers consist of isogeny classes.

$$\left(\begin{array}{c} \pi_1^{rig}(\mathbb{P}^{h-1}) \\ \downarrow \\ SL_h(\mathbb{Q}_p) \end{array} \right) \qquad \begin{tikzcd} M_0^\circ \arrow[r, phantom, "\subset"] \arrow[dr, bend right, "SL_h(\mathbb{Q}_p)"'] & M \arrow[r, "GL_h(\mathbb{Q}_p)"] \arrow[d, "GL_h(\mathbb{Q}_p)"] & M_0 \arrow[dl] \\ & \mathbb{P}^{h-1} & \end{tikzcd}$$

Thm (Strauch): M° = connected comp. of M.
Stabilizer in $GL_h(\mathbb{Q}_p)$ is $SL_h(\mathbb{Q}_p)$.

nilp. PD-thickening $\quad \theta : A \longrightarrow A_0 \quad$ p-adic rings,

$I = \ker\theta$ has PD-structure and is top. nilp.

$$x \in I \Rightarrow \frac{x^n}{n!} \in I.$$

We have been considering obvious such structures thus far:

$$W \to k, \quad \mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K, \ \ldots$$

$\mathcal{O}_K \to \mathcal{O}_K/m_K = \overline{\mathbb{F}}_p$ is one only if $e(K/\mathbb{Q}_p) \leq p-1$.

Fontaine's ring A_{cris}:

$\theta : A_{cris} \longrightarrow \mathcal{O}_{\mathbb{C}_p}$ top. nilp. PD-thickening, and is the universal one. In fact, Fontaine defines $A_{cris}(\mathcal{O})$ for any $\mathcal{O}$ satisfying:

(the Frob. map on $\mathcal{O}/p\mathcal{O}$ is surjective) "perfect"

<u>Examples:</u> 1) $A_{cris}(\mathbb{Z}_p) = \mathbb{Z}_p$

2) $A_{cris}(W) = W$

3) $\mathbb{Z}_p[\sqrt{p}]/p$ is not perfect.

4) $A_{cris}(\mathcal{O}_{\mathbb{C}_p}) = A_{cris}$.

(Fontaine): $A_{cris}(\mathcal{O}) = A_{cris}(\mathcal{O}/p\mathcal{O})$.

$\phi \circlearrowleft$ on $A_{cris}(\mathcal{O})$; $\circlearrowleft \phi : x \mapsto x^p$

- A_{cris} is a W-module.
- $\phi : A_{cris} \to A_{cris}$ is W-semilinear
- $\theta : A_{cris} \longrightarrow \mathcal{O}_{\mathbb{C}_p}$
- A_{cris} is <u>NOT</u> an $\mathcal{O}_{\mathbb{C}_p}$-module.
- ϕ and θ do not interact nicely.

$B^+_{cris} = A_{cris}[\frac{1}{p}]$ is a K_0-alg.

$\theta : B^+_{cris} \twoheadrightarrow \mathbb{C}_p$.

$\cdot (B^+_{cris})^{\phi=1} = \mathbb{Q}_p$.

$U_{\mathbb{Q}_p} := (B^+_{cris})^{\phi=p} = \{ x \in B^+_{cris} : \phi(x) = px \}$

$\simeq \{ (y^{(0)}, y^{(1)}, \dots) : y^{(n)} \in 1 + m_{\mathbb{C}_p},\ (y^{(n)})^p = y^{(n-1)} \}$

$(y_1 + y_2)^{(n)} = y_1^{(n)} y_2^{(n)}$

$a \in \mathbb{Z}_p \quad (ay)^{(n)} = (y^{(n)})^a$.

$\frac{1}{p}(y^{(0)}, y^{(1)}, \dots) = (y^{(1)}, y^{(2)}, \dots)$

So $U_{\mathbb{Q}_p}$ is a $\mathbb{Q}_p$-vector space.

$$\Theta : (B_{cris}^{+})^{\phi = p} \twoheadrightarrow \mathbb{C}_p$$
$$(y^{(0)}, y^{(1)}, \ldots) \longmapsto \log y^{(0)}$$

This map is $\mathbb{Q}_p$-linear. We have

$$0 \to \mathbb{Q}_p t \to U_{\mathbb{Q}_p} \xrightarrow{\Theta} \mathbb{C}_p \to 0$$
$$t = (1, \zeta_p, \zeta_{p^2}, \ldots) \qquad t = \text{“}2\pi i\text{''}$$

$$U_{\mathbb{Q}_p} = \{ (x^{(0)}, x^{(1)}, \ldots) : \; x^{(n)} \in \mathfrak{m}_{\mathbb{C}_p}, \; [p]_{\hat{\mathbb{G}}_m}(x^{(n)}) = x^{(n-1)} \}$$
$$x^{(n)} = y^{(n)} - 1 \qquad [p]_{\hat{\mathbb{G}}_m}(x) = (1+x)^p - 1$$
$$\simeq (B_{cris}^{+})^{\phi = p}.$$

$E/\mathbb{Q}_p$ finite extension, residue field $\mathbb{F}_q$, unif ϖ.

$$f(T) = \varpi T + T^q.$$

G = Lubin-Tate formal $\mathcal{O}_E$-module law with $[\varpi]_G = f$.

$$U_E = \{ (x^{(0)}, x^{(1)}, \ldots) : \; x^{(n)} \in \mathfrak{m}_{\mathbb{C}_p}, \; [\varpi]_G(x^{(n)}) = x^{(n-1)} \}.$$

This is a E-vector space:

- $(x_1 + x_2)^{(n)} = x_1^{(n)} +_G x_2^{(n)}$
- $a \in \mathcal{O}_E$, $(ax)^{(n)} = [a]_G(x^{(n)})$
- $\frac{1}{\varpi}(x^{(0)}, x^{(1)}, \ldots) = (x^{(1)}, x^{(2)}, \ldots)$

We have

$$\log_G : G_E \xrightarrow{\sim} (\hat{\mathbb{G}}_a)_E.$$

Define

$$\Theta : U_E \to \mathbb{C}_p \text{ by } \Theta(x) = \log_G x^{(0)}.$$

We again have an exact sequence:

$$0 \to E t_E \to U_E \xrightarrow{\Theta} \mathbb{C}_p \to 0$$

$$t_E = (0, \zeta_1, \zeta_2, \ldots)$$

ζ_1 = root of f, ..

<u>Prop:</u> $E = W(\mathbb{F}_{p^h})[\frac{1}{p}]$. Then

$$(B^+_{cris})^{\phi^h = p} \simeq U_E.$$

Let $G/\mathcal{O}_{\mathbb{C}_p}$ be a p-div. grp.

$$T_p G = \varprojlim G[p^n]$$

$$\simeq \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G)$$

$$V_p G = \mathrm{Hom}^\circ(\mathbb{Q}_p/\mathbb{Z}_p, G)$$

1-dim A_{cris}-module.

$$\mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)(A_{cris} \to \mathcal{O}_{\mathbb{C}_p}) = A_{cris} \cdot e$$

$\circlearrowleft \varphi$ $\qquad \varphi(e) = e.$

$$\mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)^\vee (A_{cris} \to \mathcal{O}_{\mathbb{C}_p}) = A_{cris} \cdot e$$

$$\varphi(e) = pe.$$

$$\begin{aligned} V(G) &\to \mathrm{Hom}(\mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)(A_{cris} \to \mathcal{O}_{\mathbb{C}_p})^\vee[\tfrac{1}{p}], \mathbb{D}(G)(A_{cris} \to \mathcal{O}_{\mathbb{C}_p})^\vee[\tfrac{1}{p}]) \\ &= (\mathbb{D}(G)(A_{cris} \to \mathcal{O}_{\mathbb{C}_p})^\vee[\tfrac{1}{p}])^{\varphi = p}. \\ &\overset{i_*^{-1}}{\simeq} \mathbb{D}(G_0)(\quad '' \quad)^{\varphi = p} \\ &= (M(G_0)^\vee \otimes_W B^+_{cris})^{\varphi = p}. \end{aligned}$$

$$M(G_0) = \bigoplus_{i=1}^h W e_i$$

$$F = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ p & & & 0 \end{pmatrix}, \quad F^h = p$$

$$(M(G_0)^\vee \otimes_W B_{cris}^+)^{\varphi=p} \simeq (B_{cris}^+)^{\varphi^h=p} = U_E.$$

$V(G) \longrightarrow U_E$

pt on $M(\mathbb{C}_p) \longleftrightarrow (G, \iota, \alpha)$ $\qquad \alpha \longleftrightarrow h$ elts of $V(G)$.

$\qquad \rightsquigarrow h$ elts of U_E

$\Pi : M \to U_E^h$

<u>Thm (Fargues):</u>

$$\begin{tikzcd}
M \arrow[r] \arrow[d] \arrow[dd, bend right=60, "G.H."'] & (U_E)^h \arrow[d, "\Theta"] & (x_1, \dots, x_h) \arrow[d, mapsto] \\
(Mat_h(\mathbb{C}_p))^{rk = h-1} \arrow[r] \arrow[d, "span"] & Mat_h(\mathbb{C}_p) & \theta(\varphi^j(x_i)) \\
\mathbb{P}^{h-1} & &
\end{tikzcd}$$